Nondegenerate linearisable centres of complex planar quadratic and symmetric cubic systems in $\mathbb{C}^2$

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CRM-2542
April 1998

This work was supported by NSERC and FCAR
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Abstract

In this paper we consider complex differential systems in the plane, which are linearisable in the neighborhood of a nondegenerate centre. Several algorithms are described which allow us to find necessary conditions for linearisability. These are applied to the class of complex quadratic systems and to the class of complex cubic systems symmetric with respect to a centre.

The sufficiency of these conditions is shown by exhibiting explicitly a linearising change of coordinates, either of Darboux type or a generalisation of it. We also apply our results to the corresponding real systems having a linearisable integrable saddle.

1991 *Mathematical Subject Classification.* 34C,58F

Résumé

Dans cet article on considère des systèmes différentiels complexes dans le plan, qui sont linéarisables au voisinage d'un centre non-dégénéré. Plusieurs algorithmes sont décrits, qui permettent de trouver des conditions nécessaires pour la linéarisabilité. Ils sont appliqués à la classe des systèmes quadratiques complexes et à la classe des systèmes complexes cubiques symétriques par rapport à l'origine.

La suffisance des conditions est montrée en construisant explicitement un changement de coordonnées linéarisant, soit de type de Darboux, soit par une généralisation des fonctions de type Darboux. On applique ces résultats aux systèmes réels quadratiques ou cubiques symétriques ayant un point de selle intégrable linéarisable avec résonance 1:1.

1. Introduction

This paper originated from the interest of the two authors in isochronous centres ([CD], [MRT] and [MMR]). The study of isochronous systsems started before the development of differential calculus when Huygens investigated the cycloidal pendulum. This pendulum has isochronous oscillations in contrast to the monotonicity of the period of the usual pendulum.

More recently several authors have made a systematic study of the isochronous centres inside certain classes of systems with centre ([CJ], [L], [P], [RT1], [De]). The first four papers study isochronous centres inside quadratic systems and cubic systems symmetric with respect to its centre. This study was made possible by the fact that the centre conditions are known in both these cases.

Unfortunately, there are very few "natural" families in which the centre conditions are known, although a number of mechanisms producing strata of centres are well known. The work of Devlin [De] deals with the centres for which there is an integrating factor $(x^2+y^2)^\alpha$.

In [CD] the Kukles family of systems was studied:

$$\begin{aligned}\dot{x} &= -y\\ \dot{y} &= x + a_1x^2 + a_2xy + a_3y^2 + a_4x^3 + a_5x^2y + a_6xy^2 + a_7y^3.\end{aligned} \tag{1.1}$$

An original feature of this work is that although the centre conditions are not known, it is still possible to find necessary and sufficient conditions for a isochronous centre in this family.

The paper [CD] brings to light some other interesting questions, one of which is the starting point for our investigations here. In looking for the necessary conditions for an isochronous centre in the Kukles system, a new set of conditions was found which cannot be satisfied for real systems in the family. However if we allow the coefficients a_i to be complex, then the conditions can be satisfied, and the origin is again isochronous. The notion of isochronicity still makes sense in this context as long as we keep to real time.

In [CD] it was shown that a singular point is a complex isochronous centre if and only if a whole punctured neighborhood of the point consists entirely of closed trajectories. This interesting result thus gives a geometric characterization of isochronicity. This is completely different from the real case, where any centre can be made isochronous by multiplying by a suitable positive function. In the complex case, multiplying by a non-constant function changes the geometry of the (real time) trajectories.

Simultaneous to these investigations, there has been a growing interest in the complex centres of systems in $\mathbb{C}^2$, a centre being a non-degenerate singular point with zero trace and a local analytic first integral. Note that integrable saddles also come within this catagory. Complex centres in quadratic systems have been studied by several mathematicians, the first being Dulac [Du]. The most general and elegant presentation is provided by Farell [F].

Putting together the results of these works it becomes natural to ask what are the complex isochronous quadratic systems. This raises the need of a good definition of a complex isochronous centre. Using a real time for such a system is somewhat artificial, especially since it excludes the natural identification of centres and saddles when working over the complex numbers. Our starting point here, therefore, is to introduce the equivalent notion of *linearisable* centre. That is, a system whose centre can be reduced to its linear parts by a change of coordinates.

In this paper we classify the linearisable complex quadratic and symmetric cubic centres. For that purpose we introduce in Section 2 three equivalent algorithms to give necessary conditions for linearisability. The calculations were made independently, the first author using REDUCE, and the second MAPLE, to provide a verification that the calculations contained no mistakes.

From the calculations emerged a set of candidates for necessary and sufficient conditions. The paper [MMR] introduces a new method to prove isochronicity via Darboux linearising change of coordinates. A refinement of this method allows us to give explicit linearising changes of coordinates for each of these conditions. Thus we obtain a complete set of necessary and sufficient conditions.

One important contribution of these investigations to the study of isochronicity is that there seems to be a far wider range of linearisable centres if we allow the coefficients to be complex. This is in distinction to the apparent case of integrability, where the corresponding strata are very similar. However, if we consider linearisable saddles instead, then the complex and real cases are comparable.

A simple reason for this situation is that for real centres, the separatrices are conjugate and there can be no independence in how a real linearising transformation can act on either one. In the complex case and in the case of a real saddle this independence is preserved.

It seems, therefore, that a better understanding of isochronicity as a phenomena should be obtained from considering linearisable saddles in more detail. As an approach to this we classify the integrable saddles in the complex case in the final section, and study their strata. [We also consider briefly in an appendix the possibility of linearising just one of the separatrices—an operation which has no analogue in the real centre case.]

In all studies of this nature, there is a need to have "visible" phenomena to spur on investigation. In the case of a real isochronous centre, it is easy to see that no finite critical point can lie on the boundary of the period anulus attached to the centre; a result which has been used in several classification problems. If we consider linearisable saddles, then this phenomena only remains valid with complex time, and consequently looses its power. However, from our investigations we conjecture the following obstruction to linearisability, which we hope will provide an impetus to further investigation along these lines: *No linearisable saddle can lie on a homoclinic loop or, more generally, on a monodromic graphic.*

The paper is organized in the following way. Section 2 contains generalities: the definitions of complex and linearisable centres; a description of three algorithms to find necessary conditions for linearisability; the method of construction of Darboux linearising change of coordinates and its refinements. In section 3 (rep. 4) we determine the necessary and sufficient conditions for a quadratic system (resp. symmetric cubic system) to have a linearisable centre and give the linearising change of coordinates. In section 5 we concentrate our attention to linearisable and integrable saddles.

2. Generalities

2.1 Complex centres and linearisable centres.

We work here with the classical definition of a centre for a complex system.

Definition 2.1.

(1) *A singular point of a complex system*

$$\begin{aligned}\dot z &= P(z,w)\\ \dot w &= Q(z,w)\end{aligned} \tag{2.1}$$

in $\mathbb{C}^2$ is a centre if the system has an analytic first integral in a neighborhood of the singular point.

(2) *A centre is nondegenerate if the system has a non-vanishing 1-jet at the singular point which is a Morse singular point of an analytic first integral.*

(3) *A nondegenerate centre is linearisable if there exists an analytic change of coordinates in the neighborhood of the singular point, bringing the system to a linear system.*

Remark 2.2. A nondegenerate centre necessarily has two opposite eigenvalues.

From now on, we will always restrict ourselves to nondegenerate centres.

Proposition 2.3. *Let us suppose that the system* (2.1) *has a nondegenerate centre at the origin and is of the form*

$$\begin{aligned}\dot z &= iz + p(z,w) = iz + o(|(z,w)|^2)\\ \dot w &= -iw + q(z,w) = -iw + o(|(z,w)|^2).\end{aligned} \tag{2.2}$$

Then the origin is linearisable if and only if there exists a neighborhood of the origin such that every trajectory inside that neighborhood is periodic.

Proof. Any trajectory of the linear system

$$\begin{aligned}\dot Z &= iZ\\ \dot W &= -iW\end{aligned} \tag{2.3}$$

is periodic, which implies the same result for the linearisable system. Conversely, if the system (2.2) is not linearisable then it can be brought to a normal form

$$\begin{aligned}\dot Z &= iZ + c_j Z^{j+1}W^j + \dots\\ \dot W &= -iW + d_j Z^j W^{j+1} + \dots,\end{aligned} \tag{2.4}$$

with at least one of c_j, d_j nonzero. Let us suppose it is c_j. Then any neighbourhood of the origin contains nonperiodic solutions. Indeed consider $R = ZW$. Then

$$\dot R = \sum_{k\geq j}(c_k + d_k)R^{k+1}. \tag{2.5}$$

If one of the $c_k + d_k \neq 0$, then it is clear that the system (2.5) has non periodic solutions. If all $c_k + d_k = 0$, then R is constant. Suppose that $R = \epsilon$, then (2.4) gives

$$\dot Z = Z(i + c_j\epsilon^j + c_{j+1}\epsilon^{j+1} + \cdots), \qquad c_j \neq 0.$$

Clearly there are values of $\epsilon \in \mathbb{C}$ arbitrarily close to 0 for which the expression in parentheses has a non-zero real part, and hence the trajectories cannot all be closed. □

2.2 Algorithms to find necessary conditions for linearisability.

First algorithm: look for a linearising change of coordinates. Let us start with a system (2.2). We look for a linearising change of coordinates of the form

$$(Z,W) = (z + f(z,w), w + g(z,w)) = (F(z,w), G(z,w)). \tag{2.6}$$

We must have

$$\begin{aligned} iF(z,w) &= \frac{\partial F}{\partial z}\dot{z} + \frac{\partial F}{\partial w}\dot{w} \\ -iG(z,w) &= \frac{\partial G}{\partial z}\dot{z} + \frac{\partial G}{\partial w}\dot{w}. \end{aligned} \tag{2.7}$$

Taking series

$$\begin{aligned} F(z,w) &= z + \sum_{j+k\geq 2} a_{jk} z^j w^k \\ G(z,w) &= w + \sum_{j+k\geq 2} b_{jk} z^j w^k \end{aligned} \tag{2.8}$$

and equating coefficients of monomials $z^j w^k$ in (2.7) allows to determine uniquely the a_{jk} and b_{kj}, when $j-k \neq 1$, as long as some compatibility conditions (the conditions of linearisability) are met. The $a_{j+1,j}$ and $b_{k,k+1}$ can be taken arbitrary. The compatibility conditions come from the $z^{j+1}w^j$ (resp. $z^j w^{j+1}$) terms in the first (resp. second) equation of (2.7).

Second algorithm: look for a normalizing change of coordinates. We bring the system (2.2) to normal form. The system is linearisable if and only if the normal form is linear. The way to bring (2.2) to normal form is well known. We look for a change of coordinates

$$(z,w) = (F(Z,W), G(Z,W)) = (Z + \sum_{j+k\geq 2} A_{jk} Z^j W^k, W + \sum_{j+k\geq 2} B_{jk} Z^j W^k) \tag{2.9}$$

satisfying the equations

$$\begin{aligned} \frac{\partial F}{\partial Z}\dot{Z} + \frac{\partial F}{\partial W}\dot{W} &= iF(Z,W) + p(F(Z,W), q(Z,W)) \\ \frac{\partial G}{\partial Z}\dot{Z} + \frac{\partial G}{\partial W}\dot{W} &= -iG(Z,W) + q(F(Z,W), q(Z,W)), \end{aligned} \tag{2.10}$$

where

$$\begin{aligned} \dot{Z} &= iZ + \sum_{j\geq 1} c_j Z^{j+1} W^j \\ \dot{W} &= -iW + \sum_{j\geq 1} d_j Z^j W^{j+1}. \end{aligned} \tag{2.11}$$

The conditions of linearisability are the conditions $c_j = d_j = 0$, for all $j \geq 1$.

Third algorithm: the algorithm of Christopher and Devlin. This algorithm first appeared in [CD]. Using the change of coordinates $(x,y) = (\frac{1}{2}(z+w), -\frac{i}{2}(z-w))$ we consider a system of the form

$$\begin{aligned} \dot{x} &= -y + p_1(x,y) = P_1(x,y) \\ \dot{y} &= x + q_1(x,y) = Q_1(x,y). \end{aligned} \tag{2.12}$$

If the system is linearisable then there exists a function $H(x,y) = x + F(x,y) = x + o(|(x,y)|)$ such that

$$\frac{d^2H}{dt^2} + H = 0. \tag{2.13}$$

Indeed the analytic linearising change of coordinates is obtained as

$$\begin{aligned} (X,Y) &= (H(x,y), -\dot{H}(x,y)) \\ &= (H(x,y), -\frac{\partial H}{\partial x}(-y + p_1(x,y)) - \frac{\partial H}{\partial y}(x + q(x,y))). \end{aligned} \tag{2.14}$$

Finding $H(x,y) = x + \sum_{j+k\geq 2} e_{jk} x^j y^k$ satisfying (2.13) amounts to solve

$$\begin{aligned} P_1^2 H_{xx} + 2P_1Q_1H_{xy} + Q_1^2 H_{yy} + (P_{1x}P_1 + P_{1y}Q_1)H_x \\ + (Q_{1x}P_1 + Q_{1y}Q_1)H_y + H = 0. \end{aligned} \tag{2.15}$$

This is solvable provided the linearisability conditions are satisfied.

2.3 Darboux linearisation and its generalisation.
We start by some definitions.

Definitions 2.4.

(1) *An invariant algebraic curve of the system* (2.2) *is a curve in* $\mathbb{C}^2$ *given by an equation* $F(z,w)=0$, *with* $F(z,w)\in\mathbb{C}[z,w]$ *such that there exists* $K(z,w)\in\mathbb{C}_{n-1}[z,w]$ *satisfying*

$$DF(z,w)=F(z,w)K(z,w). \tag{2.16}$$

Here $\mathbb{C}_{n-1}[x,y]$ *denotes the space of polynomials in* x *and* y *of degree* $\leq n-1$ *and complex coefficients.*

(2) *Any analytic function* $F(x,y)$ *satisfying* (2.16), *for some* $K(z,w)\in\mathbb{C}_{n-1}[z,w]$, *is a generalised Darboux factor. The polynomial* $K(F)=K(z,w)$ *is called the cofactor of the Darboux factor.*

(3) *A nonconstant function* $F(z,w)$ *satisfying* $DF(z,w)\equiv 0$ *is a first integral.*

(4) *A Darboux function is a function* $Z(z,w)$ *of the form*

$$Z=\prod_{j=0}^{k}F_j^{\alpha_j},\quad \alpha_j\in\mathbb{C}, \tag{2.17}$$

with $F_j\in\mathbb{C}[z,w]$, $j=0,\dots,k$.

(5) *Given a system* (2.2) *and the differential operator* D *defined by* (2.16), *a Darboux function (resp. generalised Darboux function) associated with the system* (2.2) *is a function* Z *of the form* (2.17), *with* $F_j=0$ *invariant algebraic curves (resp.* F_j *Darboux factors),* $j=0,\dots,k$.

(6) *A system is Darboux integrable if it has a first integral which is a Darboux function associated to it.*

Many of the strata of polynomial systems with a centre have a first integral which is a Darboux function, or a generalised Darboux function (cf. [C], [S1], [S2] and [Z]). In practice, the generalised Darboux factors that we are most interested in arise as limiting cases of Darboux functions, and can be expressed in the form $e^{D/E}$, where D and E are polynomials [C]. Many examples of isochronous centres having Darboux first integrals are also given in [MRT].

The concept of Darboux linearisability is introduced in [MMR], which however is only concerned with real systems. In the real context a linearising change of coordinates is hence given by a unique function $Z=F(z,\overline{z})$. Here we must adapt the definitions to the fact that we are dealing with systems in $\mathbb{C}^2$.

Definition 2.5. *The system* (2.2) *is (generalised) Darboux linearisable if there exists a (generalised) Darboux change of coordinates*

$$(Z,W)=\Big(\prod_{j=0}^{k}F_j^{\alpha_j},\prod_{j=0}^{\ell}G_j^{\beta_j}\Big), \tag{2.18}$$

regular at the origin, i.e. of the form $(Z,W)=(z+o(|(z,w)|),w+o(|(z,w)|))$ *linearising* (2.2). *Such a function* Z *is called a (generalised) Darboux linearising change of coordinates.*

Remark. A Darboux linearisable system is Darboux integrable with first integral $F(Z,W)=ZW$.

The following theorem characterizing Darboux linearisability is obtained exactly as the corresponding theorem in [MMR]:

Theorem 2.6.

(i) The system (2.2) *is Darboux linearisable if and only if there exist invariant algebraic curves* $F_0=0$ *and* $G_0=0$ *of the form* $F_0(z,w)=z+o(|(z,w)|)$, $G_0(z,w)=w+o(|(z,w)|)$ *and there exist invariant algebraic curves* $F_j=0$, $j\in J_1$, $G_j=0$, $j\in J_2$ *where* J_1 *and* J_2 *are finite subsets of* $\mathbb{N}$ *(possibly void) such that* $F_j(0,0)\neq 0$, $G_j(0,0)\neq 0$ *and*

$$\begin{aligned} K_0+\sum_{j\in J_1}\alpha_jK_j&=i\\ L_0+\sum_{j\in J_2}\beta_jL_j&=-i \end{aligned} \tag{2.19}$$

where K_j *is the cofactor of* F_j, L_j *is the cofactor* G_j *and* $\alpha_j,\beta_j\in\mathbb{C}$. *The Darboux linearising change of coordinates is then given by*

$$(Z,W)=\Big(F_0\prod_{j\in J_1}F_j^{\alpha_j},G_0\prod_{j\in J_2}G_j^{\beta_j}\Big). \tag{2.20}$$

(ii) The system (2.2) is generalised Darboux linearisable if and only if (2.19) is satisfied with the K_j and L_j cofactors of Darboux factors F_j and G_j. The linearising change of coordinates is again given by (2.20).

Proof. The proof goes exactly as in [MMR]. It follows from the fact that $(Z,W) = (F(z,w), G(z,w))$ is a linearising change of coordinates if and only if the functions F and G are Darboux factors satisfying $DF = i$ and $DG = -i$. □

It occured however in several examples of Darboux integrable systems that we could only find algebraic invariant curves or Darboux factors so that one equation of (2.19) is satisfied. In order to construct the linearising change of coordinates in that case we prove the following theorem.

Theorem 2.7. *Suppose that the system (2.2) has first integral $H(z,w) = zw + o(|(z,w)|^2)$, and that there exist (generalised) Darboux factors F_j such that the first equation of (2.19) is satisfied. Then a linearising change of coordinate is given by*

$$(Z,W) = (F_0 \prod_{j\in J_1} F_j^{\alpha_j}, \frac{H(z,w)}{F_0 \prod_{j\in J_1} F_j^{\alpha_j}}). \tag{2.21}$$

Proof. Let us call $F = F_0 \prod_{j\in J_1} F_j^{\alpha_j}$ and $G = \frac{H(z,w)}{F_0 \prod_{j\in J_1} F_j^{\alpha_j}}$. Then $H = FG$. Since H is a first integral we have $DH_0 = 0$. From $DF = i$ we then deduce that $DG = -i$. □

Remark. A common method to exhibit explicitly a first integral is to use the Darboux method (with invariant algebraic curves or Darboux factors). It may happen that all Darboux factors occuring in the expression of the first integral do not vanish at the origin. In that case the integral obtained $H(z,w)$ does not vanish at the origin. The theorem has to be applied to the first integral $\overline{H}(z,w) = H(z,w) - H(0,0)$. Even if H is a Darboux first integral it may occur that $\overline{H}$, and hence G is not a Darboux function.

3. Linearisable complex quadratic systems

Theorem 3.1. *We consider a quadratic system in $\mathbb{C}^2$:*

$$\begin{aligned} \dot z &= iz + c_{20}z^2 + c_{11}zw + c_{02}w^2 \\ \dot w &= -iw + d_{20}z^2 + d_{11}zw + d_{02}w^2. \end{aligned} \tag{3.1}$$

The system has a linearisable centre if and only if one of the following conditions is satisfied

$$I \quad c_{11} = d_{20} = d_{11} = 0 \tag{3.2}$$

$$II \quad c_{11} = c_{02} = d_{11} = 0 \tag{3.3}$$

$$III \quad c_{02} = d_{20} = 0, \quad c_{20} - d_{11} = d_{02} - c_{11} = 0 \tag{3.4}$$

$$\begin{aligned} IV \quad & 7c_{11} - 6d_{02} = 7d_{11} - 6c_{20} = 7d_{11}^2 - 12d_{20}d_{02} \\ & = 14d_{20}c_{02} - 3d_{11}d_{02} = 49d_{11}c_{02} - 18d_{02}^2 = 0 \end{aligned} \tag{3.5}$$

$$\begin{aligned} V \quad & 2c_{20} - 5d_{11} = 2d_{02} - 5c_{11} = 15d_{11}^2 + 4d_{20}d_{02} \\ & = 6d_{02}^2 + 25c_{02}d_{11} = 10d_{20}c_{02} - 9d_{11}d_{02} = 0 \end{aligned} \tag{3.6}$$

$$VI \quad c_{11} = c_{02} = d_{02} = 0 \tag{3.7}$$

$$VII \quad c_{20} = d_{20} = d_{11} = 0 \tag{3.8}$$

$$VIII \quad c_{11} = d_{20} = d_{02} = c_{20} - 2d_{11} = 0 \tag{3.9}$$

$$IX \quad c_{20} = c_{02} = d_{11} = d_{02} - 2c_{11} = 0. \tag{3.10}$$

Proof. We bring the system (3.1) to normal form

$$\begin{aligned} \dot Z &= iZ + \sum_{j=1}^{3} a_j Z^{j+1}W^j + o(|Z,W|^7) \\ \dot W &= -iW + \sum_{j=1}^{3} b_j Z^j W^{j+1} + o(|Z,W|^7) \end{aligned} \tag{3.11}$$

under a change of coordinates $(z,w)=(Z+o(|Z,W|),W+o(|Z,W|))$. If the system is linearisable then $a_1=a_2=a_3=b_1=b_2=b_3=0$. We calculate a_j and b_j, for $j=1,2,3$ (One author used Maple and the other Reduce). We then ask for the Gröbner basis of $(a_1,a_2,a_3,b_1,b_2,b_3)$. This yields the cases I-IX of the Theorem.

The sufficiency comes from the study of the different cases in (3.2)-(3.10). In each case we show that the system is linearisable and we exhibit the linearising change of coordinates. For that purpose we use the fact that scaling on z (resp. w) allows to suppose that $c_{20}=0,1$ (resp. $d_{02}=0,1$).

(1) $c_{11}=d_{20}=d_{11}=0$. Let us first suppose that $c_{20},d_{02}\neq 0$. We then scale $c_{20}=d_{02}=1$. (The case $c_{02}=0$ corresponds to the Loud system (S_1) in the notation of [MRT]). The system has four invariant lines $F_j(z,w)=0$ with cofactors K_j given by:

$$\begin{array}{rr} F_1(z,w)=w & K_1(z,w)=-i+w \\ F_2(z,w)=1+iw & K_2(z,w)=w \\ F_{3,4}=1-iz+B_{3,4}w & K_{3,4}(z,w)=z-iB_{3,4}w, \end{array} \tag{3.12}$$

where $B_{3,4}$ are the roots of $B^2-iB-c_{02}=0$. The first integral is given by

$$H(z,w)=\frac{1-iz+B_3w}{1-iz+B_4w}(1+iw)^{i(B_3-B_4)}. \tag{3.13}$$

If we rather choose $K(z,w)=i\frac{1-H(z,w)}{B_3-B_4}$, then

$$K(z,w)=zw+o(|z,w|^2). \tag{3.14}$$

The linearising change of coordinates is given by

$$\begin{aligned} Z&=\frac{K(z,w)(1+iw)}{w} \\ W&=\frac{w}{1+iw}. \end{aligned} \tag{3.15}$$

We now consider the case $d_{02}=0$ and $c_{20}=1$ (after scaling). The system has a Darboux factor $F_1(z,w)=e^w$ with cofactor $K_4(z,w)=-iw$ and three invariant lines $F_j(z,w)=0$ with cofactors $K_j(z,w)=0$ given by:

$$\begin{array}{rr} F_2(z,w)=w & K_2(z,w)=-i \\ F_3(z,w)=1-iz+w & K_3(z,w)=z-iw \\ F_4=1-iz-w & K_4(z,w)=z+iw. \end{array} \tag{3.16}$$

This yields a first integral

$$H(z,w)=e^{-2w}\frac{1-iz+w}{1-iz-w}. \tag{3.17}$$

We let $K(z,w)=\frac{1}{2i}(F(z,w)-1)=zw+o(|z,w|^2)$ yielding the linearising change of coordinates

$$\begin{aligned} Z&=\frac{K(z,w)}{w} \\ W&=w. \end{aligned} \tag{3.18}$$

The case $c_{20}=d_{02}=0$ is contained in (7) below.

(2) $c_{11}=c_{02}=d_{11}=0$ is simply the dual of (1): we just swap the variables z and w over.

(3) $c_{02}=d_{20}=0$, $c_{20}-d_{11}=d_{02}-c_{11}=0$. In the case $c_{20}d_{02}\neq 0$ we can scale $c_{20}=d_{02}=1$, yielding the system

$$\begin{aligned} \dot z&=iz+z^2+zw \\ \dot w&=-iw+zw+w^2. \end{aligned} \tag{3.19}$$

This corresponds to the Loud system (S_2) in the notation of [MRT]. This system has the three invariant lines $F_j(z,w)=0$ with cofactors $K_j(z,w)$, where

$$\begin{array}{rr} F_1(z,w)=z & K_1(z,w)=i+z+w \\ F_2(z,w)=w & K_2(z,w)=-i+z+w \\ F_3=1-iz+iw & K_3(z,w)=z+w, \end{array} \tag{3.20}$$

yielding the first integral

$$H(z,w)=\frac{zw}{(1-iz+iw)^2}, \tag{3.21}$$

and the linearising change of coordinates

$$\begin{aligned} Z&=\frac{z}{1-iz+iw}\\ W&=\frac{w}{1-iz+iw}. \end{aligned} \tag{3.22}$$

In the nonlinear case and either $c_{20}=0$ or $d_{02}=0$ we are respectively in case (7) or (6) below.

(4) $7c_{11}-6d_{02}=7d_{11}-6c_{20}=7d_{11}^2-12d_{20}d_{02}=14d_{20}c_{02}-3d_{11}d_{02}=49d_{11}c_{02}-18d_{02}^2=0$. We can easily check that either c_{20} and d_{02} vanish simultaneously, or they are both nonzero. If they do vanish, all variables vanish except one of c_{02} or d_{20}, and we have a subcase of (6) or (7). The non-vanishing case corresponds to the Loud system (S_4) (in the notation of [MRT]). It can be scaled to the system

$$\begin{aligned} \dot z&=iz+7z^2+6zw+3w^2\\ \dot w&=-iw+3z^2+6zw+7w^2. \end{aligned} \tag{3.23}$$

This system has an invariant line $F_1(z,w)=0$ and three invariant conics $F_{2,3,4}(z,w)=0$, with respective cofactors K_j given by

$$\begin{array}{cc} F_1(z,w)=1-4i(z-w) & K_1(z,w)=4(z+w)\\ F_2(z,w)=1-16i(z-w)-96(z^2+w^2)-64zw & K_2(z,w)=16(z+w)\\ F_3(z,w)=z-i(z-w)^2 & K_3(z,w)=i+8(z+w)\\ F_4(z,w)=w+i(z-w)^2 & K_4(z,w)=-i+8(z+w), \end{array} \tag{3.24}$$

yielding the first integral

$$H(z,w)=\frac{(z-i(z-w)^2)(w+i(z-w)^2)}{(1-4i(z-w))^4}, \tag{3.25}$$

and the linearising change of coordinates

$$\begin{aligned} Z&=\frac{z-i(z-w)^2}{(1-4i(z-w))^2}\\ W&=\frac{w+i(z-w)^2}{(1-4i(z-w))^2}. \end{aligned} \tag{3.26}$$

(5) $2c_{20}-5d_{11}=2d_{02}-5c_{11}=15d_{11}^2+4d_{20}d_{02}=6d_{02}^2+25c_{02}d_{11}=10d_{20}c_{02}-9d_{11}d_{02}=0$. Here again c_{20} and d_{02} vanish simultaneously. If this happens all coefficients vanish except one of c_{02} or d_{20}. Again this case is included in (6) or (7). The non-vanishing case corresponds to the Loud system (S_3) (in the notation of [MRT]) and can be scaled to the system

$$\begin{aligned} \dot z&=iz+5z^2+2zw-3w^2\\ \dot w&=-iw-3z^2+2zw+5w^2. \end{aligned} \tag{3.27}$$

This system has an invariant line $F_1(z,w)=0$ and two invariant conics $F_{2,3}(z,w)=0$, with respective cofactors K_1-K_3 given by

$$\begin{array}{cc} F_1(z,w)=1-8i(z-w) & K_1(z,w)=8(z+w)\\ F_2(z,w)=z+i(z+w)^2 & K_2(z,w)=i+4(z+w)\\ F_3(z,w)=w-i(z+w)^2 & K_3(z,w)=-i+4(z+w). \end{array} \tag{3.28}$$

This yields a first integral

$$H(z,w)=\frac{(z+i(z+w)^2)(w-i(z+w)^2)}{1-8i(z-w)}. \tag{3.29}$$

The linearising change of coordinates is given by

$$\begin{aligned} Z &= \frac{z + i(z+w)^2}{\sqrt{1-8i(z-w)}} \\ W &= \frac{w - i(z+w)^2}{\sqrt{1-8i(z-w)}}. \end{aligned} \tag{3.30}$$

(6) $c_{11} = c_{02} = d_{02} = 0$. Let us first look at the case $c_{20} \neq 0$ in which case we scale $c_{20} = 1$. Scaling on w allows to choose $d_{20} = 0, 1$. In the case $d_{20} = 0$, i.e. we have the system

$$\begin{aligned} \dot{z} &= iz + z^2 \\ \dot{w} &= -iw + d_{11}zw, \end{aligned} \tag{3.31}$$

with linearising change of coordinates

$$\begin{aligned} Z &= \frac{z}{1-iz} \\ W &= \frac{w}{(1-iz)^{d_{11}}}. \end{aligned} \tag{3.32}$$

When $d_{20} \neq 0$, scaling on w allows to take $d_{20} = 1$, i.e. to consider the system

$$\begin{aligned} \dot{z} &= iz + z^2 \\ \dot{w} &= -iw + z^2 + d_{11}zw. \end{aligned} \tag{3.33}$$

This system has two invariant lines $F_{1,2}(z,w) = 0$ and an invariant conic $F_3(z,w) = 0$, with respective cofactors $K_j(z,w)$ given by

$$\begin{array}{rlrl} F_1(z,w) &= z & K_1(z,w) &= i + z \\ F_2(z,w) &= 1 - iz & K_2(z,w) &= z \\ F_3(z,w) &= -2 + 2i(1+d_{11})z + (d_{11} + d_{11}^2)z^2 & K_3(z,w) &= (1+d_{11})z, \\ & +d_{11}(d_{11}^2 - 1)zw & & \end{array} \tag{3.34}$$

yielding first integrals

$$H(z,w) = F_3(z,w)(1-iz)^{-(1+d_{11})}. \tag{3.35}$$

and $K(z,w) = \frac{1}{d_{11}(d_{11}^2-1)}(H(z,w) - H(0,0)) = zw + o(|z,w|^2)$. The linearising change of coordinates is given by

$$\begin{aligned} Z &= \frac{z}{1-iz} \\ W &= \frac{K(z,w)(1-iz)}{z} \end{aligned} \tag{3.36}$$

(noting that z divides $K(z,w)$).

We next consider the case $c_{20} = 0$, i.e. we look at a system

$$\begin{aligned} \dot{z} &= iz \\ \dot{w} &= -iw + d_{20}z^2 + d_{11}zw. \end{aligned} \tag{3.37}$$

Let us first consider the case $d_{20} = 0$. In the nonlinear case we then can scale $d_{11} = 1$. We have the two invariant lines $z = 0$, $w = 0$ and the Darboux factor e^z, yielding the first integral $H(z,w) = zwe^{iz}$ and the linearising change of coordinates

$$\begin{aligned} Z &= z \\ W &= we^{iz}. \end{aligned} \tag{3.38}$$

In the case $c_{20} = 0$ and $d_{20} \neq 0$ we scale $d_{20} = 1$ and consider the system

$$\begin{aligned} \dot{z} &= iz \\ \dot{w} &= -iw + z^2 + d_{11}zw. \end{aligned} \tag{3.39}$$

This system has an invariant line $F_1(z,w) = z = 0$. For $d_{11} \neq 0$ it has an invariant conic $F_2(z,w) = 0$ and a Darboux factor $F_3(z,w) = e^z$ with cofactors $K_j(z,w)$ given by

$$\begin{array}{cc} F_1(z,w) = z & K_1(z,w) = i \\ F_2(z,w) = 1 - id_{11}z - \frac{d_{11}^2}{2}z^2 - \frac{d_{11}^3}{2}zw & K_2(z,w) = d_{11}z \\ F_3(z,w) = e^z & K_3(z,w) = iz. \end{array} \tag{3.40}$$

This yields the first integral

$$H(z,w) = e^{id_{11}z}(1 - id_{11}z - \frac{d_{11}^2}{2}z^2 - \frac{d_{11}^3}{2}zw). \tag{3.41}$$

We consider the first integral $K(z,w) = -\frac{2}{d_{11}^3}(H(z,w) - 1) = wz + o(|z,w|^2)$, yielding the linearising change of coordinates

$$\begin{aligned} Z &= z \\ W &= \frac{K(z,w)}{z}. \end{aligned} \tag{3.42}$$

If $d_{11} = 0$ we have a Hamiltonian system with first integral

$$H = zw - \frac{i}{3}z^3 = z(w + \frac{i}{3}z^2). \tag{3.43}$$

This yields the linearising change of coordinates

$$\begin{aligned} Z &= z \\ W &= w + \frac{i}{3}z^2. \end{aligned} \tag{3.44}$$

(7) $c_{20} = d_{20} = d_{11} = 0$. This case is the dual of (6).

(8) $c_{11} = d_{20} = d_{02} = c_{20} - 2d_{11} = 0$. If $c_{20} = 0$ then $d_{11} = 0$ and we are in case (7). Otherwise, we scale to make $c_{20} = 1$ and obtain the system

$$\begin{aligned} \dot{z} &= iz + z^2 + c_{02}w^2 \\ \dot{w} &= -iw + \frac{zw}{2}. \end{aligned} \tag{3.45}$$

This system has one invariant line $F_1(z,w) = 0$ and two invariant conics $F_{2,3}(z,w) = 0$ with cofactors K_j given by

$$\begin{array}{cc} F_1(z,w) = w & K_1(z,w) = -i + \frac{z}{2} \\ F_2(z,w) = z - \frac{ic_{02}}{3}w^2 & K_2(z,w) = i + z \\ F_3(z,w) = 1 - iz - \frac{c_{02}}{2}w^2 & K_3(z,w) = z, \end{array} \tag{3.46}$$

yielding the first integral

$$H(z,w) = \frac{w(z - \frac{ic_{02}}{3}w^2)}{(1 - iz - \frac{c_{02}}{2}w^2)^{3/2}}. \tag{3.47}$$

This yields the linearising change of coordinates

$$\begin{aligned} Z &= \frac{z - \frac{ic_{02}}{3}w^2}{1 - iz - \frac{c_{02}}{2}w^2} \\ W &= \frac{w}{(1 - iz - \frac{c_{02}}{2}w^2)^{1/2}}. \end{aligned} \tag{3.48}$$

(9) $c_{20} = c_{02} = d_{11} = d_{02} - 2c_{11} = 0$. This case is the dual of (8). $\square$

Theorem 4.1. *We consider a complex cubic symmetric system in* $\mathbb{C}^2$

$$\begin{aligned}\dot{z} &= iz + c_{30}z^3 + c_{21}z^2w + c_{12}zw^2 + c_{03}w^3\\ \dot{w} &= -iw + d_{30}z^3 + d_{21}z^2w + d_{12}zw^2 + d_{03}w^3.\end{aligned} \tag{4.1}$$

The system has a linearisable centre if and only if

$$c_{21} = d_{12} = 0 \tag{4.2}$$

and one of the following conditions is satisfied

$$I \quad c_{30} = d_{30} = d_{21} = 0 \tag{4.3}$$

$$II \quad c_{12} = c_{03} = d_{03} = 0 \tag{4.4}$$

$$III \quad c_{30} - d_{21} = c_{12} - d_{03} = d_{30} = c_{03} = 0 \tag{4.5}$$

$$IV \quad c_{12} = c_{03} = d_{30} = d_{21} = 0 \tag{4.6}$$

$$\begin{aligned}V \quad 3c_{30} - 7d_{21} &= 3d_{03} - 7c_{12} = 48c_{30}^3 + 343c_{12}d_{30}^2 = 16c_{30}c_{12} - 21d_{30}c_{03}\\ &= 49c_{12}^2d_{30} + 9c_{30}^2c_{03} = 27c_{30}c_{03}^2 + 112c_{12}^3 = 0\end{aligned} \tag{4.7}$$

$$VI \quad d_{30} = d_{03} = c_{12} = c_{30} - 3d_{21} = 0 \tag{4.8}$$

$$VII \quad c_{30} = c_{03} = d_{21} = d_{03} - 3c_{12} = 0. \tag{4.9}$$

Proof. As before the necessity of the conditions is obtained by annihilating the coefficients of the normal form (3.11) up to degree 7. In each case we prove their sufficiency by providing the linearising change of coordinates.

(1) $c_{30} = d_{30} = d_{21} = 0$. We consider a system

$$\begin{aligned}\dot{z} &= iz + c_{12}zw^2 + c_{03}w^3\\ \dot{w} &= -iw + d_{03}w^3.\end{aligned} \tag{4.10}$$

If $d_{03} \neq 0$ we can scale $d_{03} = i$. The system has the three invariant lines $F_1(z,w) = w = 0$ and has $F_{2,3}(z,w) = 1 \pm w = 0$ with respective cofactors $K_1(z,w) = -i(1-w^2)$ and $K_{2,3}(z,w) = iw(w \mp 1)$.

If $c_{03} \neq 0$ we can scale $c_{03} = 1$. We then have for $c_{12} \neq \pm i$ an invariant conic $F_4(z,w) = 1 + \frac{i}{2}(1+c_{12}^2)zw + \frac{1}{2}(-1+ic_{12})w^2$ with cofactor $K_4(z,w) = (i+c_{21})w^2$. This yields a Darboux first integral

$$H(z,w) = \frac{(1-w^2)^{1-ic_{12}}}{F_4^2}. \tag{4.11}$$

A linearising change of coordinates is given by

$$\begin{aligned}Z &= \frac{i(H(z,w)-1)(1-w^2)^{1/2}}{(1+c_{12}^2)w}\\ W &= \frac{w}{(1-w^2)^{1/2}}.\end{aligned} \tag{4.12}$$

If $c_{12} = -i$ a first integral is given by

$$H(z,w) = (1-w^2)\exp(-2izw + w^2), \tag{4.13}$$

where the exponential term has a cofactor $-2iw^2$. This gives the linearising change of coordinates

$$\begin{aligned}Z &= \frac{i(H(z,w)-1)(1-w^2)^{1/2}}{2w}\\ W &= \frac{w}{(1-w^2)^{1/2}}.\end{aligned} \tag{4.14}$$

If $c_{12} = i$ a first integral is given by

$$H(z,w) = (1-w^2)\exp\left(\frac{1+2izw}{1-w^2}\right) \tag{4.15}$$

where the exponential factor has a cofactor $-2iw^2$. A linearising change of coordinate is therefore given by

$$\begin{aligned} Z &= \frac{-i(H(z,w)-e)(1-w^2)^{1/2}}{2ew} \\ W &= \frac{w}{(1-w^2)^{1/2}}. \end{aligned} \tag{4.16}$$

If $c_{03} = 0$ the system has the invariant line $z = 0$ which has cofactor $i + c_{12}w^2$. This yields a first integral

$$H(z,w) = zw(1-w^2)^{\frac{1}{2}(-1+ic_{12})}. \tag{4.17}$$

This yields the linearising change of coordinates:

$$\begin{aligned} Z &= z(1-w^2)^{\frac{i}{2}c_{12}} \\ W &= \frac{w}{(1-w^2)^{1/2}}, \end{aligned} \tag{4.18}$$

which is also valid in the case $c_{12} = -i$.

We then come to the case $d_{03} = 0$. Here again we need distinguish several cases. If $c_{03} \neq 0$ we can scale $c_{03} = 1$. When $c_{12} \neq 0$ we have a first integral

$$H(z,w) = (1 + \frac{i}{2}c_{12}^2 zw + \frac{i}{2}c_{12}w^2)e^{-\frac{i}{2}c_{12}w^2} \tag{4.19}$$

where the exponential factor has a cofactor $c_{12}w^2$. We therefore have the linearising change of coordinates:

$$\begin{aligned} Z &= \frac{-2i(H(z,w)-1)}{c_{12}^2 w} \\ W &= w. \end{aligned} \tag{4.20}$$

In the case $c_{12} = 0$ we are in the Hamiltonian case, yielding the linearising change of coordinates

$$\begin{aligned} Z &= z - \frac{i}{4}w^3 \\ W &= w. \end{aligned} \tag{4.21}$$

If $d_{03} = c_{03} = 0$, in the nonlinear case, we can scale $c_{12} = 1$. This yields a first integral

$$H(z,w) = zw\exp(-\frac{i}{2}w^2). \tag{4.22}$$

and a linearising change of coordinates

$$\begin{aligned} Z &= z\exp(-\frac{i}{2}w^2) \\ W &= w. \end{aligned} \tag{4.23}$$

(2) $c_{12} = c_{03} = d_{03} = 0$ is the dual of (1).

(3) $c_{30} - d_{21} = c_{12} - d_{03} = d_{30} = c_{03} = 0$. When $c_{30}d_{03} \neq 0$ we recover the case (S_2^*) of [MRT]. We can scale $c_{30} = d_{03} = i$, i.e. we have four invariant lines given by $1 + irz + sw = 0$ with cofactors of the form $-(rz + sw) + i(w^2 + z^2)$ and $r,\ s = \pm 1$. This yields a linearising change of coordinates

$$\begin{aligned} Z &= \frac{z}{\sqrt{1-(iz+w)^2)}} \\ W &= \frac{w}{\sqrt{1-(iz+w)^2)}}. \end{aligned} \tag{4.24}$$

In the nonlinear case, when either $c_{30} = 0$ or $d_{03} = 0$, we are in case (1) or (2).

(4) $c_{12} = c_{03} = d_{30} = d_{21} = 0$. If $c_{30}d_{03} \neq 0$ we recover the case (S_1^*) of [MRT]. We can scale $c_{30} = -i$ and $d_{03} = i$. This yields the linearising change of coordinates

$$\begin{aligned} Z &= \frac{z}{\sqrt{1-z^2}} \\ W &= \frac{w}{\sqrt{1-w^2}}, \end{aligned} \tag{4.25}$$

and the first integral $H(z, w) = ZW$. If $c_{30} = 0$ or $d_{03} = 0$ then we are in case (1) or (2).

(5) $3c_{30} - 7d_{21} = 3d_{03} - 7c_{12} = 48c_{30}^3 + 343c_{12}d_{30}^2 = 16c_{30}c_{12} - 21d_{30}c_{03} = 49c_{12}^2d_{30} + 9c_{30}^2c_{03} = 27c_{30}c_{03}^2 + 112c_{12}^3 = 0$. When $c_{30}d_{03} \neq 0$ we recover the two cases (S_3^*) and $(\tilde{S}_3^*)$ of [MTR]. We can scale $c_{30} = 7i = -d_{03}$. This yields $c_{03} = -d_{30}$ and $d_{30}^2 = -16$. The two cases $d_{30} = \pm 4i$ are equivalent under $(z, w) \mapsto (z, -w)$. Hence we limit ourselves to the case $d_{30} = 4i$, i.e. to the system

$$\begin{aligned} \dot{z} &= iz + 7iz^3 - 3izw^2 - 4iw^3 \\ \dot{w} &= -iw + 4iz^3 + 3iz^2w - 7iw^3. \end{aligned} \tag{4.26}$$

This system has the first integral

$$H = \frac{(z + (z-w)^3)(w - (z-w)^3)}{1 + 9(z^2 + zw + w^2)}, \tag{4.27}$$

where the cofactors of the three brackets are $i + 9i(z^2 - w^2)$, $-i + 9i(z^2 - w^2)$ and $18i(z^2 - w^2)$. From this we obtain the linearising change of coordinates

$$\begin{aligned} Z &= \frac{z + (z-w)^3}{\sqrt{1 + 9(z^2 + zw + w^2)}} \\ W &= \frac{w - (z-w)^3}{\sqrt{1 + 9(z^2 + zw + w^2)}}. \end{aligned} \tag{4.28}$$

In the nonlinear case, if either $c_{30} = 0$ or $d_{03} = 0$ we are in case (1) or (2).

(6) $d_{30} = d_{03} = c_{12} = c_{30} - 3d_{21} = 0$. This case is very similar to case (8) of Theorem 3.1. If $c_{30} = 0$ then we are in case (1), otherwise we scale to make $c_{30} = 1$. We then get a first integral

$$H(z, w) = \frac{w(z - \frac{1}{4}ic_{03}w^3)}{(1 - iz^2 - c_{03}zw^3 + \frac{1}{6}ic_{03}^2w^6)^{2/3}} \tag{4.29}$$

and a linearising change of coordinates

$$\begin{aligned} Z &= \frac{z - \frac{1}{4}ic_{03}w^3}{(1 - iz^2 - c_{03}zw^3 + \frac{1}{6}ic_{03}^2w^6)^{1/2}} \\ W &= \frac{w}{(1 - iz^2 - c_{03}zw^3 + \frac{1}{6}ic_{03}^2w^6)^{1/6}}. \end{aligned} \tag{4.30}$$

(7) $c_{30} = c_{03} = d_{21} = d_{03} - 3c_{12} = 0$ is the dual of (6). □

Case (6) of Theorem 4.1 and case (8) of Theorem 3.1 lie in an infinite series of systems:

Theorem 4.2. *The system*

$$\begin{aligned} \dot{z} &= iz + z^n + aw^n \\ \dot{w} &= -iw + \frac{1}{n}z^{n-1}w. \end{aligned} \tag{4.31}$$

has a linearisable centre at the origin.

Proof. For each system, we have the invariant algebraic curves $F_1(z, w) = w = 0$ and $F_2(z, w) = z - iaw^n/(n+1) = 0$, with cofactors $-i + z^{n-1}/n$ and $i + z^{n-1}$ respectively. We show that there always exists another invariant curve of the form $F_3(z, w) = 1 + h(z, w^n)$ with cofactor $K_3(z, w) = (n-1)z^{n-1}$, where h is a homogeneous polynomial of degree $n - 1$. A Darboux linearising change of coordinates can then be constructed from these curves.

The existence of F_3 with cofactor K_3 is proved as follows. We let $W = w^n$. Then h must satisfy

$$\begin{aligned} h_z(iz + z^n + aw^n) + nw^{n-1}h_W(-iw + \frac{1}{n}z^{n-1}w) \\ - (n-1)z^{n-1}(1 + h(z, w^n)) = 0. \end{aligned} \tag{4.32}$$

From the homogeneity of $h(z, W)$ we have

$$z^n h_z + w^n z^{n-1} h_W - (n-1)z^{n-1}h = 0. \tag{4.33}$$

Hence $h(z, W)$ must satisfy

$$h_z(iz + aW) - inWh_W - (n-1)z^{n-1} = 0. \tag{4.34}$$

Letting $h(z, W) = \sum_{j=0}^{n-1} a_j z^j W^{n-1-j}$ the equation (4.34) has the following solution

$$a_{n-1} = -i, \qquad a_j = -\frac{i(j+1)a}{n(n-1-j)-j}a_{j+1} \qquad j = 0, \dots n-2. \tag{4.35}$$

(Note that for $j = 0, \dots, n-2$ we always have $n(n-1-j) - j > 0$). □

5. Real linearisable saddles

We now study real quadratic and symmetric cubic systems with a linearisable saddle at the origin. The idea is simply to change coordinates $(z, w, t) \mapsto (ix, iy, -it)$ in Theorems 3.1 and 4.1. The conditions for integrability given in those theorems thus remain unchanged. This is in marked contrast to the real centre case, where the change of coordinates $(z, w, t) \mapsto (x + iy, x - iy, t)$ forces the conditions to have complex conjugate coefficients and in this way reduces the number of branches that can be realised in $\mathbb{R}^2$.

The above observation seems to indicate that the linearisable saddles in some way give a better indication of what happens in questions of linearisability than is the case for centres. However, the disadvantage is that we lose the visible aspects of the isochronicity, since the system is isochronous with respect to pure imaginary time. Even the concept of an *integrable saddle* which corresponds to a centre is less well known, and not geometrically intuitive.

In this last section, we therefore investigate the conditions for an integrable saddle in quadratic and symmetric cubic systems. We also consider some of the generic phase portraits of linearisable saddles to see if there are any general conclusions we can draw on their topology.

It was remarked by Teixeira and Yang [TY] that analytic time-reversibility and integrability were equivalent for real centres. We adapt this criterion for our case. The converse part requires more work since we cannot simply assert that time-reversibility implies the existence of an analytic first integral.

Theorem 5.1. *A non-degenerate saddle of a real or complex analytic system is integrable if and only if there is a local analytic transformation of the system T, with $T^2 = id$ whose effect is the same as the time reversal $t \mapsto -t$.*

Proof. Suppose the system is integrable. Then there is an analytic change of coordinates which brings the system to the form

$$\dot{x} = x(\lambda + \sum a_i(xy)^i), \qquad \dot{y} = -y(\lambda + \sum a_i(xy)^i). \tag{5.1}$$

It is easy to see that the transformation $(x, y) \mapsto (y, x)$ reverses the system.

Conversely suppose that the saddle point of the system is not integrable. Modulo an analytic change of coordinates and division by a positive function we can bring the system to a normal form

$$\dot{x} = x, \qquad \dot{y} = -y + Ax^k y^{k+1} + o(|(x, y)|^{2k+1}) \tag{5.2}$$

with $A \neq 0$. We will see that this is an obstruction to find an involution. Indeed it can be argued that such an involution T has a linear part of the form $T_1(x, y) = (by, x/b)$, with $b \neq 0$. We look for the involution as a power series $T(x, y) = (X, Y) = (by + \sum_{r=2}^{\infty} h_r(x, y), x/b + \sum_{r=2}^{\infty} k_r(x, y))$, where $h_r(x, y)$ and $k_r(x, y)$ are homogeneous polynomials in x and y of degree r. However, from the hypothesis, we must have the relation $\dot{X} = -X$. However the term in $x^k y^{k+1}$ of degre $2k + 1$ in the expression $\dot{X} + X$ is always of the form $bA \neq 0$, which contradicts the existence of such a T. □

Remarks.

(1) In fact the hypothesis of this theorem can be relaxed. For example, by considering the order $2k+1$ terms of equation $\dot{Y}$ as well as $\dot{X}$ we can assume only that the linear parts of T (in the expansion about the critical

point) are involutive and that the transformed system is equal to the original system multiplied by some negative function. Details are left to the reader.

(2) It is also possible to prove the converse part of the theorem directly, without using normal forms. We sketch the idea below.

If $T : (x,y) \mapsto (T_1(x,y), T_2(x,y))$, then we take new variables $\phi(x,y) = (X,Y) = (T_1(x,y)-x), T_1(x,y)+x)$. We want to show that in the coordinates (X,Y) the involution becomes $S(X,Y) = (-X,Y)$, with $S = \phi\circ T\circ \phi^{-1}$. If we call $R(X,Y) = (-X,Y)$ the symmetry with respect to the Y-axis, this amounts to showing that $R\circ\phi\circ T(x,y) = \phi(x,y)$, which is a straightforword consequence of the fact that T is an involution.

We thus obtain a new system with a corresponding reversing transformation S, and so the system must be of the form

$$\dot X = -Y - P(X^2,Y), \qquad \dot Y = -X + XQ(X^2,Y). \tag{5.3}$$

Now such a system arises from the system

$$\dot W = -2Y - 2P(W,Y), \qquad \dot Y = -1 + Q(W,Y), \tag{5.4}$$

via the transformation $W = X^2$. Since this later system is non-singular at the origin, it has a first integral $H(W,Y) = W - Y^2 + o(W) + o(Y^2)$, which can be pulled back to a first integral $K(X,Y) = X^2 - Y^2 + o(|(X,Y)|^2)$ of the original system.

Theorem 5.2. *We consider a quadratic system in $\mathbb{R}^2$ with a saddle point at the origin with opposite eigenvalues*

$$\begin{aligned}\dot x &= x + c_{20}x^2 + c_{11}xy + c_{02}y^2\\ \dot y &= -y + d_{20}x^2 + d_{11}xy + d_{02}y^2.\end{aligned} \tag{5.5}$$

The system is linearisable at the origin if and only if one of the conditions (3.2)-(3.10) *is satisfied. The phase portraits are given in Figures 1-6.*

Proof. The system (5.5) is obtained from (3.1) by means of the transformation $(x,y,T) = (-iz,-iw,it)$. As mentioned above the conditions in (3.2)-(3.10) remain unchanged.

We have a real system whenever the c_{jk} are real. As the conditions are invariant under $(x,y)\mapsto(ax,by)$, the families I, II, VI-IX can be reduced to one-dimensional families, while the cases III-V can be reduced to 0-dimensional families. However for practical reasons it is simpler for cases I, II, VI and VII to draw the bifurcation diagram on one fourth of a 2-sphere.

For case I we suppose $c_{20}^2 + c_{02}^2 + d_{02}^2 = 1$ and $c_{20}, d_{02} \geq 0$. The bifurcation diagram appears in Figure 1. The case II can be deduced easily from it.

The phase portraits of case III, IV, V appear in Figures 2, 3, and 4 respectively.

For case VI we suppose $c_{20}^2 + d_{20}^2 + d_{11}^2 = 1$, $c_{20}, d_{20} \geq 0$. The bifurcation diagram appears on Figure 5. The case VII can be easily deduced from it.

For case VIII we can scale $d_{11}^2 + c_{02}^2 = 1$, $d_{11} \geq 0$. The bifurcation diagram appears in Figure 6. The case IX follows from it. □

A natural question for us was to see where these linearisable saddles lie inside the strata of integrable saddles. For that purpose we have recalculated the integrability conditions for a quadratic system with a saddle at the origin. These first appear in the work of Dulac [Du] in a case by case procedure. The integrable saddle points of quadratic systems are systematically studied in [DGS]. The saddle quantities have been simplified and the strata identified with the use of Gröbner bases. Here again the calculations have been done independently by the two authors using Maple and Reduce.

Theorem 5.3. *The saddle quantities of the system* (5.5) *are given by*

$$\begin{aligned}V_1 &= d_{11}d_{02} - c_{20}c_{11}\\ V_2 &= 2d_{20}c_{11}^3 + 2c_{20}^2d_{11}c_{02} - 3c_{20}c_{02}d_{11}^2 - 2c_{02}d_{11}^3 + 3c_{11}^2d_{02}d_{20} - 2c_{11}d_{20}d_{02}^2\\ V_3 &= -(2c_{02}d_{20} - d_{11}d_{02})(c_{11}^2d_{20}d_{02} + 2c_{11}d_{20}d_{02}^2 - c_{20}c_{02}d_{11}^2 - 2c_{20}^2c_{02}d_{11}).\end{aligned} \tag{5.6}$$

The strata of integrable saddles are given by

$$
\begin{array}{ll}
(A) & c_{11} = d_{11} = 0 \\
(B) & d_{11} + 2c_{20} = c_{11} + 2d_{02} = 0 \\
(C) & \begin{cases} c_{20}c_{11} - d_{11}d_{02} = 0 \\ c_{20}^3 c_{02} - d_{20}d_{02}^3 = 0 \\ c_{20}^2 c_{02} d_{11} - c_{11} d_{20} d_{02}^2 = 0 \\ c_{20}c_{02}d_{11}^2 - c_{11}^2 d_{20} d_{02} = 0 \\ c_{11}^3 d_{20} - d_{11}^3 c_{02} = 0 \end{cases} \\
(D) & 2c_{11} - d_{02} = c_{20} - 2d_{11} = c_{02}d_{20} - d_{11}c_{11} = 0
\end{array}
\tag{5.7}
$$

Geometric meaning of the different strata in (5.7)**:** the stratum (A) corresponds to systems with three invariant lines generically. Such systems have a first integral which can be constructed from the corresponding Darboux factors. Two of the lines may occur in a complex pair even if the coefficients are real. The non-generic cases in this stratum will arise from generalised Darboux factors of exponential form. The stratum (B) consists of Hamiltonian systems, with a cubic first integral.

The stratum (D) consists of systems having generically an invariant conic and an invariant cubic allowing Darboux integration. Again, the non-generic cases will involve exponential factors. For example, in the generic case $c_{11}c_{02} \neq 0$, we can scale the axes to give $c_{11} = c_{02} = 1$ which gives $d_{02} = 2$, $c_{20} = 2d_{11}$ and $d_{20} = d_{11}$. We then obtain the invariant curves

$$
\begin{aligned}
F_1 &= 1 + 2d_{11}x - 2y + d_{11}(x-y)^2 = 0 \\
F_2 &= 1 + 3d_{11}x - 3y + 3(d_{11}+1)(x-y)(d_{11}x - y)/2 \\
&\quad + d_{11}(d_{11}+1)(x-y)^3/2 = 0,
\end{aligned}
\tag{5.8}
$$

with cofactors $2K/5$ and $3K/5$, where $K = 5(d_{11}x + y)$ is the divergence of the system. There is thus a rational first integral of the form $F_1^3 F_2^{-2}$ with integrating factor $1/F_1F_2$.

The stratum (C) is the most interesting in terms of new features. It consists of reversible systems (possibly in a generalised sense defined below). Let us suppose that $c_{11}d_{11} \neq 0$ then we can scale the system so that $c_{11} = -d_{11} = 1$. The conditions of (C) now give $c_{20} = -d_{02}$ and $c_{02} = -d_{20}$. The critical point is clearly symmetric under the transformation $(x, y) \mapsto (y, x)$.

However, there are also the limiting cases where one of c_{11} or d_{11} vanish. When both vanish, we have a proper subset of the stratum (A). The remaining cases are

$$
(C_1) \quad c_{02} = d_{02} = c_{11} = 0 \tag{5.9}
$$

$$
(C_2) \quad c_{20} = d_{20} = d_{11} = 0. \tag{5.10}
$$

Because of the simplicity of the quadratic systems we can in fact find Darboux integrals for these systems, as we can for all of the stratum (C). Generically, they will consist of systems with an invariant line and an invariant conic. For example, when $c_{11} = -d_{11} = 1$, we have for $d_{20} \neq 0$ the invariant curves

$$
\begin{aligned}
F_1 &= 1 + (d_{20} - d_{02})(x+y) = 0 \\
F_2 &= 1 - (1 + d_{20} + d_{02})(x+y) \\
&+ (2d_{20}+1)(d_{20}+d_{02}+1)(d_{20}x^2 + (d_{02} - d_{20} - 1)xy + d_{20}y^2)/2d_{20} = 0
\end{aligned}
\tag{5.11}
$$

with cofactors $(d_{20} - d_{02})(x-y)$ and $-(1 + d_{20} + d_{02})(x-y)$. The function $1/F_1F_2$ is an integrating factor for this system and there is a first integral of the form $F_1^{\alpha}F_2^{\beta}$.

However in some ways this is a little artificial as we have no reason to expect that the reversible centres in higher degree systems are Darboux integrable. That is, reversible systems should give an "integrability mechanism" independent from Darboux integrability.

Although the sub-strata (C_1) and (C_2) are not reversible, we can conclude that they are integrable because they lie in the closure of a strata of integrable saddles and the saddle quantities are continuous. It would be much more satisfying, however, if we could give a direct reason for their integrability as a generalisation of reversibility, in the same way as exponential factors appear as the limits of Darboux integrals.

The effect of approaching these limiting cases within the stratum (C) is that the eigenvectors of the reversing transformation coalesce and the transformation becomes singular. Thus, in order to resolve this difficulty, we are led to consider the system as a blow-up of a simpler system.

For example, consider the stratum (C_1)

$$\begin{aligned}\dot{x} &= x + c_{20}x^2\\ \dot{y} &= -y + d_{20}x^2 + d_{11}xy.\end{aligned} \tag{5.12}$$

Taking $X = x$ and $Y = xy$, we obtain the system

$$\begin{aligned}\dot{X} &= (1 + c_{20}X)X\\ \dot{Y} &= ((c_{20} + d_{11})Y + d_{20}X^2)X.\end{aligned} \tag{5.13}$$

Removing the factor of X, we have a non critical point at the origin for this system. It therefore has a local first integral $\Phi(X,Y) = Y + o(|(X,Y)|)$ which can be pulled back to a local first integral of the original system, $\Phi(x, xy) = xy + o(|(x,y)|^2)$. Thus (C_1) is integrable. The case (C_2) is handled similarly.

This is a particular case of the following theorem:

Theorem 5.4. *The following class of integrable saddles lie in the closure of the strata of reversible systems:*

$$\begin{aligned}\dot{x} &= x + P(x,y) = x + o(|(x,y)|^2)\\ \dot{y} &= -y + Q(x,y) = -y + o(|(x,y)|^2),\end{aligned} \tag{5.14}$$

where we have either

$$P(x,y) = \sum_{i>j} c_{ij}x^iy^j, \qquad Q(x,y) = \sum_{i>j-2} d_{ij}x^iy^j, \tag{5.15}$$

with $c_{r+1,r} + d_{r,r+1} = 0$, *or the conjugate system under* $(x,y,t) \mapsto (y,x,-t)$.

Proof. The proof that they are centres follows exactly the lines above. □

Remarks 5.5.

(1) In the corresponding case for real centres, such systems do not arise except in the trivial case

$$\begin{aligned}\dot{x} &= -y(1 + \sum \alpha_i(x^2+y^2)^i)\\ \dot{y} &= x(1 + \sum \alpha_i(x^2+y^2)^i).\end{aligned} \tag{5.16}$$

(2) It would be interesting to see if the corresponding notion of a limit of rational reversible systems would also give some new integrability conditions.

Proposition 5.6. *The linearisable saddles described in Theorem 5.2 lie in the stratum* (A) *for the cases I-II, the strata* (C) *for the cases III-VII, and the strata* (D) *for the cases VIII-IX.*

Theorem 5.7. *We consider a cubic system in* $\mathbb{R}^2$ *symmetric with respect to a saddle point at the origin with opposite eigenvalues*

$$\begin{aligned}\dot{x} &= x + c_{30}x^3 + c_{21}x^2y + c_{12}xy^2 + c_{03}y^3\\ \dot{y} &= -y + d_{30}x^3 + d_{21}x^2y + d_{12}xy^2 + d_{03}y^3.\end{aligned} \tag{5.17}$$

The system is linearisable at the origin if and only if (4.2) *and one of the conditions* (4.3)-(4.9) *is satisfied. The phase portraits appear in Figures 7-12 (only the generic cases).*

Proof. The system (5.17) is invariant under $(x,y) \mapsto (ax, by)$.

For Case I we suppose $c_{12}^2 + c_{03}^2 + d_{03}^2 = 1$, $c_{03} \geq 0$, which yields a half 2-sphere. The bifurcation diagram appears in Figure 7. Case II can easily be deduced from it.

For Case III we suppose $c_{30}^2 + c_{12}^2 = 1$, which yields a circel. The bifurcation diagram appears in Figure 8.

For Case IV we suppose $c_{30}^2 + d_{03}^2 = 1$, which yields a circle. The bifurcation diagram appears in Figure 9.

Case V is zero-dimensional. We have the conditions $c_{30}c_{12} < 0$, $d_{30}c_{03} < 0$, $c_{30}d_{21} > 0$, $c_{12}d_{03} > 0$ which yields, after scaling, the two cases

$$\begin{aligned}\dot{x} &= x + 7x^3 - 3xy^2 - 4y^3\\ \dot{y} &= -y + 4x^3 + 3x^2y - 7y^3\end{aligned} \tag{5.18}$$

and

$$\begin{aligned}\dot{x} &= x - 7x^3 + 3xy^2 + 4y^3\\ \dot{y} &= -y - 4x^3 - 3x^2y + 7y^3.\end{aligned} \tag{5.19}$$

Their respective phase portraits appear in Figures 10 and 11.

For Case VI we suppose $c_{03}^2 + d_{21}^2 = 1$ and $c_{03} \geq 0$, which yields a half-circle. The bifurcation diagram appears in Figure 12. Case VII can be easily deduced from it. □

Theorem 5.8. *The saddle quantities of the system* (5.17) *are given by*

$$\begin{aligned} V_1 &= c_{21} + d_{12} \\ V_2 &= d_{21}d_{03} - c_{30}c_{12} \\ V_3 &= 3c_{30}^2c_{03} - 3c_{03}d_{21}^2 - 3c_{12}^2d_{30} + 3d_{30}d_{03}^2 - 8c_{30}c_{03}d_{21} - 8c_{12}d_{30}d_{03} \\ V_4 &= c_{21}(3c_{30}c_{03}d_{21} + 3c_{12}d_{30}d_{03} + c_{03}d_{21}^2 + d_{30}c_{12}^2) \\ V_5 &= (3d_{30}c_{03} - 4d_{21}d_{03})(3c_{30}c_{03}d_{21} + 3c_{12}d_{30}d_{03} + c_{03}d_{21}^2 + c_{12}^2d_{30}). \end{aligned} \tag{5.20}$$

The strata of integrable saddles are given by

$$\begin{aligned} &(A) \quad d_{21} + 3c_{30} = c_{21} + d_{12} = c_{12} + 3d_{03} = 0 \\ &(B) \quad c_{21} = d_{12} = c_{30} - 3d_{21} = 3c_{12} - d_{03} = 3c_{03}d_{30} - 4d_{21}d_{03} = 0 \\ &(C) \quad \begin{cases} c_{21} + d_{12} = 0 \\ c_{30}c_{12} - d_{21}d_{03} = 0 \\ c_{30}^2c_{03} + d_{30}d_{03}^2 = 0 \\ c_{30}c_{03}d_{21} + c_{12}d_{30}d_{03} = 0 \\ c_{12}^2d_{30} + c_{03}d_{21}^2 = 0. \end{cases} \end{aligned} \tag{5.21}$$

Geometric meaning of the different strata in (5.21)**:** the stratum (A) consists of Hamiltonian systems. The stratum (B) consists of systems having generically an invariant quartic and an invariant sextic which give a rational first integral. If we take the generic case $(c_{12}^3 - 4d_{21})c_{03} \neq 0$, then we can scale the axes to make $c_{03} = 4$. This gives the integral curves

$$\begin{aligned} F_1 &= 1 + 2d_{21}x^2 + (c_{12}^3 - 4d_{21})xy/c_{12} - 2c_{12}y^2 - d_{21}(c_{12}x - 2y)^4/(4c_{12}) = 0 \\ F_2 &= 1 + 3d_{21}x^2 + 48c_{12}^2\lambda xy - 3c_{12}y^2 \\ &- 6\lambda(c_{12}x - 2y)^3(2d_{21}x + c_{12}^2y) + d_{21}\lambda(c_{12}x - 2y)^6 = 0, \end{aligned} \tag{5.22}$$

where $2\lambda = -d_{21}(c_{12}^3 - 4d_{21})^{-1}$. The cofactors are $2K/5$ and $3K/5$ respectively, where K is the divergence of the system. Once again $1/F_1F_2$ is an integrating factor, and F_1^3/F_2^2 is a rational first integral.

The stratum (C) consists of reversible systems. In the generic case $c_{12}d_{21} \neq 0$ we can scale the system so that

$$(C_1) \quad c_{12} + d_{21} = c_{30} + d_{03} = c_{03} + d_{30} = c_{21} + d_{12} = 0, \tag{5.23}$$

$$(C_2) \quad c_{12} - d_{21} = c_{30} - d_{03} = c_{03} + d_{30} = c_{21} + d_{12} = 0. \tag{5.24}$$

In the non-generic cases, we also have the limiting symmetric systems considered earlier:

$$(C_3) \quad c_{21} + d_{12} = c_{30} = d_{30} = d_{21} = 0, \tag{5.25}$$

$$(C_4) \quad c_{21} + d_{12} = c_{03} = d_{03} = c_{12} = 0, \tag{5.26}$$

which are particular cases of Theorem 5.4.

Systems in the sub-stratum (C_1) are symmetric under the transformation $(x, y) \mapsto (y, x)$. In fact, these systems are also symmetric under $(x, y) \mapsto (-y, -x)$, which means that they can be obtained from a simpler system via the double folding transformation $(u, v) \mapsto ((x - y)^2, (x + y)^2)$, or equivalently $(z, w) \mapsto (x^2 + y^2, xy)$. The transformed system will be linear and thus has two invariant lines generically. These will be carried over to two invariant conics in the transformed system. A Darbouxian first integral can then be constructed.

If we work over the complex numbers we can reduce (C_2) to the case (C_1) by a complex scaling. However, over the reals the reversing transformation is complex. If we want to see what is happening to cause a centre, we can use the equivalent transformation to the double folding transformation discussed above:

$$(u, v) \mapsto (x^2 - y^2, xy). \tag{5.27}$$

The system can be shown to arise in this way from a new system

$$\begin{aligned} \dot{u} &= (u^2 + v^2)(1 + h(u, v)) \\ \dot{v} &= (u^2 + v^2)(k(u, v)). \end{aligned} \tag{5.28}$$

When we divide through by the common factor $(u^2 + v^2)$, the origin is not a critical point. Trajectories passing close to the origin approximate to the lines $v = \textit{constant}$ and are transformed to trajectories appoximating the curves $xy = \textit{constant}$. Furthermore the first integral $\phi(u, v) = v + o(|(u, v)|)$ which exists in a neighbourhood of the origin is transformed to a first integral of the original system $\phi(x^2 - y^2, xy) = xy + o(|(x, y)|^2)$ and so the origin is integrable. Visually, this transformation is equivalent to the 2-1 map $z \mapsto z^2$ of the complex plane to itself.

Proposition 5.9. *The linearisable saddles described in Theorem 5.5 lie in the strata* (C) *for the cases I-V and* (B) *for the cases VI-VII.*

Finally, we give a geometric criterion which seems to arise from considering the phase portraits given in this paper. Note first that all the examples of linearisable saddles studied in this paper have at least one separatrix which is an invariant algebraic curve.

Conjecture 5.10. *A homoclinic loop through an integrable saddle of a real vector field in the plane is an obstruction to linearisability. More generally, any integrable saddle cannot lie on a monodromic graphic.*

Acknowledgements

The authors are grateful to Dana Schlomiuk for helpful discussions. The first author thanks the Centre de Recherches Mathématiques for its hospitality in 1995.

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