Hamiltonian Dynamics, Classical R-Matrices and Isomonodromic Deformations

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Abstract

The Hamiltonian approach to the theory of dual isomonodromic deformations is developed within the framework of rational classical R-matrix structures on loop algebras. Particular solutions to the isomonodromic deformation equations appearing in the computation of correlation functions in integrable quantum field theory models are constructed through the Riemann-Hilbert problem method. The corresponding τ-functions are shown to be given by the Fredholm determinant of a special class of integral operators.

Keywords. Integrable systems, isomonodromic deformations, classical R-matrix, loop algebras, Riemann-Hilbert problem, τ-function, Fredholm determinants

Résumé

On développe l'approche hamiltonienne de la théorie des déformations isomonodromiques duales dans le cadres des structures des matrices-R sur les algèbres de lacets. Des solutions particulières aux équations de déformations isomonodromiques qui figurent dans le calcul des fonctions de corrélation dans des modèles intégrables de théorie quantique des champs sont construites à l'aide de la méthode du problème de Riemann-Hilbert matriciel. Les fonctions-tau correspondantes sont données par les déterminants de Fredholm d'une classe spéciale d'opérateurs.

1 Monodromy Preserving Hamiltonian Systems

1.1 Isomonodromic Deformation Equations

Monodromy preserving deformations of rational covariant derivative operators of the form:

$$\mathcal{D}_\lambda := \frac{\partial}{\partial \lambda} - \mathcal{N}(\lambda), \tag{1.1}$$

where

$$\mathcal{N}(\lambda) := B + \sum_{i=1}^{n} \frac{N_i}{\lambda - \alpha_i}, \tag{1.2}$$

B is the diagonal $r \times r$ matrix

$$B = \mathrm{diag}(\beta_1, \ldots, \beta_r), \tag{1.3}$$

and the matrices $\{N_i\}_{i=1\cdots n}$ are $r \times r$ matrix functions of the $n + r$ deformation parameters $\{a_i, \beta_a\}$, were studied by Jimbo *et. al.* in Jimbo et al. (1980, 1981); Jimbo and Miwa (1981). It was shown there that the most general differentable monodromy preserving deformations of such operators are determined by the integrable Pfaffian system:

$$dN_i = -\sum_{\substack{j=1 \\ j\neq i}}^{n} [N_i, N_j] d\log(\alpha_i - \alpha_j) - [N_i, d(\alpha_i B) + \Theta], \tag{1.4}$$

where Θ is the $r \times r$ matrix with elements

$$\Theta_{ab} = (1 - \delta_{ab})(\sum_{i=1}^{n} N_i)_{ab} d\log(\beta_a - \beta_b). \tag{1.5}$$

Such operators and their monodromy are of great importance in the theory of quantum integrable systems, since the computation of correlation functions in such systems very often leads to particular solutions to such systems (Its et al., 1990; Korepin et al., 1993). In the following subsection, it will be shown how these equations may be understood as a compatible set of nonautonomous Hamiltonian systems generated by commuting Hamiltonians that are spectral invariants of the matrix $\mathcal{N}(\lambda)$. In subsequent sections, the classical R-matrix approach to such systems will be explained and the computation of certain solutions related to Fredholm determinant calculations via the matrix Riemann-Hilbert problem will be decribed.

1.2 Nonautonomous Hamiltonian Structure

We begin with the Lie Poisson structure on $(\oplus_{i=1}^{n} \mathfrak{gl}(r))^*$, defined by the following Poisson brackets between the various matrix elements

$$\{(N_i)_{ab}, (N_j)_{cd}\} = \delta_{ij} \left(\delta_{bc}(N_i)_{ad} - \delta_{ad}(N_i)_{cb}\right). \tag{1.6}$$

The system (1.4) can be expressed in multi-Hamiltonian form by introducing the following Hamiltonian 1-form on the parameter space

$$\theta := \sum_{i=1}^{n} H_i d\alpha_i + \sum_{a=1}^{r} K_a d\beta_a \tag{1.7}$$

where

$$H_i := \mathrm{tr}(BN_i) + \sum_{\substack{j=1 \\ j\neq i}}^{n} \frac{\mathrm{tr}(N_i N_j)}{\alpha_i - \alpha_j}, \qquad i = 1, \ldots, n \tag{1.8}$$

$$K_a := \sum_{i=1}^{n} \alpha_i (N_i)_{aa} + \sum_{\substack{b=1 \\ b\neq a}}^{r} \frac{(\sum_{i=1}^{n} N_i)_{ab} (\sum_{j=1}^{n} N_j)_{ba}}{\beta_a - \beta_b}, \qquad a = 1, \ldots, r. \tag{1.9}$$

Equations (1.4) may then equivalently be written in multi-Hamiltonian form as

$$dN_i = \{N_i, \theta\}. \tag{1.10}$$

Involutiveness of the H_i's and K_a's, which will be explained in the following section, then implies that the differential form θ on the parameter space is in fact closed

$$d\theta = 0, \tag{1.11}$$

which allows one to introduce the τ-function (Jimbo et al., 1980, 1981, cf.) by the formula

$$\theta = d \log \tau. \tag{1.12}$$

In the next section, we show how the above Hamiltonian structure may very naturally be viewed as the restriction of the rational R-matrix structure on a loop algebra to a finite dimensional Poisson submanifold. From this the commutativity of the Hamiltonians (H_i, K_a) defined above follows.

2 Loop Algebra Moment Maps, Spectral Invariants and Isomonodromic Deformation Equations

The following discussion is based on the approach developed in Harnad (1994).

2.1 Dual Moment Maps and Split R-matrix Structure

We introduce an auxiliary symplectic vector space (M, ω), which will be referred to as the *generalized Moser space*, consisting of pairs (F, G) of complex $N \times r$ matrices whose elements are viewed as canonically conjugate variables. Thus, the symplectic form ω is just

$$\omega := \mathrm{tr}(dF \wedge dG^T). \tag{2.1}$$

We use the following notation to denote the loop algebra of $r \times r$ matrices depending on a loop parameter λ, viewed as a point on a circle S^1 in the complex λ-plane, and its splitting into negative a positive Fourier components

$$\widetilde{\mathfrak{gl}}(r) = \widetilde{\mathfrak{gl}}(r)_+ + \widetilde{\mathfrak{gl}}(r)_- \quad \sim \quad \widetilde{\mathfrak{gl}}(r)^* \tag{2.2}$$

$$\widetilde{\mathfrak{gl}}(r)^*_\pm \quad \sim \quad \widetilde{\mathfrak{gl}}(r)_\mp \tag{2.3}$$

The identification with the dual space indicated in eq. (2.2) is defined through the Ad-invariant scalar product

$$< X_1, X_2 >= \oint_{S^1} \mathrm{tr}\left(X_1(\lambda) X_2(\lambda)\right) d\lambda. \tag{2.4}$$

The rational R-matrix structure is obtained by just redefining the Lie algebra structure in such a way that the new algebra splits into a Lie algebraic direct sum of the positive and negative frequency parts, with a change of sign in the Lie product for the second summand

$$\widetilde{\mathfrak{gl}}_R(r) := \widetilde{\mathfrak{gl}}(r)_+ \ominus \widetilde{\mathfrak{gl}}(r)_- \sim \widetilde{\mathfrak{gl}}_R(r)^*. \tag{2.5}$$

The rational R-matrix structure is then just the corresponding Lie Poisson structure on $\widetilde{\mathfrak{gl}}_R(r)^*$. Expressed in terms of the individual matrix elements this gives

$$[\mathcal{N}_{ab}(\lambda),\ \mathcal{N}_{cd}(\mu)] = \frac{\delta_{ad}(\mathcal{N}_{cb}(\lambda) - \mathcal{N}_{cb}(\mu)) - (ad \leftrightarrow cb)}{\lambda - \mu}. \tag{2.6}$$

This can be expressed more succintly in the tensorial (St. Petersburg) notation as follows

$$\{\mathcal{N}(\lambda) \otimes \mathcal{N}(\mu)\} = [r(\lambda - \mu),\ \mathcal{N}(\lambda) \otimes \mathrm{I} + \mathrm{I} \otimes \mathcal{N}(\mu)]\,, \tag{2.7}$$

where $\mathcal{N}(\lambda)$ and $\mathcal{N}(\mu)$ are viewed as endomorphisms of C^n and the rational R-matrix $r(\lambda - \mu)$ is the endomorphism of $\mathrm{C}^n \otimes \mathrm{C}^n$ defined by

$$r(\lambda - \mu) := \frac{P_{12}}{\lambda - \mu} \in \mathrm{End}(\mathrm{C}^n \otimes \mathrm{C}^n), \tag{2.8}$$

where P_{12} denotes the endomorphism that interchanges the first and second factors in $\mathrm{C}^n \otimes \mathrm{C}^n$.

Now let A and B be the diagonal $N \times N$ and $r \times r$ matrices, respectively, defined by

$$A := \mathrm{diag}(\alpha_i) \in \mathfrak{gl}(N), \qquad B := \mathrm{diag}(\beta_a) \in \mathfrak{gl}(r), \tag{2.9}$$

where the eigenvalues $\{\alpha_i\}_{i=1\cdots n}$ have multiplicities $\{k_i\}_{i=1,\ldots n}$, and the eigenvalues $\{\beta_a\}_{a=1\cdots r}$ are multiplicity free. Define the Poisson subspace $\mathrm{g}^A_B \subset \widetilde{\mathfrak{gl}}_R(r)^*$ by

$$\mathrm{g}^A_B := \{\mathcal{N}(\lambda) = B + \sum_{i=1}^n \frac{N_i}{\lambda - \alpha_i},\ N_i \in \mathfrak{gl}(r)\} \sim \sum_{i=1}^n \mathfrak{gl}^*(r). \tag{2.10}$$

Then the following defines a Poisson quotient map of the symplectic space M, such that the image is identified with a Poisson submanifold of g_B^A.

$$\begin{aligned} \widetilde{J}_B^A : M &\longrightarrow \mathrm{g}_B^A \\ \widetilde{J}_B^A : (F,G) &\longmapsto B + G^T(A-\lambda I_r)^{-1}F =: \mathcal{N}(\lambda) \qquad (2.11) \\ \mathcal{N}(\lambda) &= B + \sum_{i=1}^n \frac{N_i}{\lambda-\alpha_i} \in \widetilde{\mathfrak{gl}}^*(r)_R \qquad (2.12) \\ N_i &:= -G_i^T F_i, \qquad F_i, G_i \in M^{k_i\times r}. \qquad (2.13) \end{aligned}$$

We may now apply the standard classical R-matrix theory to deduce a set of commuting Hamiltonian flows on the space M, generated by the spectral invariant Hamiltonians on $\widetilde{\mathfrak{gl}}(r)^*$, pulled back to M via the above Poisson map, for which the Hamiltonian flows are represented by Lax equations. The resulting Hamiltonian flows are therefore isospectral for the matrix $\mathcal{N}(\lambda)$. We denote by

$$\mathcal{I}_B^A := \mathcal{I}(\widetilde{\mathfrak{gl}}(r)^*)|_{\mathrm{g}_B^A} \qquad (2.14)$$

the ring of spectral invariants restricted to g_B^A. Then the classical R-matrix theory in this case tells us that:

(i) $\mathcal{I}_B^A$ is Poisson commutative.

(ii) For $H \in \mathcal{I}_B^A$, Hamilton's equations have the Lax form:

$$\begin{aligned} \frac{d\mathcal{N}}{dt} &= [\mathcal{A}_\sigma^H,\ \mathcal{N}], \qquad (2.15) \\ \mathcal{A}_\sigma^H &:= \sigma dH_+(\mathcal{N}) + (\sigma-1)dH_-(\mathcal{N}) =: \mathcal{P}_\sigma(dH(\mathcal{N})), \qquad (2.16) \end{aligned}$$

where the subscripts $\pm$ denote projections to the positive and negative Fourier components and $\sigma \in \mathrm{R}$ is arbitrary.

It follows that the spectral curve defined by the characteristic equation

$$\det(\mathcal{N}(\lambda) - zI_r) = 0, \qquad (2.17)$$

where

$$\mathcal{N}(\lambda)B + G^T(A-\lambda I_N)^{-1}, \qquad (2.18)$$

is invariant under the resulting Hamiltonian flows.

In order to apply this to the isomonodromic deformation equations considered above, we must adapt these results to the case of nonautonomous Hamiltonian systems, in which the flow parameters are reinterpreted as deformation parameters upon which the spectral invariant Hamiltonians may explicitly depend. This will be done in the next subsection.

2.2 Nonautonomous systems: isomonodromic deformations

Letting the matrices A and B depend explicitly on some deformation parameter t

$$A = A(t), \quad B = B(t), \qquad (2.19)$$

the above Lax equations must be modified to take the resulting explicit t-dependence of the matrix $\mathcal{N}(\lambda)$ into account. This gives the nonautonomous system

$$\frac{d\mathcal{N}}{dt} = [A_\sigma^H, \mathcal{N}] + \frac{\partial \mathcal{N}}{\partial t}. \qquad (2.20)$$

Suppose now that, for some σ and H, the following special condition holds:

$$\frac{\partial \mathcal{N}}{\partial t} = \frac{\partial \mathcal{A}_\sigma^H}{\partial \lambda}. \qquad (2.21)$$

It follows that Hamilton's equations become *isomonodromic deformation* equations, since eq. (2.20) then takes the form of commutativity conditions

$$[\mathcal{D}_\lambda,\ \mathcal{D}_t] = 0, \qquad (2.22)$$

where

$$\mathcal{D}_\lambda := \frac{\partial}{\partial\lambda} - \mathcal{N}(\lambda) \tag{2.23}$$

$$\mathcal{D}_t := \frac{\partial}{\partial t} - \mathcal{A}^H_\sigma. \tag{2.24}$$

These are precisely the necessary conditions for the invariance of the monodromy of the operator $\mathcal{D}_\lambda$ under deformations in the parameter t.

To apply this to the system (1.4), we choose the following set of spectral invariant Hamiltonians $\{H_i \in \mathcal{I}^A_B\}_{i=1,\dots n}$

$$H_i(\mathcal{N}) := \frac{1}{4\pi i}\oint_{\lambda=\alpha_i} \mathrm{tr}((\mathcal{N}(\lambda))^2 d\lambda = \mathrm{tr}(BN_i) + \sum_{\substack{j=1\\ j\neq i}}^{n} \frac{\mathrm{tr}(N_iN_j)}{\alpha_i-\alpha_j}. \tag{2.25}$$

The autonomous form of Hamilton's equations that result are then

$$\frac{\partial\mathcal{N}}{\partial t_i} = -[(dH_i)_-, \mathcal{N}], \tag{2.26}$$

where

$$(dH_i)_- = \frac{N_i}{\lambda-\alpha_i} \in \widetilde{\mathfrak{gl}}(r)_-. \tag{2.27}$$

Identifying the various deformation parameters now with the eigenvalues of the matrix A, $\{t_i = \alpha_i\}_{i=1,\dots,n}$, the Lax equations are modified to the following form:

$$\frac{\partial\mathcal{N}}{\partial\alpha_i} = -[(dH_i)_-, \mathcal{N}] - \frac{\partial(dH_i)_-}{\partial\lambda}. \tag{2.28}$$

These are just the commutativity conditions

$$[\mathcal{D}_\lambda, \mathcal{D}_i] = 0, \quad i = 1,\dots,n, \tag{2.29}$$

$$\mathcal{D}_\lambda := \frac{\partial}{\partial\lambda} - \mathcal{N}(\lambda) \tag{2.30}$$

$$\mathcal{D}_i := \frac{\partial}{\partial\alpha_i} + (dH_i)_- = \frac{\partial}{\partial\alpha_i} + \frac{N_i}{\lambda-\alpha_i}. \tag{2.31}$$

guaranteeing the preservation of the monodromy of the operator $\mathcal{D}_\lambda$ under the deformations generated by varying the α_i's. Evaluating residues at $\{\alpha_i\}_{i=1,\dots n}$ gives

$$\frac{\partial N_j}{\partial\alpha_i} = \frac{[N_j, N_i]}{\alpha_j-\alpha_i}, \quad j\neq i, \quad i,j = 1,\dots,n, \tag{2.32}$$

$$\frac{\partial N_i}{\partial\alpha_i} = [B + \sum_{\substack{j=1\\ j\neq i}}^{n} \frac{N_j}{\alpha_i-\alpha_j}, N_i]. \tag{2.33}$$

which are just the α_i components of the differential system (1.4).

2.3 Dual Isomonodromic System

To obtain the β_a components of this system, it is convenient to introduce another representation, in terms of a second system of rational covariant derivative operators whose monodromy will also be preserved: the *dual* isomonodromic system. Define another loop algebra $\widetilde{\mathfrak{gl}}(N)$, consisting of $N\times N$ matrices depending similarly on a loop parameter z that lies on a circle S^1 in the complex z-plane, with corresponding splitting into positive and negative Fourier components:

$$\widetilde{\mathfrak{gl}}(N) = \widetilde{\mathfrak{gl}}(N)_+ + \widetilde{\mathfrak{gl}}(N)_- \sim \widetilde{\mathfrak{gl}}(N)^*, \tag{2.34}$$

and corresponding rational R-matrix structure $\widetilde{\mathfrak{gl}}(N)^*_R$ on the dual space. We also define a corresponding Poisson subspace $\mathrm{g}^B_A \subset \widetilde{\mathfrak{gl}}(N)^*_R$ consisting of rational elements $\mathcal{M}(z)$ of the form

$$\mathrm{g}^B_A := \{\mathcal{M}(z) = -A + \sum_{a=1}^{r} \frac{M_a}{z-\beta_a}\} \sim \sum_{a=1}^{r} \mathfrak{gl}^*(N). \tag{2.35}$$

Introduce the "dual" Poisson map from M to g_A^B as:

$$\begin{aligned} \widetilde{J}_A^B : M &\longrightarrow \mathrm{g}_A^B \\ \widetilde{J}_A^B : (F,G) &\longmapsto -A - F(B - zI_N)^{-1}G^T := \mathcal{M}(z), \end{aligned} \tag{2.36}$$

and denote the dual ring of spectral invariants restricted to g_A^B as

$$\mathcal{I}_A^B := \mathcal{I}(\widetilde{\mathfrak{gl}}(N)^*)|_{\mathrm{g}_A^B}. \tag{2.37}$$

We then have the remarkable fact that the spectral rings $\mathcal{I}_A^B$ and $\mathcal{I}_B^A$ *coincide* when pulled back under the respective Poisson maps $\widetilde{J}_A^B$ and $\widetilde{J}_B^A$ to M.

Theorem 2.1 (Duality theorem). *The two spectral invariant rings $\widetilde{J}_A^{B*}(\mathcal{I}_A^B)$ and $\widetilde{J}_B^{A*}(\mathcal{I}_B^A)$ coincide.*

Proof. This essentially follows from the simple linear algebra identity

$$\begin{aligned} & \det(A - \lambda I_N)\det(B + G^T(A - \lambda I_N)^{-1}F - zI_r) \\ = \; & \det(B - zI_r)\det(A + F(B - zI_r)^{-1}G^T - \lambda I_N), \end{aligned} \tag{2.38}$$

which implies that the spectral curves of $\mathcal{N}(\lambda)$ and $\mathcal{M}(z)$ are identical. □

Now define the set of spectral invariant Hamiltonians $\{K_a \in \mathcal{I}_A^B\}_{a=1,\dots r}$, similarly to the H_i's, as:

$$K_a := \frac{1}{4\pi i}\oint_{z=\beta_a} \mathrm{tr}(\mathcal{M}(z))^2 dz. \tag{2.39}$$

On g_A^B, these similarly generate the equations:

$$\frac{\partial \mathcal{M}}{\partial \beta_a} = -[(dK_a)_-, \mathcal{M}] - \frac{\partial (dK_a)_-}{\partial z}, \tag{2.40}$$

where

$$(dK_a)_-(z) = \frac{M_a}{z - \beta_a} \in \widetilde{\mathfrak{gl}}(N)_-, \tag{2.41}$$

which imply the invariance of the monodromy of the rational covariant derivative operator

$$\mathcal{D}_z := \frac{\partial}{\partial z} - \mathcal{M}(z) \tag{2.42}$$

under the deformations generated by changes in the parameters $\{\beta_a\}_{a=1,\cdots r}$.

Using Theorem 2.1, we may also pull back the K_a's under the map $\widetilde{J}_B^A$, to determine corresponding spectral invariant Hamiltonian functions of $\mathcal{N}(\lambda)$. These again generate isomonodromic deformation equations for the operator $\mathcal{D}_\lambda$, which may be expressed

$$\frac{\partial \mathcal{N}}{\partial \beta_a} = [(dK_a)_+, \mathcal{N}] + \frac{\partial (dK_a)_+}{\partial \lambda}, \tag{2.43}$$

where

$$(dK_a)_+(\lambda) = \lambda E_a + \sum_{\substack{b=1 \\ b\neq a}}^{r}\sum_{i=1}^{n} \frac{E_a N_i E_b + E_b N_i E_a}{\beta_a - \beta_b}, \tag{2.44}$$

and E_a denotes the elementary $r \times r$ matrix with diagonal entry 1 in the aa position and zero elsewhere. Evaluating residues at $\lambda = \alpha_i$ gives

$$\frac{\partial N_i}{\partial \beta_a} = [(dK_a)_+(\alpha_i), N_i], \tag{2.45}$$

which are precisely the β_a components of the isomonodromic system (1.4). Similarly, viewing $\{H_i\}_{i=1,\dots n}$ as Hamiltonians defined on g_A^B, these generate the *dual* equations, which imply the invariance of the monodromy of the operator $\mathcal{D}_z$:

$$\frac{\partial \mathcal{M}}{\partial \alpha_i} = [(dH_i)_+, \mathcal{M}] + \frac{\partial (dH_i)_+}{\partial z}, \tag{2.46}$$

where

$$(dH_i)_+(z) = -zE_i + \sum_{\substack{j=1\\ j\neq i}}^{n}\sum_{a=1}^{r}\frac{E_iM_aE_j + E_jM_aE_i}{\alpha_i - \alpha_j}, \tag{2.47}$$

and E_i denotes the elementary $n\times n$ matrix with entry 1 in the ii position. Evaluating residues at $z=\beta_a$ gives these in the form

$$\frac{\partial M_a}{\partial \alpha_i} = \left[(dH_i)_+(\beta_a), M_a\right]. \tag{2.48}$$

3 Isomonodromic Deformations and the Riemann-Hilbert Problem

In this last section, we discuss certain specific solutions of the above isomonodromic deformation equations which can be constructed through application of the Zakharov-Shabat "dressing" method (Novikov et al., 1984). This class of solutions is of particular interest from the viewpoint of applications, since they arise in the calculation of correlation functions for quantum integrable systems (Korepin et al., 1993; Harnad and Its, 1997) and spectral distributions in the theory of random matrices (Tracy and Widom, 1994; Harnad et al., 1994). The results quoted in this section are based on joint work with A. Its; the full details may be found in the joint paper (Harnad and Its, 1997).

The particular class of solutions to the isomonodromic deformation equations in question may be constructed by applying the *dressing method*, based on the matrix Riemann-Hilbert problem, suitably adapted to this case. To do this, we first introduce a *vacuum* solution Ψ_0, which is chosen as the invertible $r\times r$ matrix function obtained by exponentiating λB

$$\Psi_0(\lambda) := e^{\lambda B}. \tag{3.1}$$

We then introduce a family of loop group elements $H_0(\lambda)$, viewed as $Gl(r)$-valued functions defined along some oriented, closed curve Γ chosen, in this case, to pass *through* the points $\{\alpha_i\}$, consecutively, with the latter ordered by their subscript labels. We also assume in the following that the number n of such points is even, and write $n=2m$. (If the number happens to be odd, we just increase it by adding $\lambda=\infty$ as the last point.) Let $\{\Gamma_j\}_{j=1\cdots m}$ denote the segment of Γ between α_{2j-1} and α_{2j} and let $\theta_j(\lambda)$ denote the characteristic function, along Γ, of the interval Γ_j. We define $H_0(\lambda)$ as the piecewise constant element of the form

$$H_0(\lambda) := \mathrm{I}_r + 2\pi i\sum_{j=1}^{n}\mathrm{f}_j\mathrm{g}_j^T\theta_j(\lambda), \tag{3.2}$$

where $\{\mathrm{f}_j,\mathrm{g}_j\}_{j=1\cdots m}$ is any fixed set of $r\times p$ complex, rectangular matrices, with $p\leq r$, satisfying the null conditions

$$\mathrm{g}_j^T\mathrm{f}_k = 0, \quad \forall j,k. \tag{3.3}$$

The relevant matrix Riemann-Hilbert problem consists of finding a nonsingular $r\times r$ matrix valued function $\chi(\lambda)$ that is analytic on the complement of Γ, extending to $\lambda=\infty$ off Γ, with asymptotic form

$$\chi(\lambda) \sim \mathrm{I}_r + \mathcal{O}\left(\lambda^{-1}\right) \tag{3.4}$$

for $\lambda\to\infty$, and has cut discontinuities across Γ given by

$$\chi_-(\lambda) = \chi_+(\lambda)H(\lambda), \quad \lambda\in\Gamma, \tag{3.5}$$

where $\chi_+(\lambda)$ and $\chi_-(\lambda)$ are the limiting values of $\chi(\lambda)$ as Γ is approached from the left and the right, respectively, and $H(\lambda)$ is the $r\times r$ invertible matrix valued function along Γ defined by

$$H(\lambda) = \Psi_0(\lambda)H_0(\lambda)\Psi_0^{-1}(\lambda). \tag{3.6}$$

Following the Zakharov-Shabat dressing method, we define the *dressed* wave function as

$$\Psi_\pm(\lambda) := \chi_\pm(\lambda)\Psi_0(\lambda), \tag{3.7}$$

with limiting values $\Psi_\pm$ on either side of the segments of Γ given by

$$\Psi_\pm(\lambda) := \chi_\pm(\lambda)\Psi_0(\lambda). \tag{3.8}$$

We then have the following result, which is quoted here from Harnad and Its (1997),

Theorem 3.1. *The wave function $\Psi(\lambda)$ defined by (3.8) satisfies the equations*

$$\frac{\partial \Psi}{\partial \lambda} - \left(B + \sum_{j=1}^{n} \frac{N_j}{\lambda - \alpha_j} \right) \Psi = 0, \tag{3.9}$$

$$\frac{\partial \Psi}{\partial \alpha_j} + \frac{N_j}{\lambda - \alpha_j} \Psi = 0, \tag{3.10}$$

with the N_j's given by

$$N_j := -\mathrm{F}_j \mathrm{G}_j^T, \tag{3.11}$$

where

$$\mathrm{F}_j := \lim_{\lambda \to \alpha_j} \mathrm{F}(\lambda) \quad \mathrm{G}_j := (-1)^j \lim_{\lambda \to \alpha_j} \mathrm{G}(\lambda). \tag{3.12}$$

This implies the commutativity

$$[\mathcal{D}_\lambda,\ \mathcal{D}_{\alpha_j}] = 0, \quad [\mathcal{D}_{\alpha_i},\ \mathcal{D}_{\alpha_j}] = 0, \quad i, j = 1, \ldots, n \tag{3.13}$$

of the operators

$$\mathcal{D}_\lambda := \frac{\partial}{\partial \lambda} - B - \sum_{j=1}^{n} \frac{N_j}{\lambda - \alpha_j} = \frac{\partial}{\partial \lambda} - \mathcal{N}(\lambda) \tag{3.14}$$

$$\mathcal{D}_{\alpha_j} := \frac{\partial}{\partial \alpha_j} + \frac{N_j}{\lambda - \alpha_j}, \tag{3.15}$$

and hence the invariance of the monodromy data of the operator $\mathcal{D}_\lambda$ under changes in the parameters $\{\alpha_j\}$.

Thus, the operators defined in eqs. (3.14)-(3.15) represent a solution to the α_j components of the isomonodromic system (1.4). The following result, also quoted from Harnad and Its (1997), shows that the same construction also provides a solution to the β_a components.

Theorem 3.2. *The wave function $\Psi(\lambda)$ also satisfies the equations*

$$\mathcal{D}_{\beta_a} \Psi = 0, \quad a = 1, \ldots, r, \tag{3.16}$$

where the operators $\{\mathcal{D}_{\beta_a}\}_{b=1,\ldots,r}$ are defined by

$$\mathcal{D}_{\beta_a} := \frac{\partial}{\partial \beta_a} - \lambda E_a - \sum_{\substack{b=1 \\ b \neq a}}^{r} \frac{E_a \left(\sum_{j=1}^{n} N_j \right) E_b + E_b \left(\sum_{j=1}^{n} N_j \right) E_a}{\beta_a - \beta_b}, \tag{3.17}$$

with the N_i's given by eqs. (3.11), (3.12). This implies the commutativity conditions

$$[\mathcal{D}_\lambda,\ \mathcal{D}_{\beta_a}] = 0, \quad [\mathcal{D}_{\beta_a},\ \mathcal{D}_{\beta_b}] = 0, \quad a, b = 1, \ldots, r, \tag{3.18}$$

and hence the invariance of the monodromy data of $\mathcal{D}_\lambda$ under the deformations parameterized by $\{\beta_a\}_{a=1,\ldots,r}$.

From the viewpoint of applications to quantum integrable systems (Its et al., 1990; Korepin et al., 1993) and the spectral theory of random matrices (Tracy and Widom, 1994; Harnad et al., 1994) this construction has particular importance, since the underlying τ-function, as defined in eq. (1.12), is just the Fredholm determinant of an integral operator that may be constructed from the same data, and which gives the correlation functions and spectral distribution generating functions in question. This result is contained in the following theorem, also quoted from Harnad and Its (1997).

Theorem 3.3. *Let K be the $p \times p$ matrix Fredholm integral operator acting on C^p-valued functions $\mathrm{v}(\lambda)$,*

$$\mathrm{K}(\mathrm{v})(\lambda) = \int_\Gamma K(\lambda, \mu) \mathrm{v}(\mu) d\mu, \tag{3.19}$$

defined along the curve Γ, with integral kernel given by

$$K(\lambda, \mu) = \frac{\mathrm{f}^T(\lambda) \mathrm{g}(\mu)}{\lambda - \mu}, \tag{3.20}$$

where f, g *are the rectangular* $r \times p$ *matrix valued functions*

$$\mathrm{f}(\lambda) \quad := \quad \Psi_0(\lambda) \sum_{j=1}^{m} \mathrm{f}_j \theta_j(\lambda) \tag{3.21}$$

$$\mathrm{g}(\lambda) \quad := \quad \left(\Psi_0^T(\lambda)\right)^{-1} \sum_{j=1}^{m} \mathrm{g}_j \theta_j(\lambda). \tag{3.22}$$

Then the logarithmic derivative of the Fredholm determinant is given by

$$d \ln \det(\mathrm{I} - \mathrm{K}) = \omega = \sum_{k=1}^{n} H_k d\alpha_k + \sum_{a=1}^{r} K_a d\beta_a, \tag{3.23}$$

where the individual factors may be expressed

$$H_k \quad = \quad \frac{\partial \ln \det(\mathrm{I} - \mathrm{K})}{\partial \alpha_k} = \mathrm{tr}(BN_k) + \sum_{j=1, j \neq k}^{n} \frac{\mathrm{tr}\, N_j N_k}{\alpha_k - \alpha_j} \tag{3.24}$$

$$\begin{aligned} K_a \quad &= \quad \frac{\partial \ln \det(\mathrm{I} - \mathrm{K})}{\partial \beta_a} \\ &= \quad \sum_{j=1}^{n} \alpha_j \left(N_j\right)_{aa} + \sum_{\substack{b=1 \\ b \neq a}}^{r} \frac{\left(\sum_{j=1}^{n} N_j\right)_{ab} \left(\sum_{k=1}^{n} N_k\right)_{ba}}{\beta_a - \beta_b}. \end{aligned} \tag{3.25}$$

Hence, $\det(\mathrm{I} - \mathrm{K})$ *may be identified as the* τ*-function defined in eq. (1.12).*

Finally, it should be mentioned that the dual isomonodromic systems defined in eqs. (2.42)-(2.48) may be derived in exactly the same way, by interchanging the rôles of the matrices A and B when defining the vacuum wave function (3.1). The corresponding curve $\widetilde{\Gamma}$ must be chosen, in the complex z-plane, so as to pass through the parameters $\{\beta_a\}$ giving the diagonal elements of the matrix B. The resulting τ-function turns out to just be given by the Fredholm determinant of the *dual* Fredholm integral operator $\widetilde{\mathrm{K}}$, which is related to the operator K appearing in Theorem 3.2 by taking a Fourier-Laplace transform along the curves Γ and $\widetilde{\Gamma}$. Full details regarding this result, as well as a number of related results, including generalizations to isomonodromic deformations of operators having higher order pole singularities, may be found in Harnad and Its (1997).

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