Genericity conditions for finite cyclicity of elementary graphics

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Abstract

In this paper we explore the Khovanskii method for proving the finite cyclicity of elementary graphics and how it can be applied in practice. In particular we discuss some of the genericity conditions which allow to prove finite cyclicity for elementary graphics with the Khovanskii method. By comparing them with other known finite cyclicity theorems for elementary graphics we see that they form a proper subset of the usual methods. However, contrary to the usual method, the additional conditions may consist of non-intrinsic conditions, allowing to conclude to finite cyclicity for a larger set of limit periodic sets than expected at first. Hence we introduce an extension of the method which treats the usual functional-Pfaffian systems together with the admissible changes of coordinates in the functional equations.

Résumé

Dans cet article nous explorons la méthode de Khovanskii pour montrer la cyclicité finie de graphiques élémentaires et son application en pratique. En particulier nous discutons de quelques conditions génériques suffisantes pour montrer la cyclicité finie de graphiques élémentaires par la méthode de Khovanskii et nous les comparons avec d'autres conditions suffisantes dans les théorèmes de cyclicité finie. Les nouvelles conditions suffisantes forment un sous-ensemble propre des conditions suffisantes usuelles. Mais, contrairement aux méthodes usuelles, les conditions additionnelles peuvent être non intrinsèques, permettant alors de montrer la cyclicité finie d'un ensemble plus vaste de graphiques qu'il n'y paraît au premier abord. Nous introduisons donc une extension de la méthode de Khovanskii qui traite des systèmes fonctionnels-pfaffiens habituels, munis des changements de coordonnées admissibles dans les équations fonctionnelles.

1. Introduction

The study of limit periodic sets and their cyclicity for planar vector fields is a very important subject. Since we cannot in general compute explicitly trajectories of vector fields it is one way to keep track of the number of limit cycles appearing in polynomial families of vector fields and to obtain partial results in the spirit of Hilbert's 16-th problem.

On the other hand Arnold [A] pointed out the importance of studying the bifurcations of codimension k, especially for small k, since they are unavoidable in k-parameter families of vector fields. Their study and the understanding of the bifurcations is a key to understand the dynamics of some global families of vector fields.

Up to now, most of the interest has been on the finite cyclicity of elementary graphics. One reason is that any non-elementary singular point can be blown-up into elementary singular points, yielding the hope that the methods for elementary graphics can be further refined to treat the non-elementary ones. The second reason comes from the method itself. Indeed, the idea underlying many proofs of finite cyclicity is to "calculate" a return map. For that purpose we write the return map as a composition of Dulac maps in the neighborhood of the singular points of the graphic with regular transitions between the Dulac sectors. The vector field in the neighborhood of elementary singular points can be brought to a nice normal form allowing, either an explicit calculation of the Dulac maps, or a proof of their analytic properties (for instance flatness).

In 1995 Il'yashenko and Yakovenko [IY2] published a beautiful paper on the finite cyclicity of elementary polycycles in generic families. The paper asserts the existence of a bound $E(k)$ for the cyclicity of elementary graphics appearing in generic k-parameter families. The idea of the proof is to replace the calculation of Dulac maps by the fact that the graphs of these maps are separating solutions of Pfaffian equations, allowing the use of the powerful method of Khovanskii ([K1], [K2], [K3]).

To our knowledge the first time that Khovanskii's method was used to discuss the Dulac maps in the neighborhood of singular points is in the paper of Moussu and Roche [MR], where they discuss a reduced version of Dulac's theorem on the finiteness of limit cycles for a polynomial vector field from the non-accumulation of limit cycles on a polycycle.

The method used by Il'yashenko and Yakovenko is the following. They first derive a polynomial normal form for "generic" elementary singular points [IY1]. The number of limit cycles in a pertubation of a graphic with n singular points is then studied as the number of solutions of a system of $2n$ equations in $2n$ unknowns. Half of these equations are equations of the form $x_{i+1} = f_i(y_i)$, where the functions f_i are C^K-diffeomorphisms, with K arbitrarily large, and the other half are equations of the form $y_i = D_i(x_i)$, where D_i is the Dulac map in the neighborhood of the i-th singular point. From the polynomial normal form of the vector field in the neighborhood of the singular points we can deduce that the graphs of the Dulac maps are separating solutions of Pfaffian equations. The Khovanskii procedure allows to give a bound on the number of solutions of such a system. The proof proceeds by induction on the number of Pfaff forms. At each step a reduction allows to get rid of one Pfaff form and to replace the original system by two systems containing one Pfaff form less and one more C^K-function. We end up with a number of systems of n C^K-functions in n-unknowns. From an upper bound to the number of local solutions to each of these systems we obtain an upper bound for the number of solutions of the original system.

Since we will thoroughly use the method used by Il'yashenko and Yakovenko (which is derived from the theory of Khovanskii) throughout the paper we will give it a name: the *Il'yashenko-Khovanskii-Yakovenko method*, which we will abbreviate to IKY method.

The IKY method works for generic systems. However the genericity conditions are not explicitly given. In fact, to our knowledge, the method has never been integrally applied to particular systems. Variants of the method have been used to prove the finite cyclicity of an important number of graphics (e.g. [DER], [EM], [MM1], [MM2]). In these cases one usually replaces only one Dulac map by the separating solution of a Pfaffian equation.

In this paper we want to explore both the genericity conditions, which make the theorem of Il'yashenko and Yakovenko work and the way they can be checked in practice. For that purpose we start with a local analysis of the Dulac map for each type of singular point. Since the graph of each Dulac map is a separating solution of a Pfaffian equation, we compare the information contained in the Dulac map with the information contained in the Pfaffian equation. We describe precisely what information is kept in the process and what information is lost. Through a rapid inspection we see that information is lost in the cases of hyperbolic saddles with an irrational hyperbolicity ratio and for the central transition through a saddle-node, while no information is lost in the case of a non-integrable hyperbolic saddle with rational hyperbolicity ratio and in the case of stable-center (center-unstable) transition through a semi-hyperbolic point. Moreover, in practical applications a system is usually brought to normal form only up to a C^k-flat remainder. This means that the sections in the neighborhood of the singular points and the coordinates on them are only defined up to a certain order K depending on k and on some data in the normal form (for instance the hyperbolicity ratio at a non-resonant hyperbolic saddle). Genericity conditions for the IKY method usually involve conditions on the derivatives up to some order ℓ of the regular transitions far from the singular points. We are interested to know what is the minimum K, call it $K(\ell)$ so that the considered conditions are intrinsic.

The normalizing changes of coordinates are not unique. It is then natural in this context to consider the different changes of coordinates which preserve the normal form (up to C^k-equivalence). Under such changes of coordinates we had the surprise to find that several conditions which were needed to prove the finite cyclicity by the IKY method were not intrinsic. An adequate change of coordinates allows one to create these conditions, which concern the regular transition maps. For instance it seems a priori that the finite cyclicity (cyclicity one) of the graphic with central transition through a unique saddle-node cannot be proved by the IKY method. Indeed this follows from the fact that the Dulac map is linear with a very small coefficient. The information concerning the smallness of the coefficient is lost when one passes from the Dulac map to the Pfaff form. To conclude by the IKY method it is necessary that the regular transition be nonlinear. However this condition is not intrinsic. It is always possible to make a change of coordinates which preserves the normal form and the Pfaff form and which sends the regular transition to a nonlinear one. Hence the IKY method allows to conclude to the finite cyclicity of a graphic with central transition through a unique saddle-node.

This consideration suggested us a new refinement of the IKY method. In this refinement the data consists of the Pfaff forms, the regular transitions $x_{i+1} = f_i(y_i)$ and the admissible changes of coordinates on x_i and y_i. It is this extended IKY method which we consider throughout the paper.

We then look to the simpler graphics for which finite cyclicity is known and we compare, both the bound for the cyclicity given by IKY method and the genericity conditions allowing to conclude. It appears in all cases that the bounds are systematically greater than the bounds obtained by the analysis of the return map and hence can never be attained. The genericity conditions for the IKY method form a proper subset of the usual genericity conditions.

For graphics through one singular point this depends heavily on the local analysis and on the information which is lost in the process of passing from the Dulac maps to the separating solution of a Pfaffian equation. In the case of a saddle loop the usual generic conditions allowing to conclude to finite cyclicity are of the following three types:

(1) the hyperbolicity ratio is different from one;
(2) the hyperblicity ratio is one and the saddle point is non integrable;
(3) the regular transition far from the singular point is not the identity map.

With the IKY method we cannot conclude exactly in the infinite codimension cases corresponding to an integrable saddle point with rational hyperbolicity ratio because there is no Pfaff form in that case.

For graphics with a semi-hyperbolic point and stable-center (center-unstable) transition the IKY method allows to conclude to finite cyclicity. In the case of central transition the refinement described above is necessary to conclude to finite cyclicity through the IKY method.

We then go to graphics with two hyperbolic non-resonant saddles and prove that we have finite cyclicity as soon as the product of the hyperbolicity ratios r_1 and r_2 is different from one. When $r_1 r_2 = 1$ the condition that the two transitions have non-vanishing second derivative is sufficient to prove the finite cyclicity by the IKY method. Here again one of the conditions may not be intrinsic. We apply these results to a particular graphic surrounding a center inside quadratic systems and show that it has finite cyclicity in the same family. The method for the center case requires an argument similar to the Bautin's method [B] introduced in the study of the cyclicity of centers in quadratic systems.

The paper [RSZ] treats the case of graphics with two semi-hyperbolic points of same codimension, the unstable manifold of one point being connected to the stable manifold of the other. Such graphics, together with conditions for finite cyclicity were studied by El Morsalani [EM]. The generic conditions of El Morsalani are not sufficient to conclude to finite cyclicity by the IKY method. However using a mixture of the IKY method and of the flatness properties of the transition map in a saddle sector of an attracting semi-hyperbolic point [DRR2] the paper [RSZ] contains an elementary proof of the finite cyclicity of several such graphics among quadratic systems.

2. Dulac maps in the neighborhood of a generic elementary singular point

We first recall the theorem from [IY1] giving the normal forms of all singular points. We then discuss the general framework and the particular cases.

Theorem 2.1. *Suppose that a generic finite-parameter family $\{\mathcal{X}_\lambda\}_{\lambda\in\Lambda}$ of smooth vector fields on the plane possesses an elementary singular point for $\lambda = 0$. If this point has at least one hyperbolic sector, then the family is finitely differentiably orbitally equivalent to a family induced from some localization of one of the families given in the second column of Table 1.*

Remark 2.2. The correspondence maps differ slightly from the ones appearing in [IY1] for the last two cases. Indeed the authors consider a transition from $\{x = -1\}$ (resp. $\{y = 1\}$) to $\{x = 1\}$ in the case (D_m^C) (resp. (D_m^H)). However it may happen, depending of the value of a, that the system has an additional singular point before $x = 1$. This explains why we rather choose sections $\{x = \pm x_0\}$. For the other cases, one can always suppose that the equivalence of $\{X_\lambda\}_{\lambda\in\Lambda}$ to the normal forms given in Table 1 are defined in a neighborhood of the origin which contains sections $\{x = 1\}$ and $\{y = 1\}$ of the invariant manifolds. This last statement is a consequence of Proposition A.1 of the appendix.

2.1 Intrinsic properties of regular transitions. Here we explain in more details what we mean by an intrinsic property. Suppose we have a smooth family $(\mathcal{X}_\lambda)|_{\lambda\in\Lambda}$ of vector fields which has a polycycle for $\lambda = 0$ and let be given near each singular point a C^k-chart bringing it to one of the normal forms in Table 1. This can be done in several ways. It is clear that it is sufficient to limit ourselves to changes of coordinates that preserve the first quadrant. Two different charts of this kind differ by what we will call a k-admissible change of coordinates.

Definition 2.3. A k-admissible change of coordinates is the germ of a family of C^k-diffeomorphisms

$$\Psi_\lambda(x, y) = (X_\lambda, Y_\lambda) \tag{2.1}$$

such that

(1) Ψ_λ is defined in a connected neighborhood $U \subset \mathbb{R}^2$ of the origin such that there exist sections $x = \pm x_0$ (resp. $y = y_0$ which belong both to the domain and the image of Ψ_λ.
(2) Ψ_λ preserves the axes and their orientation.
(3) Ψ_λ preserves the normal form in the sense of orbital equivalence.
(4) The map $(x, y, \lambda) \mapsto \Psi_\lambda(x, y)$ is of class C^k.

Consider a transversal section to one characteristic manifold of the form $\{x = x_0\}$ and a regular transition map defined in that section. Parametrizing the section with the parameter y, we obtain the expression of the regular transition map as a C^k-function $y \mapsto R_\lambda(y)$ defined in a neighborhood of zero.

A change of coordinates (2.1) yields a new regular transition map defined in the section $\{X = x_0\}$ which is described by a new C^k-function $Y \mapsto \hat{R}_\lambda(Y)$. The relation between these two functions is given by $R_\lambda = \hat{R}_\lambda \circ f_\lambda$, where f_λ is the C^k-diffeomorphism satisfying $T_\lambda \circ \Psi_\lambda(x_0, y) = (x_0, f_\lambda(y))$ and T_λ is the regular transition from $\{x = x_0\}$ to $\{X = x_0\}$ in Y-coordinate.

Definitions 2.4.

(1) The *k-exit group* of the singularity is the group of families of germs of C^k-diffeomorphisms f_λ satisfying $T_\lambda \circ \Psi_\lambda(x_0, y) = (x_0, f_\lambda(y))$, where Ψ_λ is a k-admissible change of coordinates and T_λ is the regular transition from $\{x = x_0\}$ to $\{X = x_0\}$ in Y-coordinate. The group formed by the germs of functions f_0 will be called the *initial k-exit group*.
(2) A property of a regular transition starting in $x = x_0$ is *right k-intrinsic* if it is a property of all elements of the orbit of R_0 by the action of composition on the right with elements of the initial k-exit group.

For transition maps R_λ ending in the transversal section of the stable manifold $\{y = y_0\}$ (resp. in the transversal section $\{x = -x_0\}$ for the case of a central transition through a semi-hyperbolic point (type (D_m^C)) we can make an analogous construction, obtaining that if $z \mapsto \hat{R}_\lambda(z)$ is the expression of the transition map ending in $\{y = y_0\}$ (resp. $\{x = -x_0\}$) then $\hat{R}_\lambda = g_\lambda \circ R_\lambda$.

Definitions 2.5.

(1) The *k-entrance group* of the singularity is the group of families of germs of C^k-diffeomorphisms g_λ satisfying $T_\lambda \circ \Psi_\lambda(x, y_0) = (g_\lambda(x), y_0)$, where Ψ_λ is a k-admissible change of coordinates and T_λ is the regular transition from $\{y = y_0\}$ to $\{Y = y_0\}$ in X-coordinate (resp. $T_\lambda \circ \Psi_\lambda(-x_0, y) = (-x_0, g_\lambda(y))$ and T_λ is the regular transition from $\{x = -x_0\}$ to $\{X = -x_0\}$ in Y-coordinate in the case of the type (D_m^C)). The group formed by the germs of functions g_0 will be called the *initial k-entrance group*.
(2) A property of a regular transition starting in $\{x = x_0\}$ is *left k-intrinsic* if it is a property of all elements of the orbit of R_0 by the action of composition on the left with elements of the initial k-entrance group.

Definition 2.6. A property of the regular transition map R_0 starting and ending in two sections as described above, corresponding to two singularities S and $\hat{S}$ is *k-intrinsic* if it is a property of all elements of the orbit of R_0 by the action of composition on the left with elements of the initial k-entrance group of $\hat{S}$ and on the right with elements of the initial k-exit group of S.

Remarks 2.7.

(1) It is not a priori obvious that the k-exit and k-entrance groups are indeed groups. It follows easily from the fact that an element of the k-exit (resp. k-entrance) group can be realized through an admissible change of coordinates which leaves invariant the section $\{x = x_0\}$ (resp. $\{y = y_0\}$ or $\{x = -x_0\}$).
(2) We may have to consider generic properties of finite sets of regular transition maps. For instance, if we consider a graphic with exactly n singular points which are all attracting saddle-nodes with central transition, then the value of the first derivative $R_{i,\lambda}(0)|_{\lambda=0}$ of each regular transition is not intrinsic, but the product of the first derivatives $\prod_{i=1}^n R'_{i,\lambda}(0)|_{\lambda=0}$ is indeed intrinsic. This was implicitely used in the proof that this graphic has cyclicity one ([DRR2]).

The intrinsic properties of the k-jet of R_0 depend exactly on the initial k-exit and k-entrance groups of the singularities, for k sufficiently large. Let Λ_n^k be the group of germs of C^k-functions of the form $h(s) = as + o(s^n)$ where a is a positive number and let $\Lambda_{n,1}^k$ be the subgroup of germs of C^k-functions which are the sum of the identity plus a n-flat function.

Proposition 2.8.

(1) *For $n \leq k$, the n-jet at zero of R_0 is left k-intrinsic (resp. right k-intrinsic) if and only if the k-exit (resp. k-entrance) group is contained in $\Lambda_{1,n}^k$.*

(2) *If the k-exit (resp. k-entrance) initial group is contained in Λ_n^k then the order at zero of R_0 minus its linear part is left k-intrinsic (resp. right k-intrinsic) as soon as it is not greater than n.*

We then need to calculate the exit and entrance groups for all singularities appearing in Table 1. One idea which is used systematically for characterizing the exit group is the following one. Given a family of local first integrals H_λ (not necessarily smooth) of the family of vector fields $(\mathcal{X}_\lambda)_{\lambda\in\Lambda}$ whose characteristic manifolds are the axes and such that H_λ restricted to the considered sections $\{x = \pm x_0\}$ is injective, and given a family Ψ_λ of C^k-changes of coordinates defined on a sufficiently large neighborhood of the origin, then Ψ_λ is an admissible change of coordinates if and only if there exist two families of functions $h_\lambda^\pm$ such that

$$\begin{aligned} h_\lambda^+ \circ H_\lambda = H_\lambda \circ \Psi_\lambda \qquad &\text{for} \quad x > 0 \\ h_\lambda^- \circ H_\lambda = H_\lambda \circ \Psi_\lambda \qquad &\text{for} \quad x < 0. \end{aligned} \tag{2.2}$$

In particular, if $H_\lambda(x_0, y) = y$, then h_λ^+ is in the k-exit group.

In the next subsections we specialize this general framework to the four types of singularities occuring in Table 1.

2.2 Non-resonant hyperbolic saddle (type (S_0)). The normal form is

$$\begin{aligned} \dot{x} &= x \\ \dot{y} &= -ry. \end{aligned} \tag{2.3}$$

It is invariant under changes of coordinates

$$(x, y) \mapsto (X, Y) = (Ax, By). \tag{2.4}$$

The Dulac map $D_\lambda = x^{r(\lambda)}$ is calculated from a section $\{y = 1\}$ to $\{x = 1\}$. It is easily verified that the constant 1 in front of the monomial $x^{r(\lambda)}$ is intrinsic. It is clear however that this constant is lost as soon as we use the Pfaff form. Indeed the Pfaff form is not sufficiently fine to distinguish between $y = x^{r(\lambda)}$ and $y = Cx^{r(\lambda)}$. Hence all proofs of finite cyclicity that make an essential use of the value of the constant cannot be recovered by the application of the IKY method.

Usually, in finite cyclicity proofs one has to consider the truncated Taylor series of the regular transition R starting from the sections $\{x = 1\}$ of the unstable manifold or ending in the section $\{y = 1\}$ of the stable manifold. Typical generic conditions are concerned with the non-vanishing of the higher order derivatives $R^{(\ell)}(0)$, $\ell > 1$, and by the first derivative not being equal to one $R'(0) - 1 \neq 0$. From the invariance under (2.4) it is clear that the latter is not intrinsic. To consider what happens in the former case we need to consider more general k-admissible changes of coordinates.

Theorem 2.9. *We consider a non-resonant hyperbolic saddle with hyperbolicity ratio r and normal form* (2.3)*.*

(1) *Let R be a regular transition map starting from $\{x = 1\}$ and let $\ell > 1$ be the order at zero of R minus its linear part. If $r < 1$, then ℓ is left k-intrinsic as soon as $k \geq \ell$. If $r > 1$, then ℓ is left k-intrinsic as soon as k satisfies $\ell \leq [(k-1)/r] + 1$.*

(2) *Let S be a regular transition map ending in $\{y = 1\}$ and let $\ell > 1$ be the order at zero of S minus its linear part. If $r > 1$, then ℓ is right k-intrinsic as soon as $k \geq \ell$. If $r < 1$, then ℓ is right k-intrinsic as soon as k satisfies $[(k-1)r] + 1 \geq \ell$.*

Remark 2.10. Indeed, for $r > 1$, given any numbers $b_{m+1}, \ldots, b_k$ with
$m = [(k-1)/r] + 1$ there exist a k-admissible change of coordinates such that the derivatives of the transition map $Y \mapsto \hat{R}(Y)$ defined at $\{X = 1\}$ satisfy $\hat{R}^{(i)}(0) = b_i$. The case $r < 1$ is analogous.

The proof of Theorem 2.9 and of the Remark 2.10 is an immediate consequence of the next theorem:

Theorem 2.11.
Let $r > 1$ and $m = [(k-1)/r] + 1$, then

(1) *The initial k-exit group of S_0 is contained in Λ^k_m and contains Λ^{k+1}_m.*
(2) *The initial k-entrance group of S_0 is contained in Λ^k_k.*

Let $r < 1$ and $n = [(k-1)r] + 1$.

(1) *The initial k-entrance group of S_0 is contained in Λ^k_n and contains Λ^{k+1}_n.*
(2) *The initial k-exit group of S_0 is contained in Λ^k_k.*

Proof. 1) Take f in the initial k-exit group of S_0 and $(x,y) \mapsto (X,Y)$ a k-admissible change of coordinates associated to f. Since it preserves the quadrants it is of the form $(X,Y) = (x\phi(x,y), y\psi(x,y))$ with ϕ and ψ, C^{k-1}-functions satisfying $\phi(0) > 0$ and $\psi(0) > 0$.

Notice that $H(x,y) = x^r y$ is a first integral such that $H(1,y) = y$, so, for $x > 0$, $f(x^r y) = X^r Y$. Then $f(s) = sf_1(s)$, with $f_1(x^r y) = \phi(x,y)^r \psi(x,y)$. Evaluating in $y = 1$ we see that $x \mapsto f_1(x^r)$ can be extended (for $x < 0$) as a C^{k-1}-function. Also f_1 is a C^{k-1}-function.

Let ℓ be such that

$$f_1(y) = a_0 + a_\ell y^\ell + \dots a_{k-1} y^{k-1} + o(y^{k-1}),$$

yielding

$$f_1(x^r) = a_0 + a_\ell x^{\ell r} + \cdots + a_{k-1} x^{(k-1)r} + o(|x|^{(k-1)r}). \tag{2.5}$$

This function can be a C^{k-1}-function only if $\ell r > k-1$, i.e. $\ell \geq [(k-1)/r] + 1 = m$, so f is in Λ^k_m.

Conversely, take f in Λ^{k+1}_m and let f_1 be the C^k-function such that $f(s) = sf_1(s)$. Let

$$\Psi_\lambda(x,y) = \left(x f_1^{1/r(\lambda)}(|x|^{r(\lambda)} y)\, , y \right). \tag{2.6}$$

Notice that $f \circ H_\lambda = H_\lambda \circ \Psi_\lambda$, where $H_\lambda(x,y) = |x|^{r(\lambda)} y$ is a first integral such that $H_\lambda(1,y) = y$ and Ψ_λ is a diffeomorphism in a neighborhood of the origin containing the points $(\pm 1, 0)$ and $(0, \pm 1)$ in its domain and its image; so in order to prove that f is in the initial k-exit group of S_0 we need only to check that $(x,y,\lambda) \mapsto \Psi_\lambda(x,y)$ is a C^k-map, and for this, it is enough to see that $X(x,y,r) = x f_1^{1/r}(|x|^r y)$ is a C^k-function.

Let $\xi(x,y,r) = X(x,y,r) - x f_1^{1/r}(0)$. We have that

$$\xi(x,y,r) = x\, h(\rho(x,y), r) \tag{2.7}$$

where $\rho(x,y) = |x|^r y$ and $h(s,r) = f_1^{1/r}(s) - f_1^{1/r}(0)$.

We claim that ψ is a C^k-function, in fact its derivatives are of the form:

$$\frac{\partial^k \xi}{\partial r^n \partial x^{k-n-\ell} \partial y^\ell} = x^\alpha y^\beta (e_0 + e_1 \ln x + \cdots + e_n (\ln x)^n) \tag{2.8}$$

where $\alpha > 0$, $\beta \geq 0$ and $e_i(x,y,\lambda)$ are continuous functions.

For proving this claim we will just use that:

(i) ξ is of the form (2.7), $\xi(x,y,r) = o(x^k y^m)$, and ξ is k-differentiable in y.
(ii) ρ satisfies $\frac{\partial \rho}{\partial x} = r\rho/x$, $\frac{\partial \rho}{\partial y} = \rho/y$ and $\frac{\partial \rho}{\partial r} = \rho \ln x$.
(iii) For every $i \leq k$, $\frac{\partial^i h}{\partial r^i}$ is a C^k-function of the form $\frac{\partial^i h}{\partial r^i}(s,r) = s^m \xi_i(s,r)$ for some C^k-function ξ_i, yielding that $\frac{\partial^{i+\ell} h}{\partial r^i \partial s^\ell} = s^{m-\ell} \xi_{\ell,i}(s,r)$ for some $C^{k-\ell}$-functions $\xi_{\ell,i}$.
(iv) $(m-1)r > k-1$.

2) Take g in the initial k-entrance group of S_0. Using that $D(x) = x^r$ is the Dulac map, it is obtained that for $x > 0$, $g(x) = f^{1/r}(x^r)$ for some f in the initial k-exit group and hence in Λ^k_m, yielding that $g \in \Lambda^k_k$.

For $r < 1$ the proof is analogous. In this case it is suitable to use the first integral $H(x,y) = xy^{1/r}$. □

2.3 Resonant hyperbolic saddle of finite type (type (S_m)). A first remark from the Theorem 2.1 is that there is no way to bring, by a C^k-change of coordinates, a family with an integrable saddle point with rational hyperbolicity ratio to a polynomial normal form which would be independent of k, as soon as k would be sufficiently large. However there exist "natural" polycycles with finite cyclicity and with such points, for instance a reversible hyperbolic saddle (with a symmetry axis) together with a homoclinic loop on one side of the symmetry axis. These cannot be treated by the IKY method.

Definition 2.12. A non-integrable hyperbolic saddle with rational hyperbolicity $r = p/q$ is said to be *of order* m if the family can be brought to the normal form

$$\begin{aligned}\dot{x} &= x \\ \dot{y} &= y(-r \pm u^m(1 + a(\lambda)u^m) + \sum_{i=0}^{m-1} \epsilon_i(\lambda)u^i),\end{aligned} \tag{2.9}$$

where $u = x^p y^q$. The number $a(0)$ is called the *formal invariant* of the resonant hyperbolic saddle.

Proposition 2.13. *An admissible C^k-change of coordinates, with $k \geq m(p+q)+3$ has the form*

$$(X, Y) = (Ax + o(|(x,y)|), By + o(|(x,y)|), \tag{2.10}$$

with $A(0)^p B(0)^q = 1$.

Proof. Let

$$\begin{aligned}X &= Ax + \sum_{2 \leq i+j \leq (p+q)m+3} a_{ij} x^i y^j + o(|(x,y)|^{m(p+q)+3}) \\ Y &= By + \sum_{2 \leq i+j \leq (p+q)m+3} b_{ij} x^i y^j + o(|(x,y)|^{m(p+q)+3})\end{aligned} \tag{2.11}$$

with $A, B > 0$, be a C^k-admissible change of coodinates. For $\lambda = 0$ we look at the system

$$\begin{aligned}\dot{X} &= K(x,y)X \\ \dot{Y} &= K(x,y)Y(-\frac{p}{q} \pm X^{pm}Y^{qm} + a(0)X^{2pm}Y^{2qm}).\end{aligned} \tag{2.12}$$

The function $K(x, y)$ can be eliminated in (2.12), yielding

$$\dot{X}Y(-\frac{p}{q} \pm X^{pm}Y^{qm} + a(0)X^{2pm}Y^{2qm}) - X\dot{Y} = 0. \tag{2.13}$$

The left hand side is a C^{k-1}-function in (x, y). The vanishing of the coefficient of $x^{pm+1}y^{qm+1}$ yields to $A(0)^{pm+1}B(0)^{qm+1} = A(0)B(0)$ which is equivalent to $A(0)^p B(0)^q = 1$. $\square$

The Proposition A.1 of the appendix gives a method allowing to suppose that portions of the sections $x = 1$ and $y = 1$ near the intersection with the axes are included in the neighborhood of the origin where the normal form is valid. However this construction is not explicit and hence cannot be used for particular examples. Since we are also interested to apply the method in particular cases we want to be able, in practice, to restrict ourselves to scalings $(X, Y) = (Ax, By)$. The only such scalings preserving the normal form satisfy $A^p B^q = 1$. We consider transitions maps from a section $\{y = C\}$ to a section $\{x = c\}$, with $c, C > 0$. In order that the Dulac map is given by the formula $D(x) = x^{p/q}(1 + O(\lambda) + o(x))$ we need to take $C = c^{p/q}$. The rest of this section will be concerned with such sections.

Theorem 2.14.

(1) *For $k > mq$, the elements of the initial k-exit group of the singularity have the form*

$$Y(y) = y(1 + a_q y^q + \dots a_{(m-1)q} y^{(m-1)q} + o(y^{(m-1)q})). \tag{2.14}$$

In particular they are functions in $\Lambda^k_{q,1}$.

(2) *For $k > mp$, the elements of the initial k-entrance group of the singularity have the form*

$$X(x) = x(1 + b_p x^p + \dots b_{(m-1)p} x^{(m-1)p} + o(x^{(m-1)p})). \tag{2.15}$$

In particular they are functions in $\Lambda^k_{p,1}$.

(3) *In the particular case when $p = q = 1$ and R is the transition map from $x = c$ to $y = c$, if $k > m$, then the m-jet of R is k-intrinsic.*

Corollary 2.15. *The first derivative of the transition map starting in $\{x = c\}$ (resp. ending in $\{y = c^{p/q}\}$, resp. starting in $\{x = c\}$ and ending in $\{y = c^{p/q}\}$) is left k-intrinsic (resp. right k-intrinsic, resp. k-intrinsic).*

Proof. The initial exit and entrance groups of the singularity are included in $\Lambda_{1,1}^k$. $\square$

Proof of Theorem 2.14. We limit ourselves to the singularity for $\epsilon = 0$. The system has the form

$$\begin{aligned} \dot{x} &= x \\ \dot{y} &= y(-r \pm u^m(1 + au^m)), \end{aligned} \tag{2.16}$$

with $r = p/q$. Changing y to $u = x^p y^q$ allows a separation of variables and an explicit integration of the system which has the first integral

$$F = \frac{u^m}{1 + u^m \ln\left(x^{\pm qm}\left(\frac{u^m}{1+au^m}\right)^a\right)}. \tag{2.17}$$

Let us also consider the first integral:

$$F_1 = \frac{x^{p/q} y}{\left(1 + u^m \ln\left(x^{\pm qm}\left(\frac{u^m}{1+au^m}\right)^a\right)\right)^{\frac{1}{mq}}}. \tag{2.18}$$

A C^k-change of coordinates

$$\begin{aligned} X &= x(A + \phi(x, y)) \\ Y &= y(B + \psi(x, y)) \end{aligned} \tag{2.19}$$

with $A, B > 0$ is admissible if and only if there exists a function g such that

$$F_1(X, Y) = g(F_1(x, y)) \tag{2.20}$$

with $g(v) = vg_1(v) = v + o(v)$ by Proposition 2.13. We will study the smoothness properties of g. We let $z(x, y) = (A + \phi)^{p/q}(B + \psi) = 1 + O(|(x, y)|)$. The equation (2.20) is equivalent to

$$\begin{aligned} &\frac{z}{\left(1 + u^m z^{mq} \ln\left[x^{\pm qm}(A + \phi)^{\pm qm}\left(\frac{u^m z^m}{1+au^m z^m}\right)^a\right]\right)^{1/mq}} \\ &= \frac{g_1(F_1(x, y))}{\left(1 + u^m \ln\left[x^{\pm qm}\left(\frac{u^m}{1+au^m}\right)^a\right]\right)^{1/mq}}. \end{aligned} \tag{2.21}$$

Lemma 2.16. *Let us consider the function*

$$F_1(c, y) = \frac{c^{p/q} y}{\left(1 + c^{pm} y^{qm} \ln\left(c^{\pm qm}\left(\frac{c^{pm} y^{qm}}{1+ac^{pm} y^{qm}}\right)^a\right)\right)^{1/mq}}.$$

It has the properties:

1. *$F_1(c, y)$ is a C^{qm}-diffeomorphism.*
2. *$F_1(c, y)$ has an asymptotic expansion of the form*

$$F_1(c, y) = c^{p/q} y + \sum_{i \geq qm+1} c_i y^i + \sum_{i \geq qm+1} d_i y^i \ln y, \tag{2.22}$$

with $d_{qm+1} = -ac^{(mq+1)p/q}$. In particular F_1 is a function of type I_k of Mourtada [M1].

3. *$F_1(c, y)$ has an inverse function $y = h(t)$ which is a C^k-diffeomorphism, a function of type I_k of Mourtada, with an asymptotic expansion*

$$h(t) = c^{-p/q} t + \sum_{i \geq qm+1} C_i t^i + \sum_{i \geq qm+1} D_i t^i \ln t, \tag{2.23}$$

where $D_{qm+1} = ac^{-p/q}$.

The lemma is straightforward from (2.21). From this, substituting $x = c$ in (2.21) and $y = h(t)$ we deduce the properties of g_1 and, hence, those of g.

Lemma 2.17.

(1) *$g_1(t)$ is a C^{qm}-function and $g(t)$ is a C^{qm+1}-diffeomorphism tangent to the identity.*

(2) *$g_1(t)$ is of type I_{k-1} of Mourtada with an expansion*

$$g_1(t) = 1 + \sum_{i>0} e_i t^i + \sum_{i \geq qm+1} E_i t^i \ln t. \tag{2.24}$$

(3) *$g_1(t)$ has the form*

$$g_1(t) = 1 + e_q t^q + e_{2q} t^{2q} + \dots e_{mq} t^{mq} + o(t^{mq}). \tag{2.25}$$

Proof.

(1) and (2). From Lemma 2.16 we can only deduce that $g_1(t)$ is a C^{qm-1}-diffeomorphism tangent to the identity. The expansion of $g_1(t)$ should normally have a term in $t^{qm} \ln t$, but its coefficient is $E_{qm} = A^{p/q} Ba(1 - A^{pm} B^{qm}) = 0$, from Proposition 2.13. Hence the first non-differentiable term is $E_{qm+1} t^{qm+1} \ln t$, yielding the C^{qm}-differentiability.

(3) To prove this property we also consider the first integral $F_2(x,y) = F^{\frac{1}{pm}}(x,y) = F_1^{q/p}(x,y)$. There exists a function G such that $G(F_2(x,y) = F_2(X,Y)$. Localizing at $y = c^{p/q}$ we obtain properties of G similar to the properties of g. Indeed $G(s) = sG_1(s) = s + o(s)$, where G_1 is C^{mp}. Hence

$$G_1^{p/q}(s) = 1 + b_1 s + \dots b_{mp} s^{mp} + o(s^{mp}). \tag{2.26}$$

Moreover the relation between g and G is obtained from

$$G(F_1^{q/p}(x,y)) = F_1^{q/p}(X,Y), \tag{2.27}$$

yielding

$$G_1^{p/q}(t^{q/p}) = g_1(t). \tag{2.28}$$

This yields

$$g_1(t) = 1 + b_1 t^{q/p} + b_2 t^{2q/p} \dots b_{mp} t^{mq} + o(t^{mq}), \tag{2.29}$$

which is only possible if (2.25) is satisfied. □

End of proof of Theorem 2.14.

(1) We now look at the transition $(x = c, y) \mapsto (X = c, Y)$. This yields a function $Y(y)$. Let $U = X^p Y^q$. On $X = c$ we have $U = c^p Y^q$.

We have the equation $g(F_1(c,y) = F_1(c, Y(y))$, yielding

$$Y = h(T) = h(g(F_1(c,y)) = h \circ g \circ h^{-1}(y). \tag{2.30}$$

This yields that Y and g are conjugated by $h(t) = t + o(t^{qm})$ and $Y^{(\ell)}(0) = g^{(\ell)}(0)$ for $\ell \leq mq - 1$. Hence

$$Y(y) = y + a_q y^{q+1} + \dots a_{(m-1)q} y^{(m-1)q+1} + o(y^{(m-1)q+1}). \tag{2.31}$$

(2) Similarly if we consider the transition from $\{y = c^{p/q}\}$ to $\{Y = c^{p/q}\}$ we introduce $F_2(x, c^{p/q})$ which is an invertible function with inverse H. Both the functions $F_2(x, c^{p/q})$ and H have a development of the same type as h, the first nonlinear term being of the form $x^{mp+1} \ln x$. We have

$$\begin{aligned} X(x) &= H^{-1}((g(H^{p/q}(x)))^{q/p}) \\ &= H^{-1} \circ G \circ H(x), \end{aligned} \tag{2.32}$$

since $G(t^{q/p}) = g^{q/p}(t)$. The result follows from (1).

(3) In the case $p = q = 1$, let $R : \{x = c\} \to \{y = c\}$ and $\hat{R} : \{X = c\} \to \{Y = c\}$ be the corresponding expression after application of an admissible change of coordinates. We can remark that $G(t) = g(t)$ in that case. Then

$$R(y) = X^{-1} \circ \hat{R} \circ Y(y) = H \circ (G^{-1} \circ (H^{-1} \circ \hat{R} \circ h) \circ g) \circ h^{-1}(y). \tag{2.33}$$

Since $h(t) = t + o(t^m)$ and H has the same form, the m-jet of $\hat{R}$ is the same as that of $H^{-1} \circ \hat{R} \circ h$. Is is not changed by conjugation by g which is tangent to the identity, nor by the final composition with h^{-1} and H. □

2.3 Central and stable-center transitions near a semi-hyperbolic point of finite order (types (D_m^C) and (D_m^K)). If we suppose that we have a saddle sector in the first quadrant the normal form is

$$\begin{aligned} \dot{x} &= x^{m+1}(1+ax^m) + \sum_{i=0}^{m-1} \epsilon_i(\lambda)x^i = F(x,\lambda) \\ \dot{y} &= -y. \end{aligned} \tag{2.34}$$

First we discuss central transitions near a saddle-node. The Dulac map from a section $\{x = -x_0\}$ to a section $\{x = x_0\}$ is a linear map $D_\lambda(y) = C(\lambda)y$, with

$$\lim_{\lambda\to 0} C(\lambda) = 0. \tag{2.35}$$

This last property is obviously not reflected in the Pfaff equation $ydY - Ydy = 0$ which cannot distinguish between the map $D_\lambda(y)$ and any C^k-family of linear functions! Hence all proofs which make an essential use of the property (2.35) of the constant $C(\lambda)$ cannot be recovered by the application of the IKY method.

For example, for any regular transition R, $\{x = R_0(y)\}$ is transversal to the graph of the Dulac map for $\lambda = 0$, yielding that for small λ there is only one "small" point of intersection between these two curves. Nevertheless, $\{x = R_0(y)\}$ could have a large contact order with one of the solutions of the Pfaff equation, the worst case being when R_0 is linear. In that case $x = R_0(y)$ is a solution of the Pfaff equation, yielding no information about the number of "small" intersections between $\{x = R_\lambda(y)\}$ and a solution of the Pfaff equation.

We will see that part of this phenomenon is compensated by the fact that for regular transitions defined in a section transversal to the center manifold and contained in a saddle sector, the second derivative at zero is not left k-intrinsic, allowing us to suppose that the contact order of $\{x = R_0(y)\}$ with the solutions of the Pfaff equation is at most two.

For stable center transitions the Dulac map has flatness properties (see [DRR2]). These properties will be reflected by the Pfaff form. This will become evident in the examples of Section 4 where we will be able to conclude to finite cyclicity under the same conditions when we use the IKY method as when we calculate the return map.

The properties of the initial k-entrance and exit groups are given in the next

Theorem 2.18. *If* (2.34) *has a saddle sector then*

(1) *For a section $\{x = x_0\}$ transversal to the center manifold and contained in a saddle sector, the corresponding initial k-exit group coincides with the group of germs of increasing C^k-diffeomorphisms with a zero at 0.*
(2) *For a section $\{x = x_0\}$ transversal to the center manifold and contained in a node sector, if $k \geq m+1$ the corresponding initial k-entrance group is the group of germs of linear increasing functions.*
(3) *For a section $\{y = y_0\}$ transversal to the stable manifold and $k \geq m+1$, then the initial k-entrance group is contained in $\Lambda^k_{m,1}$.*

To prove Theorem 2.18 we will use the next

Lemma 2.19. *If* (2.34) *has a saddle sector and if $k \geq m+1$, all the k-admissible changes of coordinates, for $\lambda = 0$, are of the form $(x,y) \mapsto (x\phi(x,y), y\psi(x,y))$ with ϕ and ψ two C^{k-1} positive functions in a neighborhood of the origin, satisfying:*

$$\phi(x,y) = 1 + x^m\xi(x,y) \tag{2.36}$$

$$\psi(x,y) = \begin{cases} h_1^+(y\Delta(x))\frac{\Delta(x)}{\Delta(x\phi(x,y))}, & \text{for } x > 0 \\ h_1^-(y\Delta(x))\frac{\Delta(x)}{\Delta(x\phi(x,y))}, & \text{for } x < 0. \end{cases} \tag{2.37}$$

where ξ is a C^{k-m-1}-function, $h_1^\pm$ are two non-vanishing C^{k-1}-functions, and

$$\Delta(x) = \begin{cases} \exp\left(\int_{x_0}^{x} \frac{du}{F(u,0)}\right), & \text{for } x > 0 \\ \exp\left(\int_{-x_0}^{x} \frac{du}{F(u,0)}\right), & \text{for } x < 0. \end{cases} \tag{2.38}$$

Moreover the function $(x,y) \mapsto \int_x^{x\phi(x,y)} du/F(u,0)$ is a C^{k-1}-function in a neighborhood of the origin.

Proof. The form of ψ is a direct consequence of the fact that $H(x,y) = y\Delta(x)$ is a first integral of (2.34). Since $(x,y) \mapsto (X,Y)$ preserves the normal form, there exist two functions, h^+ and h^- such that $h^+(H(x,y)) = H(X,Y)$ for $x > 0$ and $h^-(H(x,y)) = H(X,Y)$ for $x < 0$. Evaluating each of these equalities at $x = x_0$ or $x = -x_0$ one can see that h^+ and h^- are C^k-functions. Deriving with respect to y and then evaluating at 0, yields that they have a zero of order one at zero, so they are of the form $h^\pm(s) = sh_1^\pm(s)$ for some non-vanishing C^{k-1}-functions $h_1^\pm$.

The region $x > 0$ is a saddle sector. Then in $\{x > 0\}$, $\Delta(x)$ is a C^∞-function which can be extended to $\{x \geq 0\}$ as a C^∞-function with a zero at 0. From (2.37) one gets that for $x > 0$

$$\exp\left(\int_x^{x\phi(x,y)} \frac{du}{F(u,0)}\right) = \frac{\Delta(x\phi(x,y))}{\Delta(x)} = \frac{h_1^+(y\Delta(x))}{\psi(x,y)}.$$

Since the right hand side is bounded and bounded away from zero, it follows that $\int_x^{x\phi(x,y)} du/F(u,0)$ is bounded for $x > 0$. Using that

$$\int_x^{x\phi(x,y)} \frac{du}{F(u,0)} = \frac{1}{m\phi^m(x,y)} \frac{\phi^m(x,y) - 1}{x^m} + \frac{1}{m} \ln\left(\frac{(1 + ax^m)\phi^m(x,y)}{1 + ax^m\phi^m(x,y)}\right).$$

one obtains that ϕ satisfies (2.36) for $x > 0$. It is also true for $x \leq 0$ because $k \geq m + 1$.

It is clear that we have proved that $(x, y) \mapsto \int_x^{x\phi(x,y)} du/F(u,0)$ is a C^{k-1}-function in a neighborhood of the origin. $\square$

Proof of Theorem 2.18. (1) Let f be an increasing C^k-diffeomorphism. We will give a k-admissible change of coordinates such that the germ of f is its corresponding element of the k-exit group.

Take $x_0 > 0$. For each λ such that $F(\cdot, \lambda)$ has a root in $[-x_0, x_0]$, let $r(\lambda)$ be the maximum of its roots in $[-x_0, x_0]$. Let $\mathcal{A}$ be the subset of $\mathbb{R} \times \mathbb{R}^n$ formed by the pairs (x, λ) such that $F(\cdot, \lambda) > 0$ in $[-x_0, x_0]$ or $x \in (r(\lambda), x_0]$. Let Δ be the function defined by

$$\Delta(x, \lambda) = \begin{cases} \exp\left(\int_{x_0}^x \frac{du}{F(u,\lambda)}\right), & \text{for } (x, \lambda) \in \mathcal{A} \\ 0 & \text{otherwise.} \end{cases} \tag{2.39}$$

Let f_1 such that $f(s) = sf_1(s)$.

We claim that

$$\begin{aligned} X_\lambda &= x \\ Y_\lambda &= yf_1(y\Delta(x, \lambda)) \end{aligned} \tag{2.40}$$

is an admissible change of coordinates associated to f.

Note that $H_\lambda(x, y) = y\Delta(x, \lambda)$ is a family of first integrals (non trivial on $\mathcal{A}$ only) satisfying $H_\lambda(x_0, y) = y$ and $f(H_\lambda(x, y)) = H_\lambda(X, Y)$. Hence for proving the claim one only has to check the smoothness properties of (2.40), i.e of (2.39). This follows partly from Theorem 3 of [DRR2], which shows that the function $y\Delta(x)$, together with its derivatives, is arbitrarily flat at the origin on the intersection of $\mathcal{A}$ with a neighborhood of the origin. Here we define Δ on a full neighborhood of the origin. It is hence necessary to check the C^k-differentiability at boundary points of $\mathcal{A}$. This is done in the Appendix.

(2) Take m odd and consider a section $\{x = -x_0\}$, with $x_0 > 0$. All linear transformations of the form: $(X_\lambda, Y_\lambda) = (x, b(\lambda)y)$ with $b(\lambda) > 0$ are admissible changes of coordinates whose corresponding element of the k-entrance group is $g(y) = b(\lambda)y$, so we need only to check that every element of the initial k-entrance group is linear.

Take a k-admissible change of coordinates. For $\lambda = 0$, it is of the form

$$\begin{aligned} X &= x\phi(x, y) \\ Y &= y\psi(x, y) \end{aligned} \tag{2.41}$$

with ϕ and ψ like in Lemma 2.19. Let g be its associated element of the initial k-entrance group and g_1 such that $g(s) = sg_1(s)$.

Note that $g_1 = h_1^-$, because $H(x, y) = y\Delta(x)$ with Δ given by (2.38) is a first integral of (2.34) satisfying $H(-x_0, y) = y$. Hence, for $x < 0$,

$$\psi(x, y) = g_1(y\Delta(x)) \exp\left(\int_{x\phi(x,y)}^x \frac{du}{F(u,0)}\right). \tag{2.42}$$

In Lemma 2.19 we proved that $(x, y) \mapsto \int_{x\phi(x,y)}^x \frac{du}{F(u,0)}$ is a C^{k-1}-function near the origin. Using that $\lim_{x\to 0^-} \Delta(x) = \infty$, evaluating (2.42) at $(x, s/\Delta(x))$ and taking the limit as x tends to zero, we obtain

$$\psi(0, 0) = g_1(s) \lim_{x\to 0^-} \exp\left(\int_{x\phi(x,s/\Delta(x))}^x \frac{du}{F(u,0)}\right) \tag{2.43}$$

so g_1 is a constant.

(3) Let g be in the initial k-entrance group and (2.41) be a corresponding change of coordinates. Using that $(x\phi(x,1),\psi(x,1))$ and $(g(x),1)$ are on the same trajectory, the function g is defined implicitly by the equation

$$\ln(\psi(x,1)) = \int_{x\phi(x,1)}^{g(x)} \frac{du}{F(u,0)}. \tag{2.44}$$

Using that

$$\int_{x}^{g(x)} \frac{du}{F(u,0)} = \frac{1}{x^m}\int_1^{g(x)/x} \frac{ds}{s^{m+1}(1+ax^m s^m)} \tag{2.45}$$

we have that, for x sufficiently small, $(x, g(x)/x)$ are the zeroes of

$$G(x,z) = x^m\left(\ln\psi(x,1) - \int_{x\phi(x,1)}^{x} \frac{du}{F(u,0)}\right) - \int_1^z \frac{ds}{s^{m+1}(1+ax^m s^m)}, \tag{2.46}$$

around $(0, g'(0))$. In Lemma 2.19, we proved that $\int_{x\phi(x,y)}^{x} \frac{du}{F(u,0)}$ is a C^{k-1}-function. Using that (2.40) is a diffeomorphism it is easy to see that $\psi(x,1)>0$, so G is a C^{k-1}-function.

Since $\frac{\partial G}{\partial z}(0,g'(0)) \neq 0$, $G(x,z)=0$ can be solved by the implicit function theorem, yielding $z=z(x)$ as a C^{k-1}-function. Since $\frac{\partial^\ell G}{\partial x^\ell}(0,1)=0$ for $\ell<m$, we obtain that $g(x) = x + o(x^m)$. □

As a consequence we obtain the intrinsic properties of the k-jet of the regular transitions of the unperturbed equation.

Theorem 2.20.

(1) *A regular transition map defined in a transversal section $\{x=x_0\}$ to the center manifold contained in a saddle sector does not have left k-intrinsic properties. In fact, given any germ of an increasing diffeomorphism R, there exists a coordinate system such that R is the expression of the transition map defined in that section.*

(2) *For transition maps R, ending in a transversal section $\{x=-x_0\}$ to the center manifold and contained in a node sector the order of R minus its linear part is right k-intrinsic, while the value of its first derivative at zero is not right k-intrinsic.*

(3) *If $k\geq m+1$, the k-jet of the regular transition ending in a section $\{y=1\}$ transversal to the stable manifold is right k-intrinsic.*

(4) *Given an admissible change of coordinates the corresponding elements f and g of the initial k-exit and k-entrance groups, associated to transversal sections $\{x=\pm x_0\}$ of the center manifold have the same 1-jet (i.e. $f'(0)=g'(0)$).*

(5) *For a regular transition map, R_λ, defined in a section $\{x=x_0\}$ contained in a saddle sector and ending in a section $\{x=-x_0\}$ contained in a node sector, both transversal to the center manifold, the value of the first derivative of R_0 at zero is k-intrinsic, but the order of R_0 minus its linear part is not k-intrinsic.*

Proof. The first three points of the theorem follow directly from Theorem 2.18. We prove (4) and (5), assuming that m is odd and $F(x,\lambda)=x^{m+1}(1+ax^m)+\sum_{i=0}^{m-1}\epsilon_i(\lambda)x^i$. Let f and g be respectively the elements of the initial k-exit and k-entrance groups, corresponding to the same diffeomorphism $(x,y)\mapsto (X,Y)=(x\phi(x,y), y\psi(x,y))$ for $\lambda=0$ of a k-admissible change of coordinates.

Lemma 2.19 gives the form of ψ in terms of two functions $h_1^\pm$ with the property that $h^\pm(H(x,y))=H(X,Y)$, where $h^\pm(s)=s\,h_1^\pm(s)$ and $H(x,y)=y\Delta(x)$ is the first integral with Δ as in (2.39). Using that $H(\pm x_0,y)=y$ one obtains that $h^+=f$ and $h^-=g$. Hence

$$\psi(x,y) = \begin{cases} f_1(y\Delta(x))\,\exp\int_{x\phi(x,y)}^{x}\frac{du}{F(u,0)}, & \text{for } x>0\\ g_1(y\Delta(x))\,\exp\int_{x\phi(x,y)}^{x}\frac{du}{F(u,0)}, & \text{for } x<0. \end{cases} \tag{2.47}$$

For saddle-node type, in part (2) of Theorem 2.18 we proved that g is linear, so g_1 is a constant. In Lemma 2.19 we showed that $(x,y)\mapsto \int_{x\phi(x,y)}^{x}\frac{du}{F(u,0)}$ is of class C^{k-1}, and we have that $\lim_{x\to 0^+}\Delta(x)=0$. The only way that ψ can be smooth is if $f_1(0)=g_1(0)$, yielding $f'(0)=g'(0)$. The argument for saddle-type is analogous (notice that in this case $\lim_{x\to 0}\Delta(x)=0$).

For (5) this proves that $R_0'(0)$ is k-intrinsic. To see that the order of R_0 minus its linear part is not k-intrinsic, take the admissible change of coordinates given by (2.40). □

3. Application of the IKY method for the study of the homoclinic loop

We will compare the results given by the IKY method with the exact cyclicity of a generic homoclinic loop ([R1], [J], [IY1]). Let us first recall these results in the terminolgy we have introduced. We suppose that we compute the return map as a composition of the Dulac map $D_\lambda(x)$ from $\{y=1\}$ to $\{x=1\}$ for a saddle with hyperbolicity ratio r and with 1-jet of the normal form:

$$\begin{aligned} \dot{x} &= x \\ \dot{y} &= -ry, \end{aligned} \tag{3.1}$$

together with a regular transition $R_\lambda(y)$ going from $\{x=1\}$ to $\{y=1\}$.

Theorem 3.1 [R1], [J], [IY1]. *We consider a homoclinic loop through a hyperbolic saddle as just described.*

1. *If the hyperbolicity ratio r of the saddle point is different from 1, i.e. the trace of the saddle point is nonzero, then the cyclicity is one (see e.g.* [ALGM]*).*
2. *If $r=1$, the cyclicity is $2m+1$ if the saddle is of finite order m and the function $V(y)=R_0(y)-y$ is m-flat at $y=0$.*
3. *If $r=1$, the cyclicity is $2n$ if the saddle is of finite order $m\geq n$ and the function $V(y)=R_0(y)-y$ is of order n at $y=0$, i.e. $V^{(i)}(0)=0$ for $i<n$ and $V^{(n)}(0)\neq 0$.*
4. *If $r=1$, and the saddle is of infinite order, then the return map $P_\lambda(x)=R_\lambda\circ D_\lambda(x)$ is a C^K-diffeomorphism and the cyclicity is $2n$ if the map $W(x)=P_0(x)-x$ is of finite order n at $x=0$.*

From the discussion of Section 2 it is clear that the case (4) cannot be proved by the IKY method. It is however an infinite codimension case.

Let us now look at the other cases by means of the IKY method. In the case of non-integrable resonant hyperbolic points this requires an adaptation of the Pfaff form appearing in Table 1. Indeed, although the graph of the Dulac map is a solution of the Pfaff form, it is not a separating solution of the Pfaff form. To obtain a separating solution we use an artifice, already introduced in [IY2]. Indeed an adequate Pfaff form is obtained by multiplying the Pfaff form of Table 1 by either $F(x^p)$ or $F(y^q)$. This clearly increases the degree of the form and will lead to upper bounds for the cyclicity which are much higher than the exact bounds.

Proposition 3.2. *We consider a homoclinic loop through a hyperbolic saddle as just described.*

1. *If the hyperbolicity ratio r of the saddle point is irrational, then the IKY method allows to conclude that the cyclicity is ≤ 2. An additional artifice allows to conclude that the cyclicity is one.*
2. *If the hyperbolicity ratio $r=p/q\neq 1$, and the saddle is of finite order m, then the IKY method allows to conclude that the cyclicity is $\leq 2m\min(p,q)+2$.*
3. *If $r=1$, if the saddle is of finite order m and if the function $V(y)=R_0(y)-y$ is m-flat at $y=0$, then the IKY method allows to conclude that the cyclicity is $\leq 3m+2$.*
4. *If $r=1$, if the saddle is of finite order m and if the function $V(y)=R_0(y)-y$ is of order $n<m$ at $y=0$, i.e. $V^{(i)}(0)=0$ for $i<n$ and $V^{(n)}(0)\neq 0$, then the IKY method allows to conclude that the cyclicity is $\leq 2m+n+1$.*

Proof.

1. Limit cycles are isolated solutions of the system

$$\begin{aligned} 0 &= -r(\lambda)y dx + x dy \\ x &= R_\lambda(y), \end{aligned} \tag{3.2}$$

in the first quadrant $x,y>0$. By Rolle's theorem for dynamical systems [K1] and using the fact that the separating solution is in one piece in the region $x,y>0$ we can conclude that the number of solutions is at most 1 plus the number of positive solutions of

$$\begin{aligned} 0 &= \begin{vmatrix} -r(\lambda)y & x \\ 1 & -R'_\lambda(y) \end{vmatrix} \\ x &= R_\lambda(y) \end{aligned} \tag{3.3}$$

which reduces to the simple equation

$$V_\lambda(y) = r(\lambda)yR'_\lambda(y) - R_\lambda(y) = 0. \tag{3.4}$$

Since $V'_\lambda(y)\neq 0$ for (y,λ) in a neighborhood of $(0,0)$ we can conclude that $V_\lambda(x)$ has at most one small zero.

Moreover it has one zero precisely when $r-1$ and $R_\lambda(0)$ have the same sign, a situation which requires an odd number of limit cycles.

(2) Limit cycles are isolated solutions of the system

$$\begin{aligned} 0 &= yF^2(y^q)\left(-\frac{p}{q}+F(x^p)\right)dx + xF(x^p)F(y^q)dy \\ x &= R_\lambda(y), \end{aligned} \tag{3.5}$$

in the first quadrant. As before the number of solutions is at most one plus the number of positive solutions of

$$W_\lambda(y) = F(y^q)\left[yR'_\lambda(y)F(y^q)\left(-\frac{p}{q}+F((R_\lambda(y))^p)\right)+R_\lambda(y)F((R_\lambda(y))^p)\right]=0. \tag{3.6}$$

The map W_λ is a regular map. If we suppose that $R_0(y)=a_1y+o(y)$ then

$$W_0(y) = \begin{cases} -a_1\frac{p}{q}y^{2mq+1}+o(y^{2mq+1}) & \text{if} \quad q<p \\ a_1^{mp+1}y^{m(p+q)+1}+o(y^{m(p+q)+1}) & \text{if} \quad q>p. \end{cases} \tag{3.7}$$

In the case $q\neq p$ we have $W_0^{(mq+1+m\min(p,q))}(0)\neq 0$, yielding, by Rolle's theorem at most $mq+1+m\min(p,q)$ small zeros for $W_\lambda(y)$ in a neighborhood of $(y,\lambda)=(0,0)$. The other inequality is obtained by reversing the time in the system and by noting that m is invariant under such a change. Another method is to use, when $p<q$, the factor $F(x^p)$ instead of $F(y^q)$ in the first form of (3.5).

(3) and (4). Limit cycles are isolated solutions of the system

$$\begin{aligned} 0 &= yF^2(y)(-1+F(x))dx + xF(x)F(y)dy \\ x &= R_\lambda(y), \end{aligned} \tag{3.8}$$

in the first quadrant. As before the number of solutions is at most one plus the number of positive solutions of

$$W_\lambda(y) = F(y)[yR'_\lambda(y)F(y)(-1+F(R_\lambda(y)))+R_\lambda(y)F(R_\lambda(y))]=0. \tag{3.9}$$

If we suppose that $R_0(y)=a_1y+a_ny^n+o(y^n)$, then

$$W_0(y) = \begin{cases} (a_1^{m+1}-a_1)y^{2m+1}+o(y^{2m+1}) & \text{if} \quad a_1\neq 1 \\ -a_n(n-m-1)y^{2m+n}+o(y^{2m+n}) & \text{if} \quad a_1=1, 1<n\leq m \quad \text{and} \quad a_n\neq 0 \\ y^{3m+1}+o(y^{3m+1}) & \text{if} \quad a_1=1 \quad \text{and} \quad n>m. \end{cases} \tag{3.10}$$

□

4. Graphics through a semi-hyperbolic point

We can of course suppose that the semi-hyperbolic point is attracting (one negative eigenvalue). We have two types of graphics, depending whether we have a central or stable-center transition.

4.1 Central-transition. The return map can be written as a composition of two maps, the linear Dulac map $D_\lambda(y)=C(\lambda)y$ and a regular C^K-diffeomorphism $R_\lambda(y)$. The graphic is generic of codimension 1 and the cyclicity 1 is easily obtained by noting that the derivative of the first return map $P_\lambda(x)=R_\lambda(C(\lambda)y)$ is different from 1 for (y,λ) in a neighborhood of $(0,0)$. We can also obtain finite cyclicity using the IKY method.

Proposition 4.1. *We consider a graphic through a semi-hyperbolic singular point, with central transition. Then the IKY method allows to conclude that the cyclicity is* ≤ 4.

Proof. Limit cycles are isolated solutions of the system

$$\begin{aligned} 0 &= y(ydx-xdy) \\ x &= R_\lambda(y). \end{aligned} \tag{4.1}$$

Indeed, as for the resonant saddle case, the Dulac map is not a separating solution of the minimal Pfaff form $ydx-xdy=0$. It becomes a separating solution of the Pfaff form $y(ydx-xdy)=0$.

We showed in Section 2.4 that we could suppose that the coordinates (x,y) are chosen so that $R''_0(0)\neq 0$.

The number of solutions is at most one plus the number of solutions of

$$\begin{aligned} 0 &= \begin{vmatrix} y^2 & -xy \\ 1 & -R'_\lambda(y) \end{vmatrix} \\ x &= R_\lambda(y). \end{aligned} \tag{4.2}$$

The number of solutions of (4.2) is the number of solutions of

$$W_\lambda(y) = y^2R'_\lambda(y) - yR_\lambda(y) = 0. \tag{4.3}$$

If we suppose that $R_0(y)=a_1y+a_2y^2+o(y^2)$, then

$$W_0(y) = a_2y^3+o(y^3) \tag{4.4}$$

from which we can conclude that $W(y,\lambda)$ has at most 3 zeros for (y,λ) in a small neighborhood of $(0,0)$. □

4.2 Stable-center transition. Here again the return map can be written as a composition of two maps: the Dulac map $D_\lambda(y)$ and a regular C^K-diffeomorphism $R_\lambda(y)$. It was proven in [DRR2] that when the system is in the normal form (2.34) the Dulac map is flat. The graphic is generic of codimension 2 and the cyclicity 1 is easily obtained by noting that the derivative of the first return map $P_\lambda(x) = R_\lambda(D_\lambda(y))$ is different from 1 for (y, λ) in a neighborhood of $(0, 0)$. Let us now see what we can obtain using the IKY method.

Proposition 4.3. *We consider a graphic through a semi-hyperbolic singular of codimension m (i.e. multiplicity $m+1$), with stable-center transition. The IKY method allows to conclude that the cyclicity is ≤ 2. An additional artifice allows to conclude, when $m = 1$, that the cyclicity is one.*

Proof. Limit cycles are isolated solutions of the system

$$\begin{aligned} &0 = -ydx + F(x, \lambda)dy \\ &x = R_\lambda(y), \end{aligned} \tag{4.5}$$

for $y > 0$ and $F(x, \lambda) > 0$. The number of solutions is at most one plus the number of solutions of

$$\begin{aligned} &0 = \begin{vmatrix} -y & F(x,\lambda) \\ 1 & -R'_\lambda(y) \end{vmatrix} \\ &x = R_\lambda(y). \end{aligned} \tag{4.6}$$

Solutions of system (4.6) are solutions of

$$W_\lambda(y) = yR'_\lambda(y) - F(R_\lambda(y), \lambda) = 0. \tag{4.7}$$

For $\lambda = 0$ and $R_0(y) = a_1 y + o(y)$ we have

$$W_0(y) = a_1 y + o(y), \tag{4.8}$$

yielding, by Rolle's theorem, that $W_\lambda(y)$ has at most one zero for (y, λ) in a small neighborhood of $(0, 0)$.

When $m = 1$, let $F(x) = x^2(1 + ax) + \epsilon$. The equation (4.7) can only have a solution when $\epsilon > 0$, i.e. when there are no singular points. In that case the geometry forces an odd number of limit cycles, hence cyclicity one. □

5. Graphics through two hyperbolic saddles

In this section we focus on one case to illustrate how the IKY method works and what are its limitations. Let us consider for instance the following theorem of Mourtada:

Theorem [M2]**.** *For an analytic vector field, a non-trivial polycycle with two hyperbolic saddles having irrational hyperbolicity ratios r_1 and r_2 satisfying $r_1 r_2 = 1$ has finite cyclicity.*

In this theorem Mourtada gives an explicit formulation of the conditions which guarantee a non-identical return map for the polycycle, namely either

(1) the return map is of the form $P(x) = a_1 x + o(x)$, with $a_1 \neq 1$, or
(2) one of the regular transitions is nonlinear, i.e. has a non-vanishing higher derivative.

Of course, from the discussion of Section 2 it is clear that the condition (1) cannot be recovered from the Pfaffian equations at the singular points. It is then natural to consider a polycycle in which one of the regular transitions is nonlinear. In general such a condition will be seen not to be sufficient to prove finite cyclicity by the IKY method. However if we take a polycycle for which the regular transition joining the point $r_1 < 4$ to $r_2 > 4$ has a non-vanishing second derivative then the IKY method allows to prove finite cyclicity. For the sake of completeness we include the case $r_1 r_2 \neq 1$.

Theorem 5.1. *Let us consider a polycycle as in Figure 1 with two hyperbolic saddles with irrational hyperbolicity ratios r_1 and r_2. By the IKY method*

(1) *the polycycle has cyclicity less than or equal to 5 if $r_1 r_2 \neq 1$;*
(2) *if $r_1(0) < 1$, $r_2(0) > 1$ and $r_1(0)r_2(0) = 1$, if one of the regular transition map is nonlinear of order 2 and the other is nonlinear of order n, then the polycycle has finite cyclicity less than or equal to $2n + 4$;*
(3) *if $r_1(0) < 1/4$ and $r_2(0) > 4$ and if the regular transition map from the first point to the second is nonlinear of order 2, then the polycycle has finite cyclicity.*

Proof. The case (3) is a subcase of the case (2). Indeed the conditions $r_1(0) < 1/4$, $r_2(0) > 4$ are here to ensure that some derivative of order n of the second transition map is non intrinsic and hence can be made nonzero by an admissible change of coordinates at one of the singular points in the class of differentiability k needed to conclude.

The Pfaffian equations at the singular points are given by

$$\begin{aligned} xdy - r_1(\lambda)ydx &= 0 \\ udv - s_2(\lambda)vdu &= 0, \end{aligned} \tag{5.1}$$

where $s_2(\lambda) = 1/r_2(\lambda)$. Hence $s_2(0) = r_1(0)$. The functional equations are

$$\begin{aligned} u &= f_\lambda(x) \\ v &= g_\lambda(y). \end{aligned} \tag{5.2}$$

Hence we look for intersection points inside the first quadrant of two separating solutions γ_1 and γ_2 of the Pfaffian equations

$$\begin{aligned} \omega_1 &= xdy - r_1(\lambda)ydx = 0 \\ \omega_2 &= f_\lambda(x)g'_\lambda(y)dy - s_2(\lambda)f'_\lambda(x)g_\lambda(y)dx = 0. \end{aligned} \tag{5.3}$$

Between two intersection points of γ_1 and γ_2 there must be, on γ_1, a contact point of ω_1 and ω_2. The equation of contact points between ω_1 and ω_2 is

$$A_\lambda(x,y) = r_1(\lambda)f_\lambda(x)yg'_\lambda(y) - s_2(\lambda)xf'_\lambda(x)g_\lambda(y) = 0. \tag{5.4}$$

We look for intersection points of $A_\lambda(x,y) = 0$ with γ_1. This number is at most the number of contact points along $A_\lambda(x,y) = 0$ of ω_1 and $dA_\lambda = 0$, plus the number of non-connected components of $A_\lambda(x,y) = 0$ (see for instance [IY2]).

On one hand we must count the number of contact points along $A_\lambda(x,y) = 0$ of ω_1 and $dA_\lambda = 0$. On the other hand we must count the number of non-compact components of $A(x,y) = 0$. For the latter we count, for a fixed small $\epsilon > 0$ and sufficiently small values of the parameters, the number of solutions of the system

$$\begin{aligned} &A_\lambda(x,y) = 0 \\ &x^2 + y^2 - \epsilon^2 = 0. \end{aligned} \tag{5.5}$$

The number of non-compact solutions is half that number.

Number of contact points along $A_\lambda(x,y) = 0$ of ω_1 and $dA_\lambda = 0$. Here again the equation of contact points is given by (5.4) together with

$$B_\lambda(x,y) = x\frac{\partial A_\lambda}{\partial x} + r_1(\lambda)y\frac{\partial A_\lambda}{\partial y} = 0. \tag{5.6}$$

We have

$$\begin{aligned} B_\lambda(x,y)|_{A_\lambda(x,y)=0} = &\frac{r_1(\lambda)y}{f'_\lambda(x)}\left[f'_\lambda(x)g'_\lambda(y)[(1-s_2(\lambda))xf'_\lambda(x) + (r_1(\lambda)-1)f_\lambda(x)]\right. \\ &\left.-xf_\lambda(x)f''_\lambda(x)g'_\lambda(y) + r_1(\lambda)f_\lambda(x)f'_\lambda(x)yg''_\lambda(y)\right] \\ = &\frac{r_1(\lambda)y}{f'_\lambda(x)}C_\lambda(x,y). \end{aligned} \tag{5.7}$$

There is another way to simplify the equation (5.6) with the help of $A_\lambda(x,y) = 0$, namely by replacing $yf_\lambda(x) = \frac{s_2(\lambda)xf'_\lambda(x)g_\lambda(y)}{r_1(\lambda)g'_\lambda(y)}$. By this method we obtain an expression which factorizes by x:

$$\begin{aligned} B_\lambda(x,y)|_{A_\lambda(x,y)=0} = &\frac{x}{g'_\lambda(y)}[f'_\lambda(x)g'_\lambda(y)[s_2(\lambda)(r_1(\lambda)-1)g_\lambda(y) + r_1(\lambda)(1-s_2(\lambda))yg'_\lambda(y)] \\ &+ r_1(\lambda)s_2(\lambda)f'_\lambda(x)yg_\lambda(y)g''_\lambda(y) - s_2(\lambda)xf''_\lambda(x)g_\lambda(y)g'_\lambda(y)] \\ = &\frac{x}{g'_\lambda(y)}D_\lambda(x,y). \end{aligned} \tag{5.8}$$

Let

$$\begin{aligned} f_\lambda(x) &= \sum_{i=0}^{k} a_i(\lambda)x^i + o(x^k) \\ g_\lambda(y) &= \sum_{i=0}^{k} b_i(\lambda)y^i + o(y^k) \end{aligned} \tag{5.9}$$

with $a_0(0) = b_0(0) = 0$, and $a_1, b_1, a_2, b_n \neq 0$ and $b_2(0), \dots, b_{n-1}(0) = 0$.

Let us consider the equations $C_\lambda(x,y) = D_\lambda(x,y) = 0$ for $\lambda = 0$.

(1) When $r_1 r_2 \neq 1$ then

$$\frac{\partial C_\lambda}{\partial x}|_{\lambda=0}, \frac{\partial D_\lambda}{\partial y}|_{\lambda=0} \neq 0. \tag{5.10}$$

Hence the change of coordinates $(X,Y) = (C_\lambda(x,y), D_\lambda(x,y))$ is a diffeomorphism and the point $(X,Y) = (0,0)$ has a unique preimage.

(2) In this case we have $r_1(0) = s_2(0)$, which we simply denote r. The x^2- and xy^{n-1}-terms of the bracket factor of C_λ behave, for $\lambda = 0$, like $-(r+1)a_1b_1a_2x^2$ and $n(n-1)ra_1^2b_nxy^{n-1}$ respectively. Also the xy- and y^n-terms of the bracket factor of D_λ behave, for $\lambda = 0$, like $-2rb_1^2a_2xy$ and $(n-1)r(1-r+rn)a_1b_1b_ny^n$ respectively. Hence, before perturbation, the equations $C_0 = 0$ and $D_0 = 0$ behave respectively approximately as

$$a_1x[-a_2b_1(r+1)x + n(n-1)ra_1b_ny^{n-1}] = a_1x[A_1x + B_1y^{n-1}] = 0 \tag{5.11}$$

and

$$yrb_1[-2a_2b_1x + (n-1)(1-r+rn)a_1b_ny^{n-1}] = yrb_1[A_2x + B_2y^{n-1}] = 0. \tag{5.12}$$

The determinant $A_1B_2 - A_2B_1 = a_1b_1a_2b_n(n-1)(r-1)(r(1-n)+1) \neq 0$ since r is irrational. This yields that the branches of the two curves $C_0 = 0$ and $D_0 = 0$ are in sufficiently "generic positions" to ensure that no intersection points of the two curves can escape through the boundary of a neighborhood of the origin for sufficiently small λ. An explicit bound can be found by doing the equivalent of the elimination theory in algebraic geometry. The details are as follows:

Detailed elimination. The equations $C_\lambda = 0$ and $D_\lambda = 0$ have, for $\lambda = 0$, the form $C_0(x,y) = A_1(0)x^2 + B_1(0)xy^{n-1} + O(x^3) + H_1(y) = 0$ and $D_0(x,y) = A_2(0)xy + B_2(0)y^n + O(y^{n+1}) + H_2(x) = 0$, with $A_1(0)B_2(0) - A_2(0)B_1(0) \neq 0$. Also the $2(n-1)$-jet (resp. 2-jet) of H_1 (resp. H_2) vanishes for $\lambda = 0$. By the Weierstrass preparation theorem for C^k-functions (see Appendix) we can write

$$C_\lambda(x,y) = (x^2 + f_{1,\lambda}(y)x + f_{0,\lambda}(y))u(x,y) \tag{5.13}$$

with $u(x,y) \neq 0$, $f_{1,\lambda}(y) = (c_0 + c_1y + c_{n-1}y^{n-1})(1+O(y))$, where $c_0(0) = 0 = \cdots = c_{n-2}(0) = 0$, $c_{n-1} = B_1(0)/A_1(0) \neq 0$, and the $2(n-1)$-jet of $f_{0,\lambda}$ vanishes identically for $\lambda = 0$. We also need that f_0 be of class C^{2n}. For this we need $[k/2] - 1 > 2n$ (see Corollary A.3 of the Appendix).

Hence we need consider the system

$$\begin{cases} P_\lambda(x,y) = x^2 + f_{1,\lambda}(y)x + f_{0,\lambda}(y) = 0 \\ D_\lambda(x,y) = 0. \end{cases} \tag{5.14}$$

With the help of P_λ we can simplify the equation $D_\lambda(x,y) = 0$, by removing all terms which are multiples of x^2. This is done as follows. Using the division theorem of Lassalle (see Theorem A.2 of the appendix) we divide $D_\lambda(x,y)$ by the generic polynomial $x^2 + d_1x + d_0$:

$$D_\lambda(x,y) = (x^2 + d_1x + d_0)Q(x,y,\lambda,d_1,d_0) + r_1(y,\lambda,d_1,d_0)x + r_0(y,\lambda,d_1,d_0). \tag{5.15}$$

We then substitute $d_i = f_{i,\lambda}(y)$ in $r_i(y,\lambda,d_1,d_0)$, yielding functions $g_{i,\lambda}(y)$. Solutions of (5.14) are solutions of

$$\begin{cases} P_\lambda(x,y) = 0 \\ Q_\lambda(x,y) = xg_{1,\lambda}(y) + g_{0,\lambda}(y) = 0 \end{cases} \tag{5.16}$$

with $g_{1,\lambda}(y) = (\alpha_0 + \alpha_1y)(1+O(y))$ where $\alpha_0(0) = 0$, $\alpha_1(0) = A_2(0) \neq 0$, $g_{0,\lambda}(y) = (\beta_0 + \beta_1y + \cdots + \beta_ny^n)(1+O(y))$ where $\beta_0(0) = \beta_1(0) = \cdots = \beta_{n-1}(0) = 0$, $\beta_n(0) = B_2(0) \neq 0$. Since we can divide by $(1+O(y))$ in $Q_\lambda(x,y)$ we can suppose that the $(1+O(y))$-factor is not present in $g_{1,\lambda}$.

We now proceed as in the classical elimination theory for polynomials of two variables. Solutions of (5.16) are solutions of

$$\begin{cases} Q_\lambda(x,y) = 0 \\ R_\lambda(x,y) = (\alpha_0 + \alpha_1y)P_\lambda(x,y) - xQ_\lambda(x,y) = 0. \end{cases} \tag{5.17}$$

The equation $R_\lambda(x,y) = 0$ is of the form

$$R_\lambda(x,y) = h_{1,\lambda}(y)x + h_{0,\lambda}(y) \tag{5.18}$$

where the $(2n-1)$-jet of $h_{0,\lambda}$ vanishes identically for $\lambda = 0$ and $h_{1,\lambda}(y) = \gamma_0 + \gamma_1y + \cdots + \gamma_ny^n)(1+O(y))$, with $\gamma_0(0) = \gamma_1(0) = \cdots = \gamma_{n-1}(0) = 0$, $\gamma_n(0) = B_1(0)A_2(0)/A_1(0) - B_2(0) \neq 0$. As before we can divide in $R_\lambda(x,y)$ by $(1+O(y))$ and suppose that the factor is not present.

Solutions of (5.17) are solutions of

$$\begin{cases} Q_\lambda(x,y) = 0 \\ S_\lambda(y) = (\gamma_0 + \gamma_1y + \cdots + \gamma_ny^n)Q_\lambda(x,y) - (\alpha_0 + \alpha_1y)R_\lambda(x,y) = 0. \end{cases} \tag{5.19}$$

Let $S_\lambda(y) = \sum_{i=0}^{2n} \eta_i(\lambda)y^i + o(y^{2n})$. We have that $\eta_i(0) = 0$ for $i < 2n$ and $\eta_{2n}(0) = \beta_n(0)\gamma_n(0) \neq 0$. The equation $S_\lambda(y)$ has at most $2n$ small zeros in y. Since $Q_\lambda(x,y)$ is linear in x, the system (5.14) has at most $2n$ solutions.

Number of non-compact components of $A_\lambda(x,y)=0$. As mentioned above a bound for the number of non compact solutions is given by half the number of solutions of (5.5). Here again a bound can be found by imitating the classical elimination theory. Indeed, using again the division Theorem A.2,

$$A_\lambda(x,y)|_{x^2=\epsilon^2-y^2} = h_{1,\lambda}(y)x + h_{0,\lambda}(y) = P_{1,\lambda}(x,y), \tag{5.20}$$

with $h_{1,\lambda}(y) = r(n-1)a_1b_ny^n + O(\lambda) + O(y^{n+1})$ and $h_{0,\lambda}(x) = -ra_2b_1y(\epsilon^2-y^2) + O(\lambda) + O(y^5)$. Solutions of (5.5) are solutions of

$$\begin{aligned} &Q_{1,\lambda}(x,y) = xP_{1,\lambda}(x,y) - h_{1,\lambda}(y)(x^2+y^2-\epsilon^2) = h_{0,\lambda}(y)x - h_{1,\lambda}(y)(y^2-\epsilon^2) = 0\\ &P_{1,\lambda}(x,y) = 0. \end{aligned} \tag{5.21}$$

Solutions of (5.21) are solutions of

$$det = h_{1,\lambda}(y)^2(y^2-\epsilon^2) + h_{0,\lambda}^2(y) = 0. \tag{5.22}$$

For $\lambda = 0$ we have

$$det|_{\lambda=0} = r^2b_1^2a_2^2y^6 + r^2b_n^2a_1^2(n-1)^2y^{2n+2} + o(\epsilon) + o(y^6). \tag{5.23}$$

Hence for a fixed small $\epsilon > 0$ there exists a small neigborhood of $\lambda = 0$ in parameter space such that (5.23) has at most 6 solutions. This yields at most three connected components of $A_\lambda(x,y) = 0$.

(3) The derivative of order n of the second transition under a C^k-admissible change of coordinates is non intrinsic provided $n > [(k-1)/r]+1$. Moreover, we want to apply the Weierstrass preparation theorem so that the coefficients of the polynomial $P_\lambda(x,y)$ in (5.14) are at least of class C^{2n}. For that purpose we need to have at least $[k/2]-1 > 2n$. This yields $[(k-1)/r]+1 < n < \frac{1}{2}[k/2] - \frac{1}{2}$, which has a solution $n \geq 2$ if $r > 4$. $\square$

We give one application of the case (1) of Theorem 5.1 by showing that the graphic (F_2^2) (Figure 2) surrounding a center in quadratic systems has finite cyclicity, in the particular case when the two singular points are hyperbolic saddles with inverse irrational hyperbolicity ratios.

Proposition 5.2. *By the IKY method we can show that the graphic (F_2^2) with two hyperbolic saddles having irrational hyperbolicity ratios has cyclicity less than or equal to 5 inside quadratic systems.*

Proof. The method used here is similar to the method introduced in [RSZ], namely the reduction to the generic case by means of the Bautin's method.

Such a graphic occurs inside the family

$$\begin{aligned} \dot{x} &= -y + Ax^2 + By^2\\ \dot{y} &= x + xy \end{aligned} \tag{5.24}$$

with $A < 0$, $B+1 > 0$. The hyperbolicity ratios are irrational if and only if $A \notin \mathbb{Q}$. We choose a 5-parameter perturbation which keeps fixed the saddle points located at $y+1 = Ax^2+B+1 = 0$. It can be taken with variable A and B inside the family

$$\begin{aligned} \dot{x} &= -y + Ax^2 + By^2 + \delta_1x(y+1)\\ \dot{y} &= x + xy + \delta_2(-y + Ax^2 + By^2) + \delta_3(y+1)(y+2). \end{aligned} \tag{5.25}$$

We postpone the argument that this is the general unfolding. The translation parameter ϵ_1 along the horizontal connection is obviously a multiple of δ_2. Hence all further calculations can be made under the condition $\delta_2 = 0$.

Let r_1 and s_2 be the hyperbolicity ratio and inverse hyperbolicity ratios at the two saddle points $(\pm x_0, -1)$ when $\delta_2 = 0$. They are given by $-\frac{2Ax_0}{x_0 \pm \delta_3}$, yielding that

$$\epsilon_3 = r_1 - s_2 = \frac{4Ax_0\delta_3}{x_0^2 - \delta_3^2}. \tag{5.26}$$

The second translation parameter can be calculated under the condition $\epsilon_3 = 0$, i.e. $\delta_3 = 0$. The initial system has the invariant hyperbola:

$$1 + \frac{A(A-1)(2A-1)}{1+B-A}x^2 + 2Ay + \frac{AB(2A-1)}{1+B-A}y^2 = 0. \tag{5.27}$$

The δ_1 perturbation is without contact with the hyperbola (except a double contact at $x=0$), yielding that the second parameter $\epsilon_2|_{\epsilon_1=\epsilon_3=0}$ is a multiple of δ_1.

What we have done proves a posteriori that the family (5.25) is indeed the general unfolding inside quadratic systems. Indeed the system (5.24) belongs to a smooth stratum of centers of dimension 2 and the variation of A and B gives the unfolding inside that stratum. It is also clear that the δ_i are three independent parameters since they control independent aspects of the dynamics of the quadratic systems. Moreover from the movement of these dynamical aspects it is clear that there is no fold in the parameter space.

The two forms in (5.3) are identical in the center case. Hence the C^k-functions $C_\lambda(x,y)$ and $D_\lambda(x,y)$ vanish identically in the center case.

The Bautin method works as follows: we consider a pointed neighborhood V of the origin in the δ-space. Then V is a union of three cones $V = V_1 \cup V_2 \cup V_3$, where $V_i = \{|\epsilon_i| \geq |\epsilon_j|, j \neq i\} \cap V$. In the cone V_i we divide the equations $C_\lambda(x,y) = 0$ and $D_\lambda(x,y) = 0$ by ϵ_i (this introduces no small divisor). In each cone we give a bound on the cyclicity.

In V_1, $C_\lambda(x,y)$ has no small zeros, yielding no small intersection points of $C_\lambda(x,y) = 0$ and $D_\lambda(x,y) = 0$. Hence the cyclicity is at most two.

In V_3, $D_\lambda(x,y)$ has no small zeros, yielding no small intersection points of $C_\lambda(x,y) = 0$ and $D_\lambda(x,y) = 0$. Hence the cyclicity is at most two.

In V_2, we can divide $C_\lambda(x,y)$ and $D_\lambda(x,y)$ by ϵ_2 and we are reduced to the generic case described in (1) of Theorem 5.1. □

Appendix

1. Extensions of neighborhoods of the origin.

Proposition A.1. *Let $\mathcal{X}$ be a C^k-vector field with one isolated singularity at 0, Γ be a characteristic manifold of $\mathcal{X}$ at 0 and Σ_i, $(i \in \{1,2\})$ be two C^k-transversal sections to Γ such that 0 is not in the segment of Γ between Σ_1 and $\Sigma_2 = \{x = x_2\}$.*

If $\mathcal{X}$ has no other singularity in Γ, then there exists a neighborhood V of 0 and a C^k-diffeomorphism $\Psi : V \to \Psi(V)$ preserving the direction field associated to $\mathcal{X}$ such that $\Psi(\Sigma_1) \subset \Sigma_2$.

Proof. The idea of the proof is to define Ψ as the identity in a neighborhood R of 0 which does not intersect Σ_1 and then extend it satisfying the needed properties. To do the last part, we use the flow-box theorem: we apply it in an open set V_1 in which $\mathcal{X}$ has no singularities, which intersects R and which contains a connected subset of Γ containing $\Gamma \cap \Sigma_1$, $\Gamma \cap \Sigma_2$ and one point of $\Gamma \cap R$. In the coordinate system obtained after applying the flow box therem, it is easy to see that the identity can be extended with the desired properties.

The details of the proof are the following. Let R be a rectangle around 0 such that $R \cap \Sigma_i = \emptyset$. Let V_1 as stated before. The intersection of the boundary of R with V_1 is a section $\{x = x_0\}$. Let $\eta : V_1 \to \eta(V_1)$ be a C^k-diffeomorphism which maps the integral curves of $\mathcal{X}$ to horizontal lines, $\{x = x_0\}$ into a subset of $\{x = x_0\}$, and Σ_1 into a subset of Σ_2.

Let $\Phi(x,y) = (\phi(x,y), y)$ be a C^k-diffeomorphism defined in $\eta(V_1)$ satisfying

$$\begin{aligned} \phi(x,y) &= x \quad \text{for } x < x_0, \\ \phi(x_2,y) &= \alpha(y). \end{aligned} \tag{A.1}$$

where $y \mapsto (\alpha(y), y)$ is a parametrization of $\eta(\Sigma_2)$.

Notice that $\eta^{-1} \circ \Phi \circ \eta$ is a C^k-diffeomorphism defined in V_1 with the properties:

(1) It preserves the direction field associated to $\mathcal{X}$ because η transforms integral curves of $\mathcal{X}$ into horizontal lines and Φ preserves horizontal lines.
(2) $\eta^{-1} \circ \Phi \circ \eta(\Sigma_1) \subset \Sigma_2$.
(3) $\eta^{-1} \circ \Phi \circ \eta$ restricted to $R \cap V$ is the identity.

The proposition is proved if we take

$$\Psi(x,y) = \begin{cases} (x,y), & \text{if } (x,y) \in R \\ \eta^{-1} \circ \Phi \circ \eta(x,y), & \text{if } (x,y) \in V_1. \end{cases} \tag{A.2}$$

□

II. Preparation theorems for C^k-functions. The papers of Barbançon [B] and Lassalle [L] contain versions of the division theorem by the generic polynomial in finite differentiable class. It is known in C^∞-class that the division theorem induces the preparation theorem allowing to write a function, regular in one variable, as the product of a Weierstrass polynomial with an invertible function. This is done through the use of the implicit function theorem where we choose the coefficients of the polynomial so that the remainder of the division vanishes identically. This step (see for instance [P]) is still valid for C^k-functions.

We recall the theorem of Lassalle:

Theorem A.2 [L]. *Let U be an open neighborhood of the origin in $\mathbb{R}^n$ and $a_1(x), \ldots a_d(x) \in C^\infty(U, 0)$, and let $B(x,t) = t^d + \sum_{i=1}^d a_i(x)t^{d-i}$. Then there exist linear applications $Q : C^{d+1}(U \times \mathbb{R}) \to C^0(U \times \mathbb{R})$ and $R : C^{d+1}(U \times \mathbb{R}) \to [C^0(U)]^d$ such that for all $k \geq d$:*

(1) *For any function $f \in C^k(U \times \mathbb{R})$ we have on $U \times \mathbb{R}$*

$$f(x,t) = B(x,t)Q(f)(x,t) + \sum_{i=1}^{d} r_i(x)t^{d-i} \tag{A.19}$$

with $R(f) = (r_1, \ldots, r_d)$.

(2) *R induces on $C^k(U \times \mathbb{R})$ a continuous application in $[C^h(U)]^d$, where $h = [k/d] - 1$.*

(3) *Q induces on $C^k(U \times \mathbb{R})$ a continuous application in $C^\ell(U \times \mathbb{R})$, where $\ell = [(k-1)/d] - 1$.*

(4) *For any point $x \in U$ and any $f \in C^k(U \times \mathbb{R})$ the function*

$$f \mapsto \frac{\partial^{\alpha_1 + \cdots + \alpha_n}}{\partial x_1^{\alpha_1} \ldots \partial x_n^{\alpha_n}} Q(f)(x,t) \tag{A.20}$$

is of class C^s, with $s = k - 1 - d(1 + \sum_{i=1}^n \alpha_i)$.

Corollary A.3. *Let $f(x,t) \in C^k(U \times \mathbb{R})$ be regular in t of order d i.e.*

$$\frac{\partial^i f}{\partial t^i}(0,0) \begin{cases} = 0 & \text{if} \quad i < d \\ \neq 0 & \text{if} \quad i = d. \end{cases} \tag{A.21}$$

Moreover let us suppose $k \geq d^2 + d$. Let $h = [k/d] - 1$ and $\ell = [(k-1)/d] - 1$. Then there exists a polynomial $P(x,t) = t^d + \sum_{i=1}^d b_i(x)t^{d-i}$, with coefficients in $C^h(V)$, where $V \subset U$ is a neighborhood of the origin in $\mathbb{R}^n$ and an invertible function $H(x,t) \in C^\ell(V \times \mathbb{R})$ such that on $V \times \mathbb{R}$ we have

$$f(x,t) = H(x,t)P(x,t). \tag{A.22}$$

Proof. We consider the function f as a function on $U \times \mathbb{R} \times \mathbb{R}^d$, which is constant in the d additional independent variables $a_1, \ldots, a_d$, which are the coefficients of $B(x,t)$. Then the functions $Q(f)$ and $R(f)$ of the theorem depend on $a_1, \ldots, a_d$. We need to solve the equations $R(f) = (r_1(f), \ldots r_d(f)) \equiv 0$ in the variables $a_1, \ldots a_d$ by the implicit function theorem. To show that this is possible we apply the operators $\frac{\partial^j}{\partial t^{j-1}\partial a_i}$, $j = 1, \ldots d$, to the equation (A.19) and evaluate at the origin, yielding

$$0 = \frac{\partial^{j-1}}{\partial t^{j-1}} \left(B(x,t,a) \frac{\partial Q(f)}{\partial a_i} \right) + \frac{\partial^{j-1}}{\partial t^{j-1}} (t^{d-i} Q(f)) + (j-1)! \frac{\partial r_{d-j+1}}{\partial a_i}. \tag{A.23}$$

The term $\frac{\partial^{j-1}}{\partial t^{j-1}} \left(B(x,t,a) \frac{\partial Q(f)}{\partial a_i} \right)$ vanishes since $j - 1 < d$. The second term vanishes when $j - 1 < d - i$ and does not vanish if $j - 1 = d - i$, from which it follows that

$$\frac{\partial r_{d+1-j}}{\partial a_i} \begin{cases} = 0 & \text{if} \quad j < d - i + 1 \\ \neq 0 & \text{if} \quad j = d - i + 1. \end{cases} \tag{A.24}$$

This can be done since $Q(f)$ and $R(f)$ are of class C^{d-1}. This allows to solve $R(f) = 0$ for $a_1, \ldots, a_d$, yielding $a_i = b_i(x)$ as functions in $C^h(V)$. $\square$

III. Differentiability properties for functions associated to semi-hyperbolic points.

Lemma A.6. *Let $F(x,\lambda) = x^{m+1}(1 + (a + \lambda_m)x^m) + \sum_{i=0}^{m-1} \lambda_i x^i$, $x_0 \in (0, |a|^{-1/m})$ and*

$$\Delta(x,\lambda) = \begin{cases} \exp \int_{x_0}^{x} \frac{du}{F(u,\lambda)}, & \text{for } (x,\lambda) \text{ such that } F(s,\lambda) > 0 \text{ for } s \in [x, x_0] \\ 0 & \text{otherwise.} \end{cases}$$

Given any k, there exists a neighborhood of the origin $N \subset \mathbb{R} \times \mathbb{R}^{m+1}$ such that Δ is a C^k-function in N.

Proof. In [DRR2] it is proved that Δ is a C^∞-flat function at the points $(\hat{x}, \hat{\lambda})$ such that $\hat{x}$ is the greatest root of $F(\cdot, \hat{\lambda})$ and has multiciplity greater or equal to 2. Let $r(\lambda)$ be the greatest root of $F(\cdot, \lambda)$ in $[-x_0, x_0]$, whenever it exists. To cover all the boundary of $\Delta^{-1}(0)$ we need to analyse the next cases:

(1) $\hat{x} < r(\hat{\lambda})$, $\hat{x}$ is not a root of $F(\cdot, \hat{\lambda})$ and $F(x, \hat{\lambda}) \geq 0$ for $x > \hat{x}$.

(2) $\hat{x} = r(\hat{\lambda})$ is a simple root of $F(\cdot, \hat{\lambda})$ and $F(x, \hat{\lambda}) > 0$ for $x > \hat{x}$.

(3) $\hat{x} < r(\hat{\lambda})$, $\hat{x}$ is a root of $F(\cdot, \hat{\lambda})$ and $F(x, \hat{\lambda}) \geq 0$ for $x > \hat{x}$ and all the zeros of $F(\cdot, \hat{\lambda})$ in $(\hat{x}, x_0]$ are given by $x_1 < \cdots < x_n = r(\lambda)$.

Case (1). Let $I : \Delta^{-1}(\mathbb{R}^+) \to \mathbb{R}$ be defined by

$$I(x, \lambda) = \int_x^{x_0} \frac{du}{F(u, \lambda)}.$$

Let I_1 be an interval centered at $\hat{x}$ and V_1 a neighborhood of $\hat{\lambda}$ such that for $(x, \lambda) \in I_1 \times V_1$, $F(x, \lambda) > F(\hat{x}, \hat{\lambda})/2$ and take $b \in I_1 \cap [-x_0, \hat{x}]$. For each λ, let $f(\lambda)$ be the minimum value of $F(x, \lambda)$ for $x \in [b, x_0]$.

We will see that there exists a neighborhood N_1 of $(\hat{x}, \hat{\lambda})$ such that for every $(x, \lambda) \in N_1 \cap \Delta^{-1}(\mathbb{R}^+)$ and every n, there exists positive numbers M_n satisfying:

$$I(x, \lambda) > \frac{M_0}{\sqrt{f(\lambda)}} \tag{A.25}$$

and

$$\left| \frac{\partial^n I}{\partial x^i \partial \lambda^j}(x, \lambda) \right| \leq \frac{M_n}{(f(\lambda))^{n+1}} \tag{A.26}$$

where $\partial \lambda^j$ means $\partial \lambda_0^{j_0} \ldots \partial \lambda_m^{j_m}$ and $i + j_0 + \cdots + j_m = n$.

These two facts clearly imply that Δ is a C^∞-flat function in $(\hat{x}, \hat{\lambda})$.

To check (A.25) we use that for $x \in [b, x_0]$ and λ in a compact neighborhood of $\hat{\lambda}$, there exist c such that

$$F(x, \lambda) \leq f(\lambda) + c(x - x_\lambda)^2$$

where x_λ is the point where $F(\cdot, \lambda)$ takes its minimum in $[b, x_0]$, so

$$I(x, \lambda) \geq \frac{1}{\sqrt{cf(\lambda)}} \left(\arctan\left(\sqrt{c/f(\lambda)}(x_0 - x_\lambda) \right) - \arctan\left(\sqrt{c/f(\lambda)}(x - x_\lambda) \right) \right). \tag{A.27}$$

Let $I_2 \times V_2 \subset \mathbb{R} \times \mathbb{R}^m$ be a neighborhood of $(r(\hat{\lambda}), \hat{\lambda})$ such that for (y, λ) in it $|F(y, \lambda)| < F(\hat{x}, \hat{\lambda})/2$. Notice that for $(x, \lambda) \in I_1 \times (V_1 \cap V_2)$, $x < x_\lambda < x_0$ so from (A.27) we obtain (A.25).

To check (A.26), for $i \geq 1$, we use that $\frac{\partial I}{\partial x}(x, \lambda) = \frac{1}{F(x, \lambda)}$ can be extended to a C^∞- function in a compact neighborhood of $(\hat{x}, \hat{\lambda})$, so each of its derivatives are bounded. The case $i = 0$ follows from the fact that

$$\frac{\partial^{|j|} I}{\partial \lambda^j}(x, \lambda) = \int_{x_0}^{x} \frac{P_j(u)}{(F(u, \lambda))^{|j|+1}} du$$

where P_j is a polynomial.

Case (2). Let g be such that $F(x, \lambda) = (x - r(\lambda))g(x, \lambda)$. Let $A(\lambda) = 1/g(r(\lambda), \lambda)$. Then

$$\frac{1}{F(x, \lambda)} = \frac{A(\lambda)}{x - r(\lambda)} + h(x, \lambda)$$

where h is a smooth function at points where $g(x, \lambda) \neq 0$.

For λ sufficiently close to $\hat{\lambda}$ then $A(\lambda) > k > 0$ and h is smooth for $x \geq r(\lambda)$.

With this, we obtain

$$\Delta(x, \lambda) = (x - r(\lambda))^{A(\lambda)} H(x, \lambda)$$

where H is a smooth function in a neighborhood of $(\hat{x}, \hat{\lambda})$. The function $(u, v) \mapsto u^v$ is a k-flat function of class C^k at $(0, v)$, provided that $v > k$, so Δ is a C^k-flat function at $(\hat{x}, \hat{\lambda})$.

Case (3). This case is a combination of the two previous ones. Indeed we can take an intermediate point y satisfying $\hat{x} < y < x_1 < \cdots < x_n < x_0$ and consider $I(x, \lambda)$ as the sum

$$I(x, \lambda) = I_1(x, \lambda) + I_2(\lambda)$$

where $I_1(x, \lambda) = \int_x^y \frac{du}{F(u, \lambda)}$ and $I_2(\lambda) = \int_y^{x_0} \frac{du}{F(u, \lambda)}$. Then I_2 is C^k-function in λ by case (1). If $\hat{x}$ is a simple (resp. multiple) root of $F(\cdot, \lambda)$, then by case (2) (resp. by Theorem 3 of [DRR2]), I_1 has the required properties to ensure that $\Delta(x, \lambda)$ is a C^k-function. $\square$

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Table 1. Normal Forms for generic elementary singular points

Type	Normal form	Dulac map $y = D_\lambda(x)$	Pfaff form
S_0	$\dot{x} = x$ $\dot{y} = -ry$ $r(0) \in \mathbb{R}^+ \setminus \mathbb{Q}^+$	$y = x^{r(\lambda)}$ $x, y > 0$	$xdy - r(\lambda)ydx = 0$
S_m	$\dot{x} = x$ $\dot{y} = y\left(-\frac{p}{q} + F(u,\lambda)\right)$ $u = x^p y^q$ $F(u,\lambda) = \pm u^m(1 + au^m)$ $+ \sum_{i=0}^{m-1} \epsilon_i(\lambda)u^i$ $\epsilon_i(0) = 0$	$q \log x = \int_{y^q}^{x^p} \frac{du}{uF(u)}$ $x, y > 0$	$yF(y^q)\left(-\frac{p}{q} + F(x^p)\right)dx$ $+xF(x^p)dy = 0$
D_m^C	$\dot{x} = F(x,\lambda)$ $\dot{y} = -y$ $F(x,\lambda) = \pm x^{m+1}(1 + ax^m)$ $+ \sum_{i=0}^{m-1} \epsilon_i(\lambda)x^i$ $\epsilon_i(0) = 0$	$y = C(\lambda)x$ $C(\lambda) = \int_{-x_0}^{x_0} \frac{dx}{F(x,\lambda)}$ $x, y \in \mathbb{R}$	$xdy - ydx = 0$
D_m^H	$\dot{x} = F(x,\lambda)$ $\dot{y} = -y$ $F(x,\lambda) = \pm x^{m+1}(1 + ax^m)$ $+ \sum_{i=0}^{m-1} \epsilon_i(\lambda)x^i$ $\epsilon_i(0) = 0$	$\log y + \int_x^{x_0} \frac{du}{F(u,\lambda)} = 0$ $y > 0,\ x \in \mathbb{R}$	$F(x,\lambda)dy - ydx = 0$