Higher Order Riccati Equations as Bäcklund Transformations

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CRM-2469

April 1997

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Abstract

In this short note we would like to show on a few examples the special role played by higher order Riccati equations in the construction of Bäcklund transformations for integrable systems.

Résumé

Dans cette note, nous proposons de montrer, en utilisant quelques exemples, le rôle spécial que joue les équations de Riccati d'ordre supérieur dans la construction des transformations de Bäcklund pour les systèmes intégrables.

1 Introduction

The Riccati equation [1], the simplest nonlinear ordinary differential equation

$$\frac{dv}{dx}(x) = \alpha(x) + \beta(x)v(x) + v^2(x) \tag{1.1}$$

plays a very important role in the solution of integrable nonlinear partial differential equations. These equations are characterized by being the compatibility conditions between two linear partial differential equations (the Lax pair) for an auxiliary function, the so called wave function [2]. Among the consequences of the existence of a Lax pair is the fact that one can obtain for them a denumerable number of exact solutions, the so called soliton solutions. The soliton solutions and their superpositions can be obtained recursively as solutions of the appropriate Bäcklund transformation, a differential relation between two different solutions of the nonlinear equation, starting from a trivial, in general constant, solution of the nonlinear partial differential equation. The best known integrable nonlinear partial differential equation is the Korteweg-de Vries equation

$$u_t = u_{xxx} + 6uu_x \tag{1.2}$$

whose simplest Bäcklund transformation is given by (1.1). In such a case the solution of the Bäcklund transformation v is related to the solution of the KdV ($u = v_x$) and depends on the variable t in a parametric way; the coefficients α and β are given functions of x, t through $u(x, t)$.

The Riccati equation, though a nonlinear equation, is characterized by the fact of possessing a superposition formula, as it is the case for all linear equations. Moreover, the Riccati equation can be linearized, in fact we can use the Cole-Hopf transformation [3] to reduce it to a linear Schrödinger spectral problem. One can consider higher order Bäcklund transformations, which, in the case of the KdV equation [2] are given by higher order differential equations which, however, are no more linearizable by a Cole-Hopf transformation.

By the use of the Cole-Hopf transformation one can obtain a whole class of nonlinear ordinary differential equations which possesses the same kind of properties as the Riccati equation, the so called Riccati chain [4]. These are not the only Riccati equations one can construct; others can be obtained by reduction of the Matrix Riccati equation [5].

In this note we will show that also the higher order members of the Riccati chain play the role of Bäcklund transformations for nonlinear integrable partial differential equations of higher order than the KdV equation. In particular we will consider the case of some of the integrable nonlinear partial differential equations of the 5th order (i.e. the Sawada-Kotera equation [6], the Fitzhugh-Nagumo equation [8] and the Burgers hierarchy [12]).

In Section 2 we will review the known results on the Riccati equation and its chain, while Section 3 is devoted to examples of equations of the Riccati chain which appear as Bäcklund transformations for the Sawada-Kotera, the Fritzhugh- Nagumo and the Burgers hierarchy. In Section 4 a few concluding remarks and comments are presented.

2 The Riccati equation and its chain

The Riccati equation (1.1), as mentioned in the introduction, is the simplest nonlinear differential equation which, being linearizable, can be completely solved. This equation appears in many fields of applied mathematics, in many instances when we can find exact solutions of nonlinear partial differential equations. It is the only first order nonlinear ordinary differential equation which possesses the Painlevé property [1], i.e. which has no movable singularity. Moreover, as was shown by S. Lie [9], the Riccati equation is the only ordinary nonlinear differential equation of first order which possesses a (nonlinear) superposition formula

$$\hat{v}(x) = \frac{v_1(x)(v_3(x) - v_2(x)) + kv_2(x)(v_1(x) - v_3(x))}{v_3(x) - v_2(x) + k(v_1(x) - v_3(x))} \tag{2.1}$$

i.e. the solution $\hat{v}(x)$ is nonlinearly expressed in terms of $v_1(x)$, $v_2(x)$, $v_3(x)$ which are given solutions of the same Riccati equation (1.1) and k is a constant parameter. The existence of a nonlinear superposition formula allows one to construct a denumerable set of solutions starting from 3 given solutions, i.e. given v_1, v_2, v_3 we can construct $v_4 = \hat{v}(x)$; then given v_2, v_3, v_4 we can construct v_5 and so on. It was shown by Lie that equation (1.1) possesses a finite-dimensional Lie algebra

$$X_0 = \frac{d}{dv}, \quad X_1 = v\frac{d}{dv}, \quad X_2 = v^2\frac{d}{dv}$$

which generate $\mathrm{sl}(2, \mathbb{R})$ and its subalgebras.

Most of the properties of the Riccati equation are also shared by the Riccati chain. The N-order equation of the Riccati chain is given by the following formula

$$L^N v(x) + \sum_{j=1}^{N} \alpha_j(x)\big(L^{j-1}v(x)\big) + \alpha_0(x) = 0 \tag{2.2}$$

where N is an integer characterizing the order of the Riccati equation in the chain , L is the following differential operator:

$$L = \frac{d}{dx} + cv(x); \tag{2.3}$$

and $\alpha_j(x)$ $j = 0, 1, \ldots, N$ are $N + 1$ arbitrary functions. The lowest order equations in the chain after Riccati equation (1.1) are

$$N = 2 \qquad \frac{d^2v}{dx^2} + \big[\alpha_2(x) + 3cv(x)\big]\frac{dv}{dx} + c^2v^3(x) + c\alpha_2(x)v^2(x) + \alpha_1(x)v(x) + \alpha_0(x) = 0. \tag{2.4}$$

$$N = 3 \qquad \frac{d^3v}{dx^3} + \big[\alpha_3(x) + 4cv(x)\big]\frac{d^2v}{dx^2} + 3c\left(\frac{dv}{dx}\right)^2 + \big[6c^2v^2(x) + 3cv(x)\alpha_3(x) + \alpha_2(x)\big]\frac{dv}{dx}$$
$$+ c^3v^4(x) + c^2\alpha_3(x)v^3(x) + c\alpha_2(x)v^2(x) + \alpha_1(x)v(x) + \alpha_0(x) = 0. \tag{2.5}$$

Let us notice that the N-Riccati chain is a polynomial expression in $v(x)$ and its derivatives such that the coefficient of the $(N-1)$-derivative of v is linear in v, that of the $(N-2)$-derivative is quadratic in v and that of 0-order derivative is a polynomial in v of order $N + 1$.

By the following Cole-Hopf transformation

$$cv(x)\psi(x) = \frac{d\psi}{dx}(x) \tag{2.6}$$

the whole class of equations (2.2) linearizes to a linear ordinary differential equation with variable coefficients of order $N + 1$.

$$\sum_{j=0}^{N} \alpha_j(x)\frac{d^j\psi}{dx^j} + \frac{d^{N+1}\psi}{dx^{N+1}} = 0 \tag{2.7}$$

From the well known linear superposition of solutions of equation (2.7) it follows that the nonlinear superposition formula (2.1) is valid for the whole Riccati chain. It is also easy to show, that the whole Riccati chain possesses the Painlevé property, moreover the 2nd-order Riccati equation is equivalent to equation VI in the Ince classification of equations possessing the Painlevé property [[1], page 334].

When the N-Riccati equation represents a Bäcklund transformation then the coefficients α_j depend also on the "time" variable t in a parametric way and are to be expressed in terms of a known solution of the given nonlinear partial differential equation. When the given solution is constant then the linear equation (2.7) has constant coefficients and generically its solution $\psi(x)$ is written out as a combination of exponential functions. These exponential functions are the main ingredients in the construction of the soliton solutions of the N- Riccati equation (2.2). The case in which the Bäcklund transformation is given by a 1-Riccati equation, which corresponds to the case of, among other equations, the KdV has been considered already in great detail in the literature [10]. So, in the following, we will consider the case of the N-Riccati with $N > 1$.

3 Examples of applications

Here in the following we consider few examples of nonlinear evolution equations which have as Bäcklund transformations a higher Riccati equation. Generically these equations are obtained from the compatibility of a Lax pair given by linear operators of order greater than the second.

3.1 The Sawada-Kotera equation

The Sawada-Kotera equation [6] is the following nonlinear partial evolution equation

$$u_t = u_{5x} + 10(uu_{xxx} + u_x u_{xx}) + 20u^2 u_x, \quad u = u(x,t). \tag{3.1}$$

This equation has the same terms as the higher order KdV equation [2],

$$u_t = u_{5x} + 10(uu_{xxx} + 2u_x u_{xx}) + 30u^2 u_x, \quad u = u(x,t) \tag{3.2}$$

but with different constant coefficients. Rewriting equations (3.1) and (3.2) in terms of a potential

$$v(x,t) = \int_x^\infty u(y,t)\, dy$$

we can easily verify that the highest order KdV equation (3.2) has the lowest order Bäcklund transformation expressed by a 1-Riccati equation while for the Sawada-Kotera equation such Bäcklund transformation is given by a 2- Riccati equation (2.4) with

$$\begin{gathered} c = -\frac{1}{3}, \quad \alpha_2 = \hat{v}(x,t), \quad \alpha_1 = \frac{1}{3}\hat{v}^2 - \hat{v}_x \\ \alpha_0 = \hat{v}\hat{v}_x - \hat{v}_{xx} - \frac{1}{9}\hat{v}^3 - \mu \end{gathered} \tag{3.3}$$

where $\hat{v}$ is any solution of the potential Sawada-Kotera equation and μ is an arbitrary constant, the Bäcklund parameter.

The above result is associated to the fact that the "space-part" of the Lax pair is given by a third-order spectral problem. Such spectral problem can be easily obtained form the Bäcklund transformation using the following Cole-Hopf transformation

$$\hat{v} = v - 3\frac{\psi_x}{\psi} \tag{3.4}$$

and reads

$$\psi_{xxx} + \hat{v}\psi_{xx} + \left(\frac{1}{3}\hat{v}^2 - \hat{v}_x\right)\psi_x + \left(\hat{v}\hat{v}_x - \hat{v}_{xx} - \frac{1}{9}\hat{v}^3 - \mu\right)\psi = 0 \tag{3.5a}$$

or equivalently

$$\begin{gathered}\varphi_{xxx} - 2\hat{v}_x\varphi_x + \left[-\frac{4}{3}\hat{v}_{xx} + \frac{4}{3}\hat{v}\hat{v}_x - \frac{11}{54}\hat{v}^3 - \mu\right]\varphi = 0 \\ \psi = \varphi \exp\left(-\int \hat{v}(y)\,dy\right).\end{gathered} \tag{3.5b}$$

For constant $\hat{v}$, a real solution of equation (3.5) is given by

$$\psi(x,t) = A(t)\exp\left[\mu + \frac{1}{9}\hat{v}^3\right]^{3/2} x + B(t)\exp\left[-\left(\mu + \frac{1}{9}\hat{v}^3\right)^{3/2}\right] x \cdot \cos\left[\frac{(\mu + \hat{v}^3/9)\sqrt{3}}{2}x + \varphi(t)\right]. \tag{3.6}$$

From (3.4) and (3.6) one gets a soliton solution of the potential Sawada- Kotera equation

$$v = -3\alpha^2(x)\frac{\{A(t)e^{\alpha(x)} - B(t)e^{-\alpha(x)}[\cos(\sqrt{3}\alpha(x)/2 + \varphi(t)) + \sqrt{3}/2\sin(\sqrt{3}\alpha(x)/2 + \varphi(t))]\}}{A(t)e^{\alpha(x)} + B(t)e^{-\alpha(x)}\cos[\sqrt{3}\alpha(x)/2 + \varphi(t)]} \tag{3.7}$$

where $\alpha(x) := (\mu + 1/9\tilde{v}^3)^{3/2}x$. The so obtained solution (3.7) is a solution of the 2-Riccati equation. The request that solution (3.7) satisfies the potential Sawada-Kotera equation fixes the dependence of A and B on the "time" variable.

3.2 The Fitzhugh-Nagumo equation

A similar result can be obtained for the Fitzhugh-Nagumo equation [8]

$$v_t - v_{xx} + v(1-v)(a-v) = 0, \quad -1 \le a \le 1. \tag{3.8}$$

Writing the 2-Riccati equation (2.4) for the field v we can show that equation (2.4) is compatible with equation (3.8) iff:

$$\begin{gathered} c = 2^{-1/2}, \quad \alpha_2 = -2^{-1/2}(1+a), \quad \alpha_1 = 2^{-1}a \\ \alpha_0 = \hat{v}_{xx} + 3\cdot 2^{-1/2}\hat{v}\hat{v}_x + \frac{5}{2}\hat{v}^3 + 2^{-1/2}(1+a)\hat{v}_x + \frac{1}{2}\hat{v}^2(1+a) - \frac{a}{2}\hat{v}. \end{gathered} \tag{3.9}$$

2-Riccati (3.9) is associated with a third order spectral problem

$$\psi_{xxx} - 2^{-1/2}(1+a)\psi_{xx} + 2^{-1}a\psi_x + \left[-\hat{v}_{xx} + 3\cdot 2^{-1/2}\hat{v}\hat{v}_x + \frac{5}{2}\hat{v}^3 + 2^{-1/2}(1+a)\hat{v}_x + \frac{1}{2}\hat{v}\right]\psi = 0 \tag{3.10}$$

and the corresponding Darboux transformation [8] is given by

$$v = \hat{v} + 2^{1/2}\frac{\psi_x}{\psi}. \tag{3.11}$$

For $\hat{v} = 0$ we have

$$\psi = A(t)e^{-2^{-1/2}\cdot x} + B(t)e^{-2^{-1/2}ax}$$

from which we get the "soliton" solution [11]

$$v = v_0 - a^{1/2}\hat{v}x - \frac{22^{1/2}a}{a-1}(1-\hat{v}^2)^{1/2}\ln\left|\cosh\left(\frac{a-1}{22^{1/2}}x + \varphi(t)\right)\right|.$$

Let us notice, however, that both the Bäcklund transformation (3.9) and the spectral problem (3.10) are free of a spectral parameter, usually an indication of nonintegrability.

3.3 The Burgers equation

In the case of the Burgers equation there is a one to one correspondence between the hierarchy of Bäcklund transformation [12] and the hierarchy of Riccati equations. In fact the Burgers equation

$$u_t - u_{xx} - 2uu_x \tag{3.12}$$

admits a hierarchy of Bäcklund transformations

$$\left[\sum_{k=0}^{N} \mathcal{A}^k \alpha_k\right]_{,x} - \hat{u} \sum_{k=0}^{N-1} \mathcal{A}^k \alpha_k = 0 \tag{3.13}$$

where $\mathcal{A} = \partial_x + u$, which, for any N, can be for any N, recast in the form (2.2). In this case, the simplest Bäcklund transformation is given by a 1-Riccati equation.

4 Conclusions

In this short note we have shown that there is a strict relationship between Riccati equations and Bäcklund transformations for integrable nonlinear partial differential equations. As it has been established in many of the well known cases the simplest Bäcklund transformation is given by the classical first order Riccati equation. There are, however, few well known cases in which the simplest Bäcklund transformation is given by a higher order differential equation. We showed here on a few examples that in such a case the Bäcklund transformation is given by a higher Riccati equation, higher in the so called Riccati chain. Not all known cases of fifth order equations have Bäcklund transformations having the form of one of the Riccati chain equations. For example the Kaup-Kupershmidt equation [7] has a Bäcklund transformation [13] given by a second order differential equation which is not equivalent to the 2nd-order equation of the Riccati chain.

Bäcklund transformations can be thought of as conditional symmetries for the equation at study [14], i.e. symmetries of the overdetermined system obtained by adding to the given differential equation under investigation differential constraints for which the symmetry criterion is identically satisfied. Up to now the only set of conditions for which the symmetry criterion is identically satisfied is given by first order differential equations [15]. The fact, shown here, that also higher order Riccati equations may play a role in the construction of Bäcklund transformations for some nonlinear partial differential equations indicates the possibility of introducing higher order conditional symmetries. This result can open the way to the construction of new classes of exact solutions for many physically important differential equations [16, 17]. Work on the extension of these results to the case of matrix Riccati chains and their reduction in application to nonlinear partial differential equations is in progress.

Acknowledgements

The research reported in this paper was partly supported by NSERC of Canada and by the research funds of the Italian Ministry of Education.

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