Universal Feedback Control via Proximal Aiming in Problems of Control under Disturbance and Differential Games

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1 Introduction

The first question arising in the study of infinitesimal properties of lower semicontinuous functions is the question about the concept of gradient or derivative for such functions which could replace the classical concept of derivative of differentiable function. The answer to this question can be given in the framework of proximal calculus by the introduction of the notion of proximal subgradient of function. Proximal calculus is an important component of nonsmooth analysis [4, 5]. Its development was originally motivated by the needs of optimization theory, particularly dynamic optimization [4]. This approach applied to nonsmooth problems of calculus of variations, control problems for differential inclusions, nonsmooth optimal control problems provides the unified characterization of optimal trajectories in terms of Euler-Lagrange and Hamilton inclusions, Pontryagin maximum principle. It appeared that methods of proximal calculus are useful in other fields of control theory too. In particular, they provide very helpful and in certain aspects indispensable tools for the study of generalized solutions of Hamilton-Jacobi equation.

In this paper methods of proximal calculus are used for the construction of universal feedback control for problems of control under disturbance. These results generalize the dynamic programming method for the case of lower semicontinuous functions which are proximal solutions of Hamilton-Jacobi inequality.

Let control system be described by differential equation

$$\dot{x}(t) = f(x(t), u(t), v(t)), \quad (1.1)$$

where $x(t) \in \mathbb{R}^n$ is a state vector, $u(t)$ is control with values in compact set $P \subset \mathbb{R}^p$, $v(t)$ is a disturbance with values in compact set $Q \subset \mathbb{R}^q$, $f(x, u, v)$ is a continuous function.

The quality of control process starting in moment $t_0$ from initial point $x_0$ is evaluated by the cost functional

$$\Lambda(x(\cdot), u(\cdot), v(\cdot))(t_0, x_0) = l(x(T)) - \int_{t_0}^{T} L(x(t), u(t), v(t))dt, \quad (1.2)$$

where $T$ is a fixed terminal time, $l, L$ are continuous functions, control $u(\cdot)$ and disturbance $v(\cdot)$ are measurable functions of time.

It is assumed that control $u(t)$ is formed by the following feedback rule

$$u(t) = u_f(t, x(t)), \quad (1.3)$$

where $u_f(t, x)$ is some feedback control law. The problem of optimal control under disturbance is the problem of finding feedback control which provides minimal guaranteed value of functional in spite of disturbance.

To make the statement of the problem more clear let us consider a special case of continuous feedback control $u_f(t, x)$, i.e. let us assume that every $x$ the function $u_f(t, x)$ is measurable in $t$ and for almost all (a.a.) $t$ it is continuous as function of $x$.

Then for any measurable disturbance $v(\cdot)$ there exists a solution $x(\cdot)$ of the equation

$$\dot{x} = f(x, u_f(t, x), v(t)), \quad x(t_0) = x_0. \quad (1.3)$$

In this case the quality of feedback control $u_f$ is evaluated by the functional

$$\Lambda_f(u_f)(t_0, x_0) = \sup_{x(\cdot), v(\cdot)} \Lambda(x(\cdot), u(\cdot), v(\cdot))(t_0, x_0),$$
where \( u(t) = u_f(t, x(t)), x(\cdot) \) is a solution of (1.3) and supremum is taken on the set of all disturbances \( v(\cdot) \) and corresponding solutions \( x(\cdot) \) of (1.3).

Thus the problem of optimal control under disturbance is the problem of minimization of a functional \( \Lambda_f \) on the set of feedback controls \( u_f \).

Here we postpone the discussion of the fact that in general it is possible to obtain essentially better values of a cost functional by using discontinuous feedback controls in order to describe the traditional method for solving the problem of control under disturbance — the dynamic programming method.

Assume that there exists such differentiable function \( \phi(t, x) \) that for all \( t \leq T, x \in \mathbb{R}^n \)

\[
\phi_t(t, x) + H^+(x, \phi_x(t, x)) \leq 0 \tag{1.4}
\]

and

\[
\phi(T, x) = l(x), \tag{1.5}
\]

where

\[
H^+(x, p) = \min_{u \in P} \max_{v \in Q} [\langle p, f(x, u, v) \rangle - L(x, u, v)]. \tag{1.6}
\]

Let us suppose that there exists a continuous feedback \( u_f(t, x) \) such that

\[
u_f(t, x) \in \text{Arg min}_{u \in P} \{\max_{v \in Q} \langle \phi_x(t, x), f(x, u, v) \rangle - L(x, u, v)\}, \tag{1.7}
\]

where \( \text{Arg min}_{u \in P} g(u) \) denotes a set of all \( u \) from \( P \) which minimize \( g \):

\[
\text{Arg min}_{u \in P} g(u) = \{\hat{u} \in P : g(\hat{u}) = \min_{u \in P} g(u)\}.
\]

Then for any solution \( x(\cdot) \) of the equation (1.3) for any measurable function \( v(\cdot) \) we have the following relation:

\[
\frac{d}{dt}\phi(t, x(t)) - L(x(t), u(t), v(t)) \leq 0,
\]

or, by integrating it

\[
\phi(t, x(t)) - \int_t^{t_0} L(x(s), u(s), v(s)) \mathrm{d}s \leq \phi(t_0, x_0), \tag{1.8}
\]

where \( u(t) = u_f(t, x(t)) \).

By taking \( t = T \) and using (1.5), we obtain:

\[
\Lambda(x(\cdot), u(\cdot), v(\cdot))(t_0, x_0) \leq \phi(t_0, x_0),
\]

which implies the following inequality for feedback control \( u_f \) defined by (1.7):

\[
\Lambda_f(u_f)(t_0, x_0) \leq \phi(t_0, x_0). \tag{1.9}
\]

Note that this inequality is valid for all initial conditions \((t_0, x_0), t_0 \leq T\), and in this sense \( u_f \) is a universal feedback control. Let us remark that universal feedback \( u_f \) is optimal if the equality holds in (1.4) instead of inequality.

However in the application of the dynamic programming method we immediately meet the well-known difficulties.

The easiest one is the absence of continuous feedback control satisfying (1.7). To overcome this difficulty one can replace discontinuous feedback control by an upper semicontinuous multifunction defined by the well-known design based on tools of the theory of differential inclusions (see, e.g.,
The other way is to use discontinuous feedback control and implementation schemes for them (see Section 2). But the main difficulty is non-existence, in general, of differentiable function \( \phi \) satisfying Hamilton-Jacobi inequality (1.4) and providing the estimate (1.9) which is close to the optimal one.

For such functions we replace classical derivatives \( \phi_t, \phi_x \) by the well-known in nonsmooth analysis \([4, 5]\) proximal subgradients \((\zeta_t, \zeta_x) \in \partial^p \phi(t, x)\) of lower semicontinuous function \( \phi \). We shall consider functions \( \phi \) satisfying the relation analogous to Hamilton-Jacobi inequality (1.4): for all \((t, x) \in (-\infty, T] \times \mathbb{R}^n\), \((\zeta_t, \zeta_x) \in \partial^p \phi(t, x)\)

\[
\zeta_t + H^+(x, \zeta_x) \leq 0. \tag{1.10}
\]

Under general assumptions on control system (1.1) and functional (1.2) this inequality is satisfied by the optimal value function

\[
w(t, x) = \inf_{u_f} \Lambda_f(u_f)(t, x) \tag{1.11}
\]

(the definition of the functional \( \Lambda_f \) for discontinuous feedback controls will be given in Section 2).

The value function \( w \) is continuous and satisfies the condition

\[
w(T, x) = l(x).
\]

The considered problem of feedback control under disturbance is closely connected with the theory of differential games. It is known (see, e.g., \([12]\)) that the above function \( w \) coincides with the value function of a differential game defined in the class of feedback controls \( u_f \) of the minimizing player and the class of the so-called counterstrategies \( v_c \) of the other (maximizing) player. Recall that counterstrategies are defined as functions \( v_c(t, x, u) \) (possibly discontinuous). It is assumed that for any \( t \leq T, x \in \mathbb{R}^n \) the map \( P \ni u \mapsto v_c(t, x, u) \in Q \) is Borel measurable.

The value function satisfies also the following inequality

\[
\zeta_t + H^+(x, \zeta_k) \geq 0, \tag{1.12}
\]

for all \((\zeta_t, \zeta_k) \in \partial_p \phi(t, x)\), where \( \partial_p \phi(t, x) \) denotes the proximal superdifferential of function \( \phi \). It is known that a continuous function satisfying simultaneously Hamilton-Jacobi inequalities (1.10), (1.12) and the terminal condition \( \phi(T, x) = l(x) \) is unique.

This fact reminds us about the connection between value function of differential game and viscosity solution of Hamilton-Jacobi equation \([6]\). The equivalent characterization of value function of differential game was given earlier in \([14]\) in terms of differential inequalities for directional derivatives of \( w \).

In conclusion of this brief survey of related facts, let us remind a construction of suboptimal feedback control suggested by N.N. Krasovskii \([10, 11]\) (see also english edition of the book \([11]\)). To simplify this description we will assume that the integrand \( L \) in the cost functional will equal to zero. Let \( w(t, x) \) be the value function, and let \( u^*_f(t, x) \) be an arbitrary function such that

\[
u^*_f(t, x) \in \arg \min_{u \in P} \{ \max_{y \in Q} \langle x - y_\alpha(t, x), f(x, u, v) \rangle \}.
\]

Here

\[y_\alpha(t, x) \in \arg \min \{ w(t, y) : \| x - y \|^2 \leq r(t, \alpha) \},\]

where \( \alpha \) is a positive parameter, \( r(t, \alpha) \) is some positive function, for its exact definition we refer to the cited works, noting only that \( r(t, \alpha) \to 0 \) as \( \alpha \downarrow 0 \). It is known that for any \( \epsilon > 0 \) and for any compact set \( D_0 \subset \mathbb{R}^n \) a parameter \( \alpha > 0 \) can be chosen such that

\[
\Lambda_f(u^*_f)(t_\alpha, x_0) \leq w(t_0, x_0) + \epsilon, \quad \forall (t_0, x_0) \in D_0.
\]
Thus, the feedback control $u^*_\alpha$ is $\epsilon$-optimal. Since this estimate holds for all initial conditions $(t_0, x_0)$ from $D_0$ this feedback control is called \textit{universal} for the region $D_0$.

In the present paper we develop constructions of universal feedback controls with the emphasis on the ideas and techniques of proximal calculus. This approach leads to the quite natural generalization of dynamic programming method for continuous or even lower semicontinuous solutions of (1.10). According to the construction proposed in this paper, universal feedback control can be defined by relations of the form (1.7), in which classical derivatives $\phi_x$ is replaced by proximal subgradients $\zeta_x$.

Now we outline briefly the contents of this paper. Section 2 contains notations, main assumptions, descriptions of the implementation schemes for discontinuous feedback control, the statement of the problem of control with uncertainty.

Basic concepts of proximal calculus (proximal subgradients, Iosida-Moreau type of regularization of functions, their properties) are included in Section 3. Section 4 is devoted to the construction of the universal discontinuous feedback control for continuous function $\phi$ satisfying Hamilton-Jacobi inequality (1.10). We remind here that the optimal value function (1.11) is a continuous function. The construction of feedback control in this case is simpler than in general case of lower semicontinuous function $\phi$ which is considered in Section 5.

2 Main Assumptions and Problem Formulation

Let $\langle x, y \rangle$ denote a scalar product of $x$ and $y$ in $R^n$, $|x| = \langle x, x \rangle^{1/2}$ be euclidean norm, $B$ be closed unit ball in $R^n$, $\|(t, x)\| = (t^2 + |x|^2)^{1/2}$.

For given partition $\pi = \{t_k\}_{i=0}^{m}$ of the interval $[t_0, T]$,

$$t_0 < t_1 < ... < t_m = T$$

the quantity

$$|\pi| = \max_{0 \leq k \leq m-1} (t_{k+1} - t_k)$$

is called a diameter of the partition.

The main assumptions about control system (1.1) and (1.2) are the following:

\textbf{Hypothesis A.} Functions $f(x, u, v), L(x, u, v)$ and $l(x)$ are continuous.

\textbf{Hypothesis B.} Functions $x \rightarrow f(x, u, v)$ and $x \rightarrow L(x, u, v)$ are locally Lipschitz uniformly with respect to $u \in P, v \in Q$.

\textbf{Hypothesis C.} For some constant $a > 0$

$$\langle x, f(x, u, v) \rangle \leq a(1 + |x|^2)$$

for all $(x, u, v) \in R^n \times P \times Q$.

The last Hypothesis C is called a growth condition. It provides for any compact set $D_0 \subset (-\infty, T] \times R^n$ the existence of compact set $D \subset (-\infty, T] \times R^n$ such that for any solution $x(\cdot)$ of (1.1) with the initial condition $(t_0, x_0) \in D_0$ we have the following inclusion:

$$(t, x(t)) \in D, \ t \in [t_0, T].$$

The set

$$\{(t, x) : \|(t - \tau, x - y)\| \leq 1 \text{ for some } (\tau, y) \in D, \ t \leq T\} \quad (2.1)$$

4
is denoted by \( D_1 \).

It follows from Hypothesis A,B that there exists some constant \( C_1 \) on \( D_1 \times P \times Q \)

\[
|f(x, u, v)| \leq C_1, \quad |L(x, u, v)| \leq C_1
\]

\[
|f(x_1, u, v) - f(x_2, u, v)| \leq C_1|x_1 - x_2|, \quad |L(x_1, u, v) - L(x_2, u, v)| \leq C_1|x_1 - x_2|.
\]

In particular, it implies that

\[
|H^+(x^1, p) - H^+(x^2, p)| \leq C_1|x^1 - x^2|(1 + |p|)
\]

for all \((t, x^i, p) \in D_1 \times R^n, \ i = 1, 2\).

The notation \( D_0, D, D_1 \) will be used in the next Sections. For example, function \( \phi_\alpha \) defined for lower semicontinuous function \( \phi \) and set \( D_1 \)

\[
\phi_\alpha(t, x) = \min_{(\tau, y) \in D_1} (\phi(\tau, y) + \frac{1}{2\alpha^2} \| (t - \tau, x - y) \|^2)
\]

will play an essential role in the following consideration. It reminds us Iosida-Moreau regularization of convex function in convex analysis.

Now we consider a concept of discontinuous feedback control \( u(t, x) \). There are examples showing that such feedback control performs essentially better than continuous feedback control. Here we place an example which demonstrates that universal discontinuous feedback control provides better values of the cost functional than universal continuous one.

**Example 2.1.** Let us consider control system

\[
\dot{x} = u,
\]

where \( x \in R^n \), control parameter \( u \) has values in unit closed ball \( B \). The functional (1.2) has form

\[
\Lambda(x(\cdot), u(\cdot))(t_0, x_0) = -|x(1)|.
\]

We claim that for any continuous feedback control there exists point \( \bar{x}_0 \in B \) such that

\[
\Lambda_f(u_f)(0, \bar{x}_0) = 0.
\]

To prove it we can consider continuous feedback control \( u(t, x) \) which is Lipschitz in \( x \). Then for any point \( x_0 \in B \) there is unique solution \( x(t; x_0) \) of the equation

\[
\dot{x} = u(t, x), \quad x(0) = x_0,
\]

which continuously depends upon \( x_0 \).

Thus continuous mapping

\[
F(x_0) = -\int_0^1 u(t, x(t; x_0))dt
\]

from \( B \) to \( B \) has a fixed point \( \bar{x}_0 \) which satisfies (2.5).

It follows from below that discontinuous feedback control

\[
u(t, x) = \begin{cases} 
\frac{x}{|x|}, & x \neq 0 \\
0, & x = 0
\end{cases}
\]
where $e$ is some unit vector, provides the following estimate for functional (2.7)

$$
\Lambda_f(u_f)(0, x_0) \leq -1 \quad \text{for all } x_0 \in B.
$$

By comparing this estimate with (2.5) we see that universal discontinuous feedback control performs better than universal continuous one.

It is clear also that in this simple example, for any fixed initial position, an optimal result attained in the class of continuous feedback controls (even in the class of constant controls) coincides with the optimal result in the class of discontinuous strategies. Let us remark that examples are known (a bit more complicated than the above one) in which for fixed initial condition an optimal result in the class of discontinuous feedbacks is neither attainable in the class of continuous feedbacks, nor approximated by these strategies (see, e.g. [12]).

Of course, in the case of discontinuous feedback control it is not possible define directly a corresponding trajectory of control system as a solution of differential equation (1.3) since its right hand side is discontinuous. So we need some implementation scheme for discontinuous feedback control which determines the value of control $u(t)$. Such scheme was suggested by N.N.Krasovskii and A.I.Subbotin in the context of differential game.

Let discontinuous feedback control $u_f$ be an arbitrary function $u_f(t, x)$ with values in $P$. Then for any partition $\pi = \{t_k\}_{k=0}^m$ of the interval $[t_0, T]$ and any disturbance $v(\cdot)$ the trajectory $x_\pi(\cdot)$ is a solution of equation (1.1) with control

$$
u_\pi(t) \equiv u_f(t_k, x_\pi(t_k)), \quad t \in [t_k, t_{k+1}], \quad k = 0, \ldots, m - 1,
$$

and initial condition $x_\pi(t_0) = x_0$.

The cost functional evaluating the quality of discontinuous feedback control is defined as follows

$$
\Lambda_f(u_f)(t_0, x_0) = \lim_{\delta \downarrow 0} \sup_{|\pi| < \delta, v(\cdot)} \Lambda(x_\pi(\cdot), u_\pi(\cdot), v(\cdot))(t_0, x_0),
$$

where supremum is taken on the set of all partitions $\pi$ with diameter less than $\delta$ and all disturbances.

Let lower semicontinuous function $\phi$ satisfies Hamilton-Jacobi inequality (1.10) and condition (1.5).

The main result of this paper is the construction for any compact set $D_0$ and any $\epsilon > 0$ a universal feedback control $u_f$ such that

$$
\Lambda_f(u_f)(t_0, x_0) \leq \phi(t_0, x_0) + \epsilon \quad \text{for all } (t_0, x_0) \in D_0
$$

Since optimal value function (1.11) satisfies Hamilton-Jacobi inequality (1.10) and (1.5) this result implies the existence of $\epsilon$-optimal universal feedback control $\hat{u}_f$ such that

$$
\Lambda_f(\hat{u}_f)(t_0, x_0) \leq \inf_{u_f} \Lambda_f(u_f)(t_0, x_0) + \epsilon
$$

for all $(t_0, x_0) \in D_0$.

The main distinction of the proposed design from the known constructions of universal feedback controls (see, e.g. [2, 10, ?, 9]) is that it appears to be the most natural generalization of the classical method of dynamic programming. Indeed, according to this design, Hamilton-Jacobi inequality and relations, defining required feedback controls, will have the same form as in the dynamic programming method, but instead of classical derivatives, which fails in general to exist, we will use proximal subgradients. Let us emphasize once more that this demonstrates that methods of proximal calculus provide adequate tools for solving problems of feedback control under uncertainty.
The universal feedback control providing the estimate (2.8) will be constructed with the aid of some Lyapunov-like function \( \eta(t, x) \) determined by \( \phi \). In order to give definition of \( \eta \) we need some additional notation.

Let \( u_f \) be discontinuous feedback control, \( v(\cdot) \) be arbitrary disturbance, \((t, x)\) be point from \( D \), \( \Delta > 0 \), \( t_\Delta = t + \Delta \). Consider the solution \( x(\tau) \) of (1.1) corresponding an initial condition \( x(t) = x \), the disturbance \( v(\cdot) \) and the control

\[
u(\tau) \equiv u_f(t, x) \quad \tau \in [t, t_\Delta).
\]

We say that such solution is implemented by feedback \( u_f \) and disturbance \( v(\cdot) \).

**Definition 2.1.** Function \( \eta(t, x) \) is called a Lyapunov function for a feedback \( u_f \), compact set \( D_0 \), constants \( \gamma_1 \geq 0 \), \( \gamma_2 \geq 0 \) and monotone increasing function \( \gamma_3(\Delta) \geq 0 \) if there exists set \( F \subset D \) such that \( \{(T, x) : (T, x) \in D\} \subset F \) and

**D1.** For any \((t, x) \in F \) and solution \( x(\cdot) \) of (1.1) with initial condition \( x(t) = x \) corresponding arbitrary control \( u(\cdot) \) and disturbance \( v(\cdot) \)

\[
l(x(T)) - \int_t^T L(x(s), u(s), v(s))ds \leq \eta(t, x) + \gamma_1.
\]

**D2.** For any \((t, x) \in D \setminus F \), \( \Delta > 0 \) such that \( t_\Delta \leq T \), and for the solution \( x(\cdot) \) implemented by \( u_f \) and an arbitrary disturbance \( v(\cdot) \)

\[
\eta(t_\Delta, x(t_\Delta)) - \int_t^{t_\Delta} L(x(s), u_f(t, x), v(s))ds \leq \eta(t, x) + \Delta(\gamma_2 + \gamma_3(\Delta)).
\]

The following Proposition will be used in the proof of universality of feedback controls in both cases of continuous and lower semicontinuous function \( \phi \).

**Proposition 2.1.** Let \( \eta \) be a Lyapunov function for feedback control \( u_f \) then for any \( \delta > 0 \), \((t_0, x_0) \in D_0 \), any partition \( \pi = \{t_k\}_0^m \) of \([t_0, T] \) with diameter \( |\pi| \leq \delta \) and any disturbance \( v(\cdot) \)

\[
\Lambda(x(\cdot), u(\cdot), v(\cdot))(t_0, x_0) \leq \eta(t_0, x_0) + \gamma_1 + (T - t_0)(\gamma_2 + \gamma_3(\delta)).
\]

We have the obvious Corollary to this Proposition.

**Corollary.** Let \( \gamma_3(\delta) \) decreasing to 0 while \( \delta \downarrow 0 \) then

\[
\Lambda_f(u_f)(t_0, x_0) \leq \eta(t_0, x_0) + \gamma_1 + (T - t_0)\gamma_2.
\]

**Proof.** Let us fix \((t_0, x_0) \in D_0 \) and consider some partition \( \pi = \{t_k\}_0^m \) of \([t_0, T] \) with diameter \( |\pi| \) less than \( \delta \). Then implementation scheme for a feedback \( u_f \) and arbitrary disturbance \( v(\cdot) \) produces a piecewise constant control \( u_\pi(\cdot) \) (2.6) and a trajectory \( x_{\pi}(\cdot) \).

It follows from **D2** that for all \( k \) such that \((t_k, x_\pi(t_k)) \notin F \)

\[
\eta(t_{k+1}, x_\pi(t_{k+1})) - \int_{t_k}^{t_{k+1}} L(x_\pi(t), u_\pi(t), v(t))dt \leq \eta(t_k, x_\pi(t_k)) + (t_{k+1} - t_k)(\gamma_2 + \gamma_3(\delta)).
\]

We used here that \( |\pi| \leq \delta \) and function \( \gamma_3(\Delta) \) is monotone in \( \Delta \).

Let us denote by \( t_N \) the first moment from \( \pi \) such that \((t_N, x_\pi(t_N)) \in F \). Then we obtain from the previous inequality that

\[
\eta(t_N, x_\pi(t_N)) - \int_{t_0}^{t_N} L(x_\pi(t), u_\pi(t), v(t))dt \leq \eta(t_0, x_0) + (T - t_0)(\gamma_2 + \gamma_3(\delta)).
\]
It follows from D1 that
\[ l(x_\pi(T)) - \int_{t_0}^{T} L(x_\pi(t), u_\pi(t), v(t))dt \leq \eta(t_0, x_0) + \gamma_1 + (T - t_0)(\gamma_2 + \gamma_3(\delta)). \]

It implies that in accordance with the definition of \( \Lambda \) we have the relation (2.11).
Proposition is proved.

**Remark 2.1.** Note that it follows from the proof of Proposition that we need to verify property D2 of the Lyapunov function \( \eta \) only for points \((t, x)\) of some subset \( M \) of \( D \) such that \( D_0 \subset M \) and the implemented control in D2 provides inclusion \((t_\Delta, x(t_\Delta)) \in M\).

## 3 Proximal Subgradients

Let \( \phi(x) \) be a lower semicontinuous function with values in \((-\infty, +\infty]\). The following concept of the proximal subgradient [4], [5] is the replacement of the classical concept of gradient which is not suitable for such function.

Vector \((\zeta_t, \zeta_x) \in R^1 \times R^n\) is called a proximal subgradient at point \((t, x)\) if there exist a constant \( \sigma \) such that for all \((t', x')\) near \((t, x)\)
\[ \phi(t', x') \geq \phi(t, x) + \zeta_t(t' - t) + \langle \zeta_x, x' - x \rangle - \sigma \| (t' - t, x' - x) \|^2. \]

The set of all proximal subgradients at \((t, x)\) is denoted by \( \partial^p \phi(t, x) \). This set is called proximal subdifferential. If \( \phi(t, x) = +\infty \) then we assume that \( \partial^p \phi(t, x) = \emptyset \).

Proximal supergradients \((\zeta_t, \zeta_x)\) of upper semicontinuous function \( \psi(t, x) \) are defined as an element of the set \( -\partial^p(-\psi)(t, x) \).

Proximal subdifferential \( \partial^p \phi(t, x) \) is non-empty on a set which is dense in \( \text{Dom} \phi = \{ (t, x) : \phi(t, x) < +\infty \} \). This fact is easily proved by means of function \( \phi_\alpha \) determined by (2.4) for \( \alpha > 0 \) and compact set \( D_1 \) (2.1) with non-empty interior.

Let \((\tau_\alpha, y_\alpha)\) be a point in \( D_1 \) at which minimum in (2.4) is attained. Vectors
\[ \zeta_t^\alpha = \frac{t - \tau_\alpha}{\alpha^2}, \quad \zeta_x^\alpha = \frac{x - y_\alpha}{\alpha^2} \quad (3.1) \]
and point \((\tau_\alpha, y_\alpha)\) depend upon \((t, x)\).

The next Lemma follows clearly from the definition of \( \phi_\alpha, \tau_\alpha, y_\alpha \).

**Lemma 3.1.** Let for given \((t, x)\) the point \((\tau_\alpha, y_\alpha)\) is the interior point of \( D_1 \). Then
\[ (\zeta_t^\alpha, \zeta_x^\alpha) \in \partial^p \phi(\tau_\alpha, y_\alpha). \quad (3.2) \]

By choosing an appropriate value of parameter \( \alpha \) it is possible to localize \((\tau_\alpha, y_\alpha)\) with respect to \((t, x)\).

Denote by \( k_2 \) the lower bound of function \( \phi \) on compact set \( D_1 \). For \( k_1 \geq k_2 \) we use notation
\[ C_2 = \sqrt{2(k_1 - k_2)} \quad (3.3) \]

**Lemma 3.2.** Let for some constant \( k_1 \)
\[ \phi_\alpha(t, x) \leq k_1. \quad (3.4) \]

Then
\[ \| (t - \tau_\alpha, x - y_\alpha) \| \leq C_2 \alpha. \quad (3.5) \]
It is easy to see that (3.5) follows immediately from the definition of \((\tau_\alpha, y_\alpha)\). Note that since for any \((t, x)\)
\[
\phi_\alpha(t, x) \leq \phi(t, x),
\]  
then condition (3.4) may be replaced by the following one:
\[
\phi(t, x) \leq k_1.
\]  
But if \(\phi\) is continuous on \(D_1\) then (3.7) holds on \(D_1\) for some constant \(k_1\). It implies that for continuous \(\phi\) the estimate (3.4) is valid with some \(k_1\) for all \((t, x)\) in \(D_1\).

We assume here that \(C_2\) in (3.3) is defined by this \(k_1\).

**Lemma 3.3.** Let \(\phi(t, x)\) be continuous on \(D_1\). Then
\[
\frac{|x - y_\alpha|^2}{2\alpha^2} \leq \omega_\phi(C_2\alpha)
\]  
for all \((t, x) \in D_1\) where \(\omega_\phi\) is a modulus of continuity of function \(\phi\) on \(D_1\).

To prove this assertion, we use the definition of the point \((\tau_\alpha, y_\alpha)\) to get
\[
\frac{|x - y_\alpha|^2}{2\alpha^2} \leq \phi(t, x) - \phi(\tau_\alpha, y_\alpha)
\]  
and the estimate (3.5) for continuous \(\phi\).

Let us define subset
\[
F_\alpha = \{(t, x) \in D: \tau_\alpha = T\}
\]  

**Lemma 3.4.** For given constant \(k_1\), arbitrary \(\alpha \in (0, 1/C_2)\) and \((t, x) \in D\) satisfying (3.4), one of the following two relations holds: vector \((\zeta^\alpha_t, \zeta^\alpha_x)\) is a proximal subgradient of function \(\phi\) at point \((\tau_\alpha, y_\alpha)\), or \((t, x) \in F_\alpha\) and
\[
|T - t| \leq C_2\alpha, \quad |x - y_\alpha| \leq C_2\alpha.
\]  

To prove this Lemma, we note that if \((t, x) \in F_\alpha\) then (3.10) holds. If \((t, x)\) is not in \(F_\alpha\) then in view of Lemma 3.2 and choice of \(\alpha\) we have that \((\tau_\alpha, y_\alpha)\) is an interior point of \(D_1\), which implies that \((\zeta^\alpha_t, \zeta^\alpha_x)\) satisfies (3.4).

For \(C^1\) function \(\phi\) Taylor expansion formula gives the following estimate of the difference of values of \(\pi\) along some direction
\[
\phi(t + \Delta, x + \Delta f) = \phi(t, x) + \Delta(\phi_t(t, x) + \langle \phi_x(t, x), f \rangle) + o(\Delta).
\]  

The following analogue of Taylor expansion formula for function \(\phi_\alpha\) will play an essential role in in next Sections.

**Lemma 3.5.** For any \(\Delta > 0\) and for any vector \(f \in R^n\)
\[
\phi_\alpha(t + \Delta, x + \Delta f) \leq \phi_\alpha(t, x) + \Delta(\zeta^\alpha_t + \langle \zeta^\alpha_x, f \rangle) + \frac{\Delta^2}{2\alpha^2}(1 + |f|^2).
\]  

We have that
\[
\phi_\alpha(t + \Delta, x + \Delta f) \leq \phi(\tau_\alpha, y_\alpha) + \frac{1}{2\alpha^2}\|(t + \Delta - \tau_\alpha, x + \Delta f - y_\alpha)\|^2.
\]  

Expanding the quadratic term we obtain (3.11).
4 Universal feedback control for continuous functions satisfying Hamilton-Jacobi inequality

Let continuous function $\phi(t, x)$ satisfies the following Hamilton-Jacobi inequality: for all $(t, x) \in (-\infty, T) \times \mathbb{R}^n$

$$\zeta_t + H^+(x, \zeta_x) \leq 0 \text{ for all } (\zeta_t, \zeta_x) \in \partial^p \phi(t, x)$$

(4.1)

and condition

$$\phi(T, x) = l(x).$$

(4.2)

It was mentioned before that such functions exist. In particular, optimal value function (1.11) is continuous and satisfies (4.1)–(4.2).

For fixed compact set $D_0 \subset (-\infty, T] \times \mathbb{R}^n$ we define compact sets $D$, $D_1$ (2.1) and function $\phi_\alpha$ (2.4). Vector $(\zeta^{\alpha}_t, \zeta^{\alpha}_x)$ determined by (3.1) depends upon $(t, x)$.

Let $u(t, x, p)$ be an arbitrary function such that

$$u(t, x, p) \in \text{Arg min}_{u \in P} \{\max_{v \in Q} [\langle p, f(x, u, v) \rangle - L(x, u, v)]\}$$

(4.3)

Then feedback control, corresponding parameter $\alpha$ and a set of initial conditions $D_0$, is defined as follows

$$u^{\alpha}_f(t, x) = u(t, x, \zeta^{\alpha}_x).$$

(4.4)

Before stating the main result of this section we introduce the following notation.

Because of continuity of $\phi$ on $D_1$ it is bounded on $D_1: k_1 \geq \phi \geq k_2$. Constant $C_2$ is given by (3.3), constant $C_1$ is defined as in (2.2), (2.3). Functions

$$\gamma_1(\alpha) = C_1 C_2 \alpha + \omega_l(C_1 C_2 \alpha) + \omega_\phi(C_2 \alpha)$$

(4.5)

$$\gamma_2(\alpha) = C_1 C_2 \alpha + 2 \omega_\phi(C_2 \alpha),$$

are defined by means of modulus of continuity $\omega_\phi, \omega_l$ of functions $\phi$ and $l$ on $D_1$,

$$\gamma_3(\alpha, \Delta) = C_1^2 C_2 \Delta/\alpha + C_1^2 \Delta + (1 + C_1^2) \Delta/2 \alpha^2.$$  

(4.6)

**Theorem 4.1.** Let continuous function $\phi(t, x)$ satisfies (4.1)–(4.2). Then for any compact set $D_0$ and $\alpha \in (0, 1/C_2)$

$$\Lambda_\alpha(u^{\alpha}_f)(t_0, x_0) \leq \phi(t_0, x_0) + (T - t_0) \gamma_1(\alpha) + \gamma_2(\alpha)$$

(4.7)

for all $(t_0, x_0) \in D_0$.

Note that for any $\epsilon \geq 0$ there exists such $\alpha_\epsilon$ that for all $\alpha \in (0, \alpha_\epsilon)$ and all $(t_0, x_0) \in D_0$

$$(T - t_0) \gamma_1(\alpha) + \gamma_2(\alpha) < \epsilon.$$  

(4.8)

It is clear that if $\phi$ and $l$ are Lipschitz functions we may assume that $\alpha_\epsilon = O(\epsilon)$. We obtain from (4.7) that for any such $\alpha$ feedback control $u^{\alpha}_f$ (4.4) is universal and provides (2.8).

**Proof.** Let us fix $\alpha \in (0, C_2^{-1})$ and define a set

$$F = \{(T, x) : (T, x) \in D\} \cup F_\alpha,$$

where $F_\alpha$ is given by (3.9).
To prove Theorem it is sufficient to show that $\phi_\alpha$ (2.4) is a Lyapunov function for feedback control $u_\alpha^F$ (4.4), set $D_0$, constants $\gamma_1(\alpha), \gamma_2(\alpha)$, function $\gamma_3(\alpha, \Delta)$ and set $F$. Then relation (4.7) will follow from Proposition 2.1 and its Corollary.

At first we prove that $\phi_\alpha, F$ satisfy $\textbf{D1}$ in Definition 2.1.
Let $u(\cdot), v(\cdot)$ be arbitrary control and disturbance, $x(\cdot)$ be the corresponding solution of (1.1) with initial condition $x(t) = x$.

**Lemma 4.1.** Let $(t, x) \in F$ then

$$l(x(T)) \leq \phi_\alpha(t, x) + \int_t^T L(x(s), u(s), v(s)) ds + \gamma_2(\alpha).$$ (4.8)

It is clear that if $(t, x) \in F_\alpha$ then

$$l(x) = \phi(T, x) \leq \phi_\alpha(t, x).$$

Because of (2.2) and (3.10)

$$|x(T) - x| \leq C_1(T - t), \quad (T - t) \leq C_2\alpha,$$

which implies that

$$l(x(T)) \leq \phi_\alpha(t, x) + \omega_1(C_1C_2\alpha).$$

By using boundness of $|L|$ (2.2) we obtain

$$l(x(T)) - \int_t^T L(x(s), u(s), v(s)) ds \leq \phi_\alpha(t, x) + C_1C_2\alpha + \omega_1(C_1C_2\alpha),$$

but it means (4.9).

If $t = T$ then $x(T) = x$ and

$$\phi_\alpha(T, x(T)) \geq \phi(\tau_\alpha, y_\alpha) \geq l(x(T)) - \omega_1(C_2\alpha),$$

which means (4.9) again. Lemma is proved.

Now we verify that $\phi_\alpha$ satisfies $\textbf{D2}$ in Definition 2.1.
In view of Lemma 3.4 for all $\alpha \in (0, 1/C_2)$ and $(t, x) \in D \setminus F_\alpha$ (see (3.9))

$$(\zeta_t^\alpha, \zeta_x^\alpha) \in \partial^p \phi(\tau_\alpha, y_\alpha).$$

**Lemma 4.2.** Let $(t, x) \in D \setminus F_\alpha$, then

$$\zeta_t^\alpha + H^+(x, \zeta_x^\alpha) \leq \gamma_1(\alpha).$$ (4.9)

To prove (4.9) we use (4.1) and previous inclusion to obtain

$$\zeta_t^\alpha + H^+(y_\alpha, \zeta_x^\alpha) \leq 0.$$

Since $H^+(x, p)$ satisfies (2.3), it follows that

$$\zeta_t^\alpha + H^+(x, \zeta_x^\alpha) \leq C_1|x - y_\alpha|(1 + |\zeta_x^\alpha|).$$

We use Lemma 3.2 and 3.3 to derive

$$\zeta_t^\alpha + H^+(x, \zeta_x^\alpha) \leq C_1C_2\alpha + C_1\omega_p(C_2\alpha),$$
which means (4.9).

Now we fix \((t, x) \in D \setminus F_{\alpha}\), arbitrary \(\Delta > 0\), measurable disturbance \(v(\cdot)\). Then there exists a unique solution \(x(\cdot)\) of the equation

\[
\dot{x}(s) = f(x(s), u^\alpha_f(t), v(s)), \quad x(t) = x, \quad s \in [t, t_\Delta),
\]

where \(t_\Delta = t + \Delta\), \(u^\alpha_f\) is defined by (4.4). It is easy to see that this construction represents the element of the implementation scheme for feedback control \(u^\alpha_f\) (4.4).

Below we use the following notation

\[
f = \frac{1}{\Delta} \int_{t}^{t+\Delta} f(x(s), u^\alpha_f(t), v(s))ds, \quad L = \frac{1}{\Delta} \int_{t}^{t+\Delta} L(x(s), u^\alpha_f(t), v(s))ds.
\]

Because of the boundness of \(|f|, |L|\) and Lipschitzness of \(f, L\) in \(x\) (2.2), we have the representation

\[
f = \frac{1}{\Delta} \int_{t}^{t+\Delta} f(x, u^\alpha_f(t), v(s))ds + b, \quad L = \frac{1}{\Delta} \int_{t}^{t+\Delta} L(x, u^\alpha_f(t), v(s))ds + c, \tag{4.10}
\]

where

\[
|b| \leq C_1^2 \Delta, \quad |c| \leq C_1^2 \Delta. \tag{4.11}
\]

**Lemma 4.3.** Let \((t, x) \in D \setminus F_{\alpha}\) then

\[
\phi_\alpha(t_\Delta, x(t_\Delta)) \leq \phi_\alpha(t, x) + \int_{t}^{t+\Delta} L(x(s), uu^\alpha_f(t), v(s))ds + \Delta(\gamma_1(\alpha) + \gamma_3(\alpha, \Delta)) \tag{4.12}
\]

We use the representation

\[
x(t_\Delta) = x + \Delta f
\]

and Lemma 3.5 to obtain the following inequality:

\[
\phi_\alpha(t_\Delta, x(t_\Delta)) \leq \phi_\alpha(t, x) + \Delta(\zeta^\alpha_t + \langle \zeta^\alpha_x, f \rangle) + \Delta^2(1 + |f|^2)/2\alpha^2.
\]

Then

\[
\phi_\alpha(t_\Delta, x(t_\Delta)) - \int_{t}^{t+\Delta} L(x(s), u^\alpha_f(t), v(s))ds \leq \phi_\alpha(t, x) + \Delta(\zeta^\alpha_t + \langle \zeta^\alpha_x, f \rangle - L) + \Delta^2(1 + C_1^2)/2\alpha^2.
\]

By using (4.10), (4.11), the definition of \(u^\alpha_f(t, x)\) and (4.4), we get

\[
\zeta^\alpha_t + \langle \zeta^\alpha_x, f \rangle - L \leq \int_{t}^{t+\Delta} [\zeta^\alpha_t + \langle \zeta^\alpha_x, f(x, u^\alpha_f(t), v(s)) \rangle - L(x, u^\alpha_f(t, x), v(s))]ds + |\zeta^\alpha_x||b|\Delta + |c|\Delta
\]

\[
\leq \int_{t}^{t+\Delta} [\zeta^\alpha_t + H^+(x, \zeta^\alpha_x)]ds + \frac{C_1^2 C_2 \Delta^2}{\alpha} + C_1^2 \Delta^2.
\]

The last term is less then

\[
\gamma_1(\alpha)\Delta + C_1^2 C_2 \Delta^2/\alpha + C_1^2 \Delta^2
\]

in view of (4.9).

Comparing three last relations we obtain (4.12).

Thus, function \(\phi_\alpha\) is Lyapunov function. We use Proposition 2.1 and its Corollary for this function and relation (3.6) to obtain (4.7). Theorem is proved.
5 Universal feedback control for lower semicontinuous functions satisfying Hamilton-Jacobi inequality.

Let lower semicontinuous function \( \phi : (-\infty, T] \times \mathbb{R}^n \to (-\infty, +\infty] \) satisfy Hamilton-Jacobi inequality (4.1) and the condition (4.2). In this Section universal feedback control \( u_f \) providing the estimate (2.8) for compact set \( D_0 \) of initial conditions \((t_0, x_0)\) is constructed. The main difference in this construction from the case of continuous function \( \phi \) lays in the fact that function \( \phi_\alpha \) cannot now play a role of Lyapunov function \( \eta \) from Proposition 2.1. To explain this fact we note that key Lemma 4.3 stating that \( \phi_\alpha \) has property \( D_2 \) of Lyapunov function was based on Lemma 4.2 in which proof the continuity of \( \phi \) was used.

In order to overcome this difficulty, a new virtual control system

\[
\dot{x}' = g_\alpha(x', u, w, v)
\]

(5.1)
is considered, where \( x' \) is virtual state, \( u \) is control, \( v \) is disturbance, \( \alpha > 0 \) is some fixed parameter,

\[
g_\alpha(x, u, w, v) = f(x, u, v) + \alpha w(v).
\]

Function \( w(v) \) is some continuous function of \( v \).

Virtual system (5.1) satisfies the condition which is analogous to (4.9) and some of its solutions satisfy the relation analogous to (4.12). Then it is sufficient to apply a feedback control which provides tracking of this virtual trajectory by the trajectory of the original system (1.1).

This approach has as its predecessor the method of control with guide [12].

Now we introduce some notation used in this Section.

Let us fix a compact set \( D_0 \) of initial points \((t_0, x_0)\). Without loss of generality we may assume that

\[
D_0 \subset [0, T] \times \mathbb{R}^n
\]

which means that for all \((t_0, x_0)\) we have a lower bound

\[
t_0 \geq 0.
\]

Compact sets \( D \) and \( D_1 \) are determined by \( D_0 \) as in (2.1). Constant \( C_1 \) is defined by (2.2), \( k_2 \) denotes a lower bound for \( \phi \) on \( D_1 \).

To clarify the exposition we start the construction of universal feedback control under the following assumption

for some constant \( k'_1 \geq k_2 \)

\[
\phi(t, x) \leq k'_1 \text{ on } D_0.
\]

(5.3)

Let \( k_1 = k'_1 + C_1 T \) and constant \( C_2 \) be defined by (3.3), \( C_3 = C_2 \exp(C_1 T) \).

It was told that we can not use directly function \( \phi_\alpha \) as a Lyapunov function in the case of lower semicontinuous function \( \phi \). In this Section the role of Lyapunov function is played by function

\[
\psi_\alpha(t, x) = \min_{x' \in R_\alpha(t, x)} \phi_\alpha(t, x'),
\]

(5.4)

where

\[
R_\alpha(t, x) = \{ x' : |x' - x| \leq \alpha \int_0^t e^{C_1 s} C_2 ds, \}
\]

(5.5)

function

\[
\phi_\alpha(t, x') = \min_{(\tau, y) \in D_1} \{ \phi(\tau, y) + \frac{1}{2\alpha^2} \|(t - \tau, x' - y)\|^2 \}
\]

(5.6)
was defined and used in previous Sections.

We use the following notation: for any given \((t, x)\) the point \(x'\) denotes the point of \(R_\alpha(t, x)\) at which the minimum in (5.8) is attained; \((\tau', y')\) denotes a point at \(D_1\) at which the minimum at (5.9) is attained for this \(x'\). Thus, everywhere points \(x', \tau', y'\) are met they defined by \((t, x)\) as it was mentioned before. Analogously

\[
\zeta'_t = \frac{t - \tau'}{\alpha^2}, \quad \zeta'_x = \frac{x' - y'}{\alpha^2}
\]

(5.7)

depend upon \((t, x)\)

Let us define sets

\[
M_\alpha = \{(t, x) \in D : \psi_\alpha(t, x) \leq k'_1 + C_1 t\}, \\
F_\alpha = \{(t, x) \in D : \tau' = T\}.
\]

(5.8)

It is easy to see from (5.2) that \(D_0 \subset M_\alpha\). We demonstrate that for the implementation of the feedback control defined below pair \((t, x, \pi(t))\) will stay on \(M_\alpha\). This implies in accordance with Remark 2.1 to the Proposition 2.1 that we need to verify the property \(D2\) for function \(\psi_\alpha\) only for points from \(M_\alpha\).

Functions

\[
\chi_1(\alpha) = \omega_l((C_1 C_2 + C_2 + C_3)\alpha) + \omega_l\phi((C_2 + C_3)\alpha) + C_1 C_2 \alpha, \\
\chi_2(\alpha) = (C_1 C_2 + C_1 C_3)\alpha, \\
\chi_3(\alpha, \Delta) = C_1^2 (1 + C_2)\alpha + C_1^2 (1 + C_2)\Delta/\alpha + [1 + C_1^2] (1 + C_2)^2 \Delta/2\alpha^2
\]

(5.9)

are used in the statement and proof of the next Theorem 5.1. Here

\[
\omega_{l, \phi}(\Delta) = \sup\{l(x) - \phi(\tau, y) : |T - \tau| \leq \Delta, |x - y| \leq \Delta, (T, x) \in D_1, (\tau, y) \in D_1\}.
\]

Because of condition (4.2), continuity of \(l\) and lower semicontinuity of \(\phi\)

\[
\lim_{\Delta \to 0} \omega_{l, \phi}(\Delta) = 0.
\]

It implies that

\[
\lim_{\alpha \to 0} \chi_1(\alpha) = 0.
\]

The universal feedback control

\[
u^\alpha_f := u^\alpha(t, x)
\]

corresponding to a parameter \(\alpha\) and a compact set of initial condition \(D_0\) is defined as follows:

\[
u^\alpha(t, x) = u(t, x', \zeta'_x)
\]

(5.10)

where \(u(t, x, p)\) is given by (4.3), \(y', \zeta'_x\) correspond to \((t, x)\) as it was pointed out before.

**Theorem 5.1** Let lower semicontinuous function \(\phi(t, x)\) satisfy Hamilton-Jacobi inequality (4.1)-(4.2) and be bounded on a compact set \(D_0 \subset [0, T] \times R^n\) (5.3). Then for all \(\alpha \in (0, (C_1 + c_3)^{-1})\) small enough

\[
\Lambda\nu(u^\alpha_f)(t_0, x_0) \leq \phi(t_0, x_0) + \chi_1(\alpha) + (T - t_0)\chi_2(\alpha)
\]

(5.11)

for all \((t_0, x_0) \in D_0\).
Remark that since $\chi_1(\alpha), \chi_2(\alpha)$ are converging to 0 when $\alpha \downarrow 0$ the inequality (5.11) implies that for any $\epsilon > 0$ there exist $\alpha_\epsilon > 0$ such that for all $\alpha \in (0, \alpha_\epsilon)$

$$\chi_1(\alpha) + T\chi_2(\alpha) < \epsilon,$$

which implies (2.8) for universal feedback control $u^\alpha_f$.

**Proof.** Let parameter $\alpha$ be as in the statement of Theorem. We may assume that $\alpha \leq 1$.

We need to verify that $\psi_\alpha$ (5.4) is a Lyapunov function for feedback control $u^\alpha_f$ (5.10), set $D_0$, constants $\chi_1(\alpha), \chi_2(\alpha)$, function $\chi_3(\alpha, \Delta)$ and set $F$ (5.8).

At first we show that $\psi_\alpha, F$ satisfy $D_1$ in Definition 2.1.

Let $u(\cdot), v(\cdot)$ be an arbitrary control and a disturbance, $x(\cdot)$ be the corresponding solution of (1.1) with the initial condition $x(t) = x$.

**Lemma 5.1** Let $(t, x) \in F'$, then

$$l(x(T)) - \int_t^T L(x(s), u(s), v(s))ds \leq \psi_\alpha(t, x) + \chi_1(\alpha)$$    \hspace{1cm} (5.12)

Because of the definition (5.8) of $F$ we need to consider two cases: $(t, x) \in F_\alpha$, and $t = T$.

Let $(t, x) \in F_\alpha$ then

$$\psi_\alpha(t, x) = \phi_\alpha(t, x') \geq \phi(T, y') = l(y').$$

We have

$$|x(T) - x| \leq C_1(T - t) \quad \text{(from (2.2))},$$

$$|T - t| \leq C_2\alpha, |y' - x| \leq C_2\alpha, \quad \text{(from (5.8) and Lemma 3.2)}$$

$$|x - x'| \leq C_3\alpha, \quad \text{(from (5.5))}.$$

It follows from the above relations that

$$|x(T) - y'| \leq (C_1C_2 + C_2 + C_3)\alpha,$$

which implies

$$\psi_\alpha(t, x) \geq l(x(T)) - \omega_l((C_1C_2 + C_2 + C_3)\alpha).$$

Taking into account boundness of $|L|$ on $D$ by $C_1$, we obtain (5.12).

Let $t = T$ then $x(T) = x$ and

$$\psi_\alpha(T, t) = \phi_\alpha(T, x') \geq \phi(\tau', y'),$$

where $\tau', y', x'$ correspond to $(T, x)$.

It follows from Lemma 3.2 and (5.5), that

$$\|T - \tau', y' - x'\| \leq C_2\alpha, \quad |x' - x| \leq C_3\alpha.$$

We obtain from these relations that

$$\phi(\tau', y') \geq l(x) - \omega_{l,\phi}(\|T - \tau', x - y'\|) \geq l(x) - \omega_{l,\phi}((C_2 + C_3)\alpha),$$
which implies (5.12).Lemma is proved.

Now we verify that $\psi_\alpha$ satisfies D2 in Definition 2.1 on $M_\alpha \setminus F_\alpha$.

Let $(t, x) \in M_\alpha$, then we have from Lemma 3.2 that

$$\|t - \tau', x' - y'\| \leq C_2 \alpha.$$  \hfill (5.13)

Since $x' \in R_\alpha(t, x)$ (5.5), we have:

$$|x' - x| \leq C_3 \alpha.$$

It follows from the choice of $\alpha$ that

$$\|(t - \tau', x - y') < 1.$$

So $(\tau', y')$ is an interior point of $D_1$ if $(t, x)$ does not belong to $F_\alpha$. In view of Lemma 3.1 it proves the following

**Lemma 5.2** For any $(t, x) \in M_\alpha \setminus F_\alpha$

$$(\zeta_t', \zeta_x') \in \partial^p \phi(\tau', y').$$

This Lemma is used in the proof of the fact that function $\psi_\alpha$ has the property D2 in the Definition 2.1 of Lyapunov function which is an assertion of the following Lemma 5.3.

Let us fix a point $(t, x) \in M_\alpha \setminus F_\alpha$, an arbitrary $\Delta > 0$, a measurable disturbance $v(\cdot)$, and define a constant control $u(\tau) \equiv u'$ on $[t, t + \Delta]$, where $t + \Delta = t + \Delta$, and

$$u' = u(t, y', \zeta_t'),$$  \hfill (5.14)

where $u(t, x, p)$ is defined by (4.3).

Then there exists a unique solution $x(\cdot)$ of (1.1) with the control $u(\cdot)$ the disturbance $v(\cdot)$ and the initial condition $x(t) = x$. It is obvious that the control $u(\cdot)$ on $[t, t + \Delta]$ is implemented by discontinuous feedback control (5.10).

**Lemma 5.3** Let $(t, x) \in M_\alpha \setminus F_\alpha$. Then $(t + \Delta, x(t + \Delta)) \in M_\alpha$ and

$$\psi_\alpha(t + \Delta, x(t + \Delta)) \leq \psi_\alpha(t, x) + \int_t^{t + \Delta} L(x(s), u(s), v(s)) ds + \Delta (\chi_2(\alpha) + \chi_3(\alpha, \Delta)).$$  \hfill (5.15)

The proof of this Lemma is given in several steps.

**Step 1.** It follows from Lemma 5.2, Hamilton-Jacobi inequality (4.1) and (4.3) that

$$\zeta_t' + \langle \zeta_x', f(y', u', v) \rangle - L(y', u', v) \leq 0.$$  \hfill (5.16)

Let us define a virtual control system (5.1) by determining function $w(v)$ as follows:

$$w(v) = 1/\alpha (f(y', u', v) - f(x', u', v)).$$

Then from (5.16) we obtain the following inequality:

$$\zeta_t' + \langle \zeta_x', g_\alpha(x', u', v) - L(x', u', v) \rangle \leq C_1 C_2 \alpha,$$  \hfill (5.17)

which is analogous to the relation (4.9).

It follows from (5.13) that

$$|w(v)| \leq C_1 C_2.$$
Particularly, it implies that function $g_\alpha$ has the same properties as function $f$: $|g_\alpha|$ is bounded on $D_1$ by the constant $C_1(1 + C_2)$, $g_\alpha$ is Lipschitz in $z$ with the constant $C_1$.

**Step 2.** This fact in combination with the inequality (5.17) means that we can apply Lemma 4.3 to the solution $z(\cdot)$ of the virtual control system (5.1), the corresponding control $u(\cdot) \equiv u'$, an arbitrary disturbance $v(\cdot)$ and the initial condition $z(t) = x'$. Then taking into account properties of $g_\alpha$, we obtain that

$$
\phi_\alpha(t_\Delta, z(t_\Delta)) \leq \phi_\alpha(t, x') + \int_t^{t_\Delta} L(z(s), u', v(s))ds + \Delta(C_1C_2\alpha + \chi_3(\alpha, \Delta)).
$$

(5.18)

Here function $\chi_3$ is obtained by substituting $C_1$ in $\gamma_3(\alpha, \Delta)$ by $C_1(C_2 + 1)$, where $C_1$ appeared as an upper bound for $|f|$.

**Step 3.** Now we show that the trajectory $x(\cdot)$ in the statement of Lemma is tracking the trajectory $z(\cdot)$. For any $t' \in [t, t_\Delta]$ we have the following:

$$|x(t') - z(t')| \leq |x - x'| + \int_t^{t'} |f(x(s), u', v(s)) - f(z(s), u', v(s))| + C_1C_2|ds \leq |x - x'| + \int_t^{t'} [C_1|x(s) - z(s)| + C_1C_2]ds.
$$

It implies that

$$|x(t') - z(t')| \leq e^{C_1(t' - t)}|x - x'| + \int_t^{t'} e^{C_1(t' - s)}C_1C_2\alpha ds.
$$

Since $x' \in R_\alpha(t, x) (5.5)$ it follows

$$z(t') \in R_\alpha(t', x(t'))
$$

(5.19)

for all $t' \in [t, t_\Delta]$.

Thus, we obtain from the above relation and (5.5) that

$$|x(t') - z(t')| \leq C_3\alpha,
$$

and

$$|L(x(s), u', v(s)) - L(z(s), u', v(s))| \leq C_1C_3\alpha.
$$

(5.20)

Because of the definition of $x'$ and (5.19) for $t' = t_\Delta$

$$\psi_\alpha(t, x) = \phi_\alpha(t, x'), \quad \psi_\alpha(t_\Delta, x(t_\Delta)) \geq \phi_\alpha(t_\Delta, z(t_\Delta)).
$$

Using these relations, (5.20) and the inequality (5.18) we obtain (5.15).

It follows from (5.15) that

$$\psi_\alpha(t_\Delta, x(t_\Delta)) \leq \psi_\alpha(t, x) + C_1\Delta \leq k'_1 + C_1t_\Delta.
$$

which implies that $(t_\Delta, x(t_\Delta))$ stays on $M_\alpha$. So we have from Lemma 5.1 and 5.3 that function $\psi_\alpha$ is Lyapunov function for the feedback control $u'_f (5.10)$.

Thus the assertion of the Theorem and the estimate (5.11) follows from Proposition 2.1, its Corollary, (3.6) and definition of $\psi_\alpha$. 

17
References


