Universal Feedback Control via Proximal Aiming in Problems of Control under Disturbance and Differential Games

F. H. Clarke[*] Yu. S. Ledyaev[†] A. I. Subbotin[‡]

CRM-2386

April 1994

[*]Centre de recherches mathématiques, Université de Montréal, Montréal, Québec H3C 3J7, Canada

[†]Steklov Institute of Mathematics, Moscow 117966, Russia

[‡]Institute of Mathematics and Mechanics, Ural Branch of Russian Academy of Sciences, Ekaterinburg 620219, Russia

1 Introduction

The first question arising in the study of infinitesimal properties of lower semicontinuous functions is the question about the concept of gradient or derivative for such functions which could replace the classical concept of derivative of differentiable function. The answer to this question can be given in the framework of proximal calculus by the introduction of the notion of proximal subgradient of function. proximal calculus is an important component of nonsmooth analysis [4, 5]. Its development was originally motivated by the needs of optimization theory, particularly dynamic optimization [4]. This approach applied to nonsmooth problems of calculus of variations, control problems for differential inclusions, nonsmooth optimal control problems provides the unified characterization of optimal trajectories in terms of Euler-Lagrange and Hamilton inclusions, Pontryagin maximum principle. It appeared that methods of proximal calculus are useful in other fields of control theory too. In particular, they provide very helpful and in certain aspects indispensable tools for the study of generalized solutions of Hamilton-Jacobi equation.

In this paper methods of proximal calculus are used for the construction of universal feedback control for problems of control under disturbance. These results generalize the dynamic programming method for the case of lower semicontinuous functions which are proximal solutions of Hamilton-Jacobi inequality.

Let control system be described by differential equation

$$\dot{x(t)} = f(x(t), u(t), v(t)), \tag{1.1}$$

where $x(t) \in R^n$ is a state vector, $u(t)$ is control with values in compact set $P \subset R^p$, $v(t)$ is a disturbance with values in compact set $Q \subset R^q$, $f(x, u, v)$ is a continuous function.

The quality of control process starting in moment t_0 from initial point x_0 is evaluated by the cost functional

$$\Lambda(x(\cdot), u(\cdot), v(\cdot))(t_0, x_0) = l(x(T)) - \int_{t_0}^{T} L(x(t), u(t), v(t))dt, \tag{1.2}$$

where T is a fixed terminal time, l, L are continuous functions, control $u(\cdot)$ and disturbance $v(\cdot)$ are measurable functions of time.

It is assumed that control $u(t)$ is formed by the following feedback rule

$$u(t) = u_f(t, x(t)),$$

where $u_f(t, x)$ is some feedback control law. The problem of optimal control under disturbance is the problem of finding feedback control which provides minimal guaranteed value of functional in spite of disturbance.

To make the statement of the problem more clear let us consider a special case of continuous feedback control $u_f(t, x)$, i.e. let us assume that every x the function $u_f(t, x)$ is measurable in t and for almost all (a.a.) t it is continuous as function of x.

Then for any measurable disturbance $v(\cdot)$ there exists a solution $x(\cdot)$ of the equation

$$\dot{x} = f(x, u_f(t, x), v(t)), \quad x(t_0) = x_0. \tag{1.3}$$

In this case the quality of feedback control u_f is evaluated by the functional

$$\Lambda_f(u_f)(t_0, x_0) = \sup_{x(\cdot), v(\cdot)} \Lambda(x(\cdot), u(\cdot), v(\cdot))(t_0, x_0),$$

where $u(t) = u_f(t, x(t))$, $x(\cdot)$ is a solution of (1.3) and supremum is taken on the set of all disturbances $v(\cdot)$ and corresponding solutions $x(\cdot)$ of (1.3).

Thus the problem of optimal control under disturbance is the problem of minimization of a functional Λ_f on the set of feedback controls u_f.

Here we postpone the discussion of the fact that in general it is possible to obtain essentially better values of a cost functional by using discontinuous feedback controls in order to describe the traditional method for solving the problem of control under disturbance — the dynamic programming method.

Assume that there exists such differentiable function $\phi(t, x)$ that for all $t \leq T$, $x \in R^n$

$$\phi_t(t, x) + H^+(x, \phi_x(t, x)) \leq 0 \tag{1.4}$$

and

$$\phi(T, x) = l(x), \tag{1.5}$$

where

$$H^+(x, p) = \min_{u \in P} \max_{v \in Q} [\langle p, f(x, u, v)\rangle - L(x, u, v)]. \tag{1.6}$$

Let us suppose that there exists a continuous feedback $u_f(t, x)$ such that

$$u_f(t, x) \in \operatorname{Arg}\min_{u \in P} \{\max_{v \in Q} \langle \phi_x(t, x), f(x, u, v)\rangle - L(x, u, v)\}, \tag{1.7}$$

where $\operatorname{Arg}\min_{u \in P} g(u)$ denotes a set of all u from P which minimize g:

$$\operatorname{Arg}\min_{u \in P} g(u) = \{\hat{u} \in P : g(\hat{u}) = \min_{u \in P} g(u)\}.$$

Then for any solution $x(\cdot)$ of the equation (1.3) for any measurable function $v(\cdot)$ we have the following relation:

$$\frac{d}{dt}\phi(t, x(t)) - L(x(t), u(t), v(t)) \leq 0,$$

or, by integrating it

$$\phi(t, x(t)) - \int_{t_0}^{t} L(x(s), u(s), v(s))ds \leq \phi(t_0, x_0), \tag{1.8}$$

where $u(t) = u_f(t, x(t))$.

By taking $t = T$ and using (1.5), we obtain:

$$\Lambda(x(\cdot), u(\cdot), v(\cdot))(t_0, x_0) \leq \phi(t_0, x_0),$$

which implies the following inequality for feedback control u_f defined by (1.7):

$$\Lambda_f(u_f)(t_o, x_0) \leq \phi(t_0, x_0). \tag{1.9}$$

Note that this inequality is valid for all initial conditions (t_0, x_0), $t_0 \leq T$, and in this sense u_f is a universal feedback control. Let us remark that universal feedback u_f is optimal if the equality holds in (1.4) instead of inequality.

However in the application of the dynamic programming method we immediately meet the well-known difficulties.

The easiest one is the absence of continuous feedback control satisfying (1.7). To overcome this difficulty one can replace discontinuous feedback control by an upper semicontinuous multifunction defined by the well-known design based on tools of the theory of differential inclusions (see, e.g.,

[7]). The other way is to use discontinuous feedback control and implementation schemes for them (see Section 2). But the main difficulty is non-existence, in general, of differentiable function ϕ satisfying Hamilton-Jacobi inequality (1.4) and providing the estimate (1.9) which is close to the optimal one.

For such functions we replace classical derivatives ϕ_t, ϕ_x by the well-known in nonsmooth analysis [4, 5] proximal subgradients $(\zeta_t, \zeta_x) \in \partial^p \phi(t, x)$ of lower semicontinuous function ϕ. We shall consider functions ϕ satisfying the relation analogous to Hamilton-Jacobi inequality (1.4) : for all $(t, x) \in (-\infty, T] \times R^n$, $(\zeta_t, \zeta_x) \in \partial^p \phi(t, x)$

$$\zeta_t + H^+(x, \zeta_x) \leq 0. \tag{1.10}$$

Under general assumptions on control system (1.1) and functional (1.2) this inequality is satisfied by the optimal value function

$$w(t, x) = \inf_{u_f} \Lambda_f(u_f)(t, x) \tag{1.11}$$

(the definition of the functional Λ_f for discontinuous feedback controls will be given in Section 2). The value function w is continuous and satisfies the condition

$$w(T, x) = l(x).$$

The considered problem of feedback control under disturbance is closely connected with the theory of differential games. It is known (see, e.g., [12]) that the above function w coincides with the value function of a differential game defined in the class of feedback controls u_f of the minimizing player and the class of the so-called counterstrategies v_c of the other (maximizing) player. Recall that counterstrategies are defined as functions $v_c(t, x, u)$ (possibly discontinuous). It is assumed that for any $t \leq T$, $x \in R^n$ the map $P \ni u \mapsto v_c(t, x, u) \in Q$ is Borel measurable.

The value function satisfies also the following inequality

$$\zeta_t + H^+(x, \zeta_k) \geq 0, \tag{1.12}$$

for all $(\zeta_t, \zeta_k) \in \partial_p \phi(t, x)$, where $\partial_p \phi(t, x)$ denotes the proximal superdifferential of function ϕ. It is known that a continuous function satisfying simultaneously Hamilton-Jacobi inequalities (1.10), (1.12) and the terminal condition $\phi(T, x) = l(x)$ is unique.

This fact reminds us about the connection between value function of differential game and viscosity solution of Hamilton-Jacobi equation [6]. The equivalent characterization of value function of differential game was given earlier in [14] in terms of differential inequalities for directional derivatives of w.

In concusion of this brief survey of related facts, let us remind a construction of suboptimal feedback control suggested by N.N. Krasovskii [10, 11] (see also english edition of the book [12]). To simplify this description we will assume that the integrand L in the cost functional will equal to zero. Let $w(t, x)$ be the value function, and let $u_f^\alpha(t, x)$ be an arbitrary function such that

$$u_f^\alpha(t, x) \in \operatorname{Arg}\min_{u \in P} \{\max_{v \in Q} \langle x - y_\alpha(t, x), f(x, u, v) \rangle\}.$$

Here

$$y_\alpha(t, x) \in \operatorname{Arg}\min\{w(t, y) : \|x - y\|^2 \leq r(t, \alpha)\},$$

where α is a positive parameter, $r(t, \alpha)$ is some positive function, for its exact definition we refer to the cited works, noting only that $r(t, \alpha) \to 0$ as $\alpha \downarrow 0$. It is known that for any $\epsilon > 0$ and for any compact set $D_0 \subset (-\infty, T] \times R^n$ a parameter $\alpha > 0$ can be chosen such that

$$\Lambda_f(u_f^\alpha)(t_o, x_0) \leq w(t_0, x_0) + \epsilon, \quad \forall (t_0, x_0) \in D_0.$$

Thus, the feedback control u_f^α is ϵ-optimal. Since this estimate holds for all initial conditions (t_0, x_0) from D_0 this feedback control is called *universal* for the region D_0.

In the present paper we develop constructions of universal feedback controls with the emphasis on the ideas and techniques of proximal calculus. This approach leads to the quite natural generalization of dynamic programming method for continuous or even lower semicontinuous solutions of (1.10). According to the construction proposed in this paper, universal feedback control can be defined by relations of the form (1.7), in which classical derivatives ϕ_x is replaced by proximal subgradients ζ_x.

Now we outline briefly the contents of this paper. Section 2 contains notations, main assumptions, descriptions of the implementation schemes for discontinuous feedback control, the statement of the problem of control with uncertainty.

Basic concepts of proximal calculus (proximal subgradients, Iosida-Moreau type of regularization of functions, their properties) are included in Section 3. Section 4 is devoted to the construction of the universal discontinuous feedback control for continuous function ϕ satisfying Hamilton-Jacobi inequality (1.10). We remind here that the optimal value function (1.11) is a continuous function. The construction of feedback control in this case is simpler than in general case of lower semicontinuous function ϕ which is considered in Section 5.

2 Main Assumptions and Problem Formulation

Let $\langle x, y\rangle$ denote a scalar product of x and y in R^n, $|x| = \langle x, x\rangle^{1/2}$ be euclidean norm, B be closed unit ball in R^n, $\|(t, x)\| = (t^2 + |x|^2)^{1/2}$.

For given partition $\pi = \{t_k\}_{i=0}^m$ of the interval $[t_0, T]$,

$$t_0 < t_1 < ... < t_m = T$$

the quantity

$$|\pi| = \max_{0\le k\le m-1}(t_{k+1} - t_k)$$

is called a diameter of the partition.

The main assumptions about control system (1.1) and (1.2) are the following:

Hypothesis A. *Functions $f(x, u, v), L(x, u, v)$ and $l(x)$ are continuous.*

Hypothesis B. *Functions $x \to f(x, u, v)$ and $x \to L(x, u, v)$ are locally Lipschitz uniformly with respect to $u \in P, v \in Q$.*

Hypothesis C. *For some constant $a > 0$*

$$\langle x, f(x, u, v)\rangle \le a(1 + |x|^2)$$

for all $(x, u, v) \in R^n \times P \times Q$.

The last Hypothesis C is called a growth condition. It provides for any compact set $D_0 \subset (-\infty, T] \times R^n$ the existence of compact set $D \subset (-\infty, T] \times R^n$ such that for any solution $x(\cdot)$ of (1.1) with the initial condition $(t_0, x_0) \in D_0$ we have the following inclusion:

$$(t, x(t)) \in D, \quad t \in [t_0, T].$$

The set

$$\{(t, x) : \|(t - \tau, x - y)\| \le 1 \text{ for some } (\tau, y) \in D,\ t \le T\} \tag{2.1}$$

is denoted by D_1.

It follows from Hypothesis A,B that there exists some constant C_1 on $D_1 \times P \times Q$

$$|f(x,u,v)| \le C_1, \qquad |L(x,u,v)| \le C_1$$
$$|f(x_1,u,v) - f(x_2,u,v)| \le C_1|x_1 - x_2|, \quad |L(x_1,u,v) - L(x_2,u,v)| \le C_1|x_1 - x_2|. \tag{2.2}$$

In particular, it implies that

$$|H^+(x^1,p) - H^+(x^2,p)| \le C_1|x^1 - x^2|(1+|p|) \tag{2.3}$$

for all $(t, x^i, p) \in D_1 \times R^n,\ i = 1, 2$.

The notation D_0, D, D_1 will be used in the next Sections. For example, function ϕ_α defined for lower semicontinuous function ϕ and set D_1

$$\phi_\alpha(t,x) = \min_{(\tau,y)\in D_1} (\phi(\tau,y) + \frac{1}{2\alpha^2}\|(t-\tau, x-y)\|^2) \tag{2.4}$$

will play an essential role in the following consideration. It reminds us Iosida-Moreau regularization of convex function in convex analysis.

Now we consider a concept of discontinuous feedback control $u(t,x)$. There are examples showing that such feedback control performs essentially better than continuous feedback control. Here we place an example which demonstrates that universal discontinuous feedback control provides better values of the cost functional than universal continuous one.

Example 2.1. Let us consider control system

$$\dot{x} = u,$$

where $x \in R^n$, control parameter u has values in unit closed ball B. The functional (1.2) has form

$$\Lambda(x(\cdot), u(\cdot))(t_0, x_0) = -|x(1)|.$$

We claim that for any continuous feedback control there exists point $\bar{x}_0 \in B$ such that

$$\Lambda_f(u_f)(0, \bar{x}_0) = 0. \tag{2.5}$$

To prove it we can consider continuous feedback control $u(t,x)$ which is Lipschitz in x. Then for any point $x_0 \in B$ there is unique solution $x(t; x_0)$ of the equation

$$\dot{x} = u(t,x), \quad x(0) = x_0,$$

which continuously depends upon x_0.

Thus continuous mapping

$$F(x_0) = -\int_0^1 u(t, x(t; x_0))dt$$

from B to B has a fixed point $\bar{x}_0$ which satisfies (2.5).

It follows from below that discontinuous feedback control

$$u(t,x) = \begin{cases} \dfrac{x}{|x|}, & x \ne 0 \\ e, & x = 0 \end{cases}$$

where e is some unit vector, provides the following estimate for functional (2.7)

$$\Lambda_f(u_f)(0,x_0) \le -1 \quad \text{for all } x_0 \in B.$$

By comparing this estimate with (2.5) we see that universal discontinuous feedback control performs better than universal continuous one.

It is clear also that in this simple example, for any *fixed* initial position, an optimal result attained in the class of continuous feedback controls (even in the class of constant controls) coincides with the optimal result in the class of discontinuous strategies. Let us remark that examples are known (a bit more complicated than the above one) in which for *fixed* initial condition an optimal result in the class of discontinuous feedbacks is neither attainable in the class of continuous feedbacks, nor approximated by these strategies (see, e.g. [12]).

Of course, in the case of discontinuous feedback control it is not possible define directly a corresponding trajectory of control system as a solution of differential equation (1.3) since its right hand side is discontinuous. So we need some implementation scheme for discontinuous feedback control which determines the value of control $u(t)$. Such scheme was suggested by N.N.Krasovskii and A.I.Subbotin in the context of differential game.

Let discontinuous feedback control u_f be an arbitrary function $u(t,x)$ with values in P. Then for any partition $\pi = \{t_k\}_{k=0}^m$ of the interval $[t_0,T]$ and any disturbance $v(\cdot)$ the trajectory $x_\pi(\cdot)$ is a solution of equation (1.1) with control

$$u_\pi(t) \equiv u_f(t_k, x_\pi(t_k)), \qquad t \in [t_k, t_{k+1}], \quad k = 0,\dots,m-1, \tag{2.6}$$

and initial condition $x_\pi(t_0) = x_0$.

The cost functional evaluating the quality of discontinuous feedback control is defined as follows

$$\Lambda_f(u_f)(t_0,x_0) = \lim_{\delta\downarrow 0} \sup_{|\pi|<\delta,\ v(\cdot)} \Lambda(x_\pi(\cdot), u_\pi(\cdot), v(\cdot))(t_0,x_0), \tag{2.7}$$

where supremum is taken on the set of all partitions π with diameter less than δ and all disturbances.

Let lower semicontinuous function ϕ satisfies Hamilton-Jacobi inequality (1.10) and condition (1.5).

The main result of this paper is the construction for any compact set D_0 and any $\epsilon > 0$ a universal feedback control u_f such that

$$\Lambda_f(u_f)(t_0,x_0) \le \phi(t_0,x_0) + \epsilon \quad \text{for all } (t_0,x_0) \in D_0 \tag{2.8}$$

Since optimal value function (1.11) satisfies Hamilton-Jacobi inequality (1.10) and (1.5) this result implies the existence of ϵ-optimal universal feedback control $\hat{u}_f$ such that

$$\Lambda_f(\hat{u}_f)(t_0,x_0) \le \inf_{u_f} \Lambda_f(u_f)(t_0,x_0) + \epsilon$$

for all $(t_0,x_0) \in D_0$.

The main distinction of the proposed design from the known constructions of universal feedback controls (see, e.g. [2, 10, **?**, 9]) is that it appears to be the most natural generalization of the classical method of dynamic programming. Indeed, according to this design, Hamilton-Jacobi inequality and relations, defining required feedback controls, will have the same form as in the dynamic programming method, but instead of classical derivatives, which fails in general to exist, we will use proximal subgradients. Let us emphasize once more that this demonstrates that methods of proximal calculus provide adequate tools for solving problems of feedback control under uncertainty.

The universal feedback control providing the estimate (2.8) will be constructed with the aid of some Lyapunov-like function $\eta(t,x)$ determined by ϕ. In order to give definition of η we need some additional notation.

Let u_f be discontinuous feedback control, $v(\cdot)$ be arbitrary disturbance, (t,x) be point from D, $\Delta > 0$, $t_\Delta = t + \Delta$. Consider the solution $x(\tau)$ of (1.1) corresponding an initial condition $x(t) = x$, the disturbance $v(\cdot)$ and the control

$$u(\tau) \equiv u_f(t,x) \quad \tau \in [t, t_\Delta).$$

We say that such solution is *implemented* by feedback u_f and disturbance $v(\cdot)$.

Definition 2.1. *Function $\eta(t,x)$ is called a Lyapunov function for a feedback u_f, compact set D_0, constants $\gamma_1 \geq 0$, $\gamma_2 \geq 0$ and monotone increasing function $\gamma_3(\Delta) \geq 0$ if there exists set $F \subset D$ such that $\{(T,x) : (T,x) \in D\} \subset F$ and*

D1. *For any $(t,x) \in F$ and solution $x(\cdot)$ of (1.1) with initial condition $x(t) = x$ corresponding arbitrary control $u(\cdot)$ and disturbance $v(\cdot)$*

$$l(x(T)) - \int_t^T L(x(s), u(s), v(s))ds \leq \eta(t,x) + \gamma_1. \tag{2.9}$$

D2. *For any $(t,x) \in D \backslash F$, $\Delta > 0$ such that $t_\Delta \leq T$, and for the solution $x(\cdot)$ implemented by u_f and an arbitrary disturbance $v(\cdot)$*

$$\eta(t_\Delta, x(t_\Delta)) - \int_t^{t_\Delta} L(x(s), u_f(t,x), v(s))ds \leq \eta(t,x) + \Delta(\gamma_2 + \gamma_3(\Delta)). \tag{2.10}$$

The following Proposition will be used in the proof of universality of feedback controls in both cases of continuous and lower semicontinuous function ϕ.

Proposition 2.1. *Let η be a Lyapunov function for feedback control u_f then for any $\delta > 0$, $(t_0, x_0) \in D_0$, any partition $\pi = \{t_k\}_0^m$ of $[t_0, T]$ with diameter $|\pi| \leq \delta$ and any disturbance $v(\cdot)$*

$$\Lambda(x_\pi(\cdot), u_\pi(\cdot), v(\cdot))(t_0, x_0) \leq \eta(t_0, x_0) + \gamma_1 + (T - t_0)(\gamma_2 + \gamma_3(\delta)). \tag{2.11}$$

We have the obvious Corollary to this Proposition.

Corollary. *Let $\gamma_3(\delta)$ decreasing to 0 while $\delta \downarrow 0$ then*

$$\Lambda_f(u_f)(t_0, x_0) \leq \eta(t_0, x_0) + \gamma_1 + (T - t_0)\gamma_2. \tag{2.12}$$

Proof. Let us fix $(t_0, x_0) \in D_0$ and consider some partition $\pi = \{t_k\}_0^m$ of $[t_0, T]$ with diameter $|\pi|$ less than δ. Then implementation scheme for a feedback u_f and arbitrary disturbance $v(\cdot)$ produces a piecewise constant control $u_\pi(\cdot)$ (2.6) and a trajectory $x_\pi(\cdot)$.

It follows from **D2** that for all k such that $(t_k, x_\pi(t_k)) \notin F$

$$\begin{gathered}\eta(t_{k+1}, x_\pi(t_{k+1})) - \textstyle\int_{t_k}^{t_{k+1}} L(x_\pi(t), u_\pi(t), v(t))dt \leq \\ \eta(t_k, x_\pi(t_k)) + (t_{k+1} - t_k)(\gamma_2 + \gamma_3(\delta)).\end{gathered}$$

We used here that $|\pi| \leq \delta$ and function $\gamma_3(\Delta)$ is monotone in Δ.

Let us denote by t_N the first moment from π such that $(t_N, x_\pi(t_N)) \in F$. Then we obtain from the previous inequality that

$$\begin{gathered}\eta(t_N, x_\pi(t_N)) - \textstyle\int_{t_0}^{t_N} L(x_\pi(t), u_\pi(t), v(t))dt \leq \\ \eta(t_0, x_0) + (t_N - t_0)(\gamma_2 + \gamma_3(\delta)).\end{gathered}$$

It follows from **D1** that

$$l(x_\pi(T)) - \int_{t_0}^{T} L(x_\pi(t), u_\pi(t), v(t))dt \leq \eta(t_0, x_0) + \gamma_1 + (T - t_0)(\gamma_2 + \gamma_3(\delta)).$$

It implies that in accordance with the definition of Λ we have the relation (2.11).

Proposition is proved.

Remark 2.1. Note that it follows from the proof of Proposition that we need to verify property **D2** of the Lyapunov function η only for points (t, x) of some subset M of D such that $D_0 \subset M$ and the implemented control in **D2** provides inclusion $(t_\Delta, x(t_\Delta) \in M$.

3 Proximal Subgradients

Let $\phi(x)$ be a lower semicontinuous function with values in $(-\infty, +\infty]$. The following concept of the proximal subgradient [4], [5] is the replacement of the classical concept of gradient which is not suitable for such function.

Vector $(\zeta_t, \zeta_x) \in R^1 \times R^n$ is called a proximal subgradient at point (t, x) if there exist a constant σ such that for all $(t^{'}, x^{'})$ near (t, x)

$$\phi(t^{'}, x^{'}) \geq \phi(t, x) + \zeta_t(t^{'} - t) + \langle \zeta_x, x^{'} - x)\rangle - \sigma\|(t^{'} - t, x^{'} - x)\|^2.$$

The set of all proximal subgradients at (t, x) is denoted by $\partial^p\phi(t, x)$. This set is called proximal subdifferential. If $\phi(t, x) = +\infty$ then we assume that $\partial^p\phi(t, x) = \emptyset$.

Proximal supergradients (ζ_t, ζ_x) of upper semicontinuous function $\psi(t, x)$ are defined as an element of the set $-\partial^p(-\psi)(t, x)$.

Proximal subdifferential $\partial^p\phi(t, x)$ is non-empty on a set which is dense in Dom $\phi = \{(t, x) : \phi(t, x) < +\infty\}$. This fact is easily proved by means of function ϕ_α determined by (2.4) for $\alpha > 0$ and compact set D_1 (2.1) with non-empty interior.

Let (τ_α, y_α) be a point in D_1 at which minimum in (2.4) is attained. Vectors

$$\zeta_t^\alpha = \frac{t - \tau_\alpha}{\alpha^2}, \quad \zeta_x^\alpha = \frac{x - y_\alpha}{\alpha^2} \tag{3.1}$$

and point (τ_α, y_α) depend upon (t, x).

The next Lemma follows clearly from the definition of $\phi_\alpha, \tau_\alpha, y_\alpha$.

Lemma 3.1. *Let for given (t, x) the point (τ_α, y_α) is the interior point of D_1. Then*

$$(\zeta_t^\alpha, \zeta_x^\alpha) \in \partial^p\phi(\tau_\alpha, y_\alpha). \tag{3.2}$$

By choosing an appropriate value of parameter α it is possible to localize (τ_α, y_α) with respect to (t, x).

Denote by k_2 the lower bound of function ϕ on compact set D_1. For $k_1 \geq k_2$ we use notation

$$C_2 = \sqrt{2(k_1 - k_2)} \tag{3.3}$$

Lemma 3.2. *Let for some constant* k_1

$$\phi_\alpha(t, x) \leq k_1. \tag{3.4}$$

Then

$$\|(t - \tau_\alpha, x - y_\alpha)\| \leq C_2\alpha. \tag{3.5}$$

It is easy to see that (3.5) follows immediately from the definition of (τ_α, y_α). Note that since for any (t, x)

$$\phi_\alpha(t, x) \le \phi(t, x), \tag{3.6}$$

then condition (3.4) may be replaced by the following one:

$$\phi(t, x) \le k_1. \tag{3.7}$$

But if ϕ is continuous on D_1 then (3.7) holds on D_1 for some constant k_1. It implies that for continuous ϕ the estimate (3.4) is valid with some k_1 for all (t, x) in D_1.

We assume here that C_2 in (3.3) is defined by this k_1.

Lemma 3.3. *Let $\phi(t, x)$ be continuous on D_1. Then*

$$\frac{|x - y_\alpha|^2}{2\alpha^2} \le \omega_\phi(C_2\alpha) \tag{3.8}$$

for all $(t, x) \in D_1$ where ω_ϕ is a modulus of continuity of function ϕ on D_1 .

To prove this assertion, we use the definition of the point (τ_α, y_α) to get

$$\frac{|x - y_\alpha|^2}{2\alpha^2} \le \phi(t, x) - \phi(\tau_\alpha, y_\alpha)$$

and the estimate (3.5) for continuous ϕ.

Let us define subset

$$F_\alpha = \{(t, x) \in D : \tau_\alpha = T\} \tag{3.9}$$

Lemma 3.4. *For given constant k_1, arbitrary $\alpha \in (0, 1/C_2)$ and $(t, x) \in D$ satisfying (3.4), one of the following two relations holds: vector $(\zeta_t^\alpha, \zeta_x^\alpha)$ is a proximal subgradient of function ϕ at point (τ_α, y_α), or $(t, x) \in F_\alpha$ and*

$$|T - t| \le C_2\alpha, \ |x - y_\alpha| \le C_2\alpha. \tag{3.10}$$

To prove this Lemma, we note that if $(t, x) \in F_\alpha$ then (3.10) holds. If (t, x) is not in F_α then in view of Lemma 3.2 and choice of α we have that (τ_α, y_α) is an interior point of D_1, which implies that $(\zeta_t^\alpha, \zeta_x^\alpha)$ satisfies (3.4).

For C^1 function ϕ Taylor expansion formula gives the following estimate of the difference of values of π along some direction

$$\phi(t + \Delta, x + \Delta f) = \phi(t, x) + \Delta(\phi_t(t, x) + \langle \phi_x(t, x), f\rangle) + o(\Delta).$$

The following analogue of Taylor expansion formula for function ϕ_α will play an essential role in in next Sections.

Lemma 3.5. *For any $\Delta > 0$ and for any vector $f \in R^n$*

$$\phi_\alpha(t + \Delta, x + \Delta f) \le \phi_\alpha(t, x) + \Delta(\zeta_t^\alpha + \langle \zeta_x^\alpha, f\rangle) + \frac{\Delta^2}{2\alpha^2}(1 + |f|^2). \tag{3.11}$$

We have that

$$\phi_\alpha(t + \Delta, x + \Delta f) \le \phi(\tau_\alpha, y_\alpha) + \frac{1}{2\alpha^2}\|(t + \Delta - \tau_\alpha, x + \Delta f - y_\alpha)\|^2.$$

Expanding the quadratic term we obtain (3.11).

4 Universal feedback control for continuous functions satisfying Hamilton-Jacobi inequality

Let continuous function $\phi(t,x)$ satisfies the following Hamilton-Jacobi inequality: for all $(t,x) \in (-\infty,T) \times R^n$

$$\zeta_t + H^+(x,\zeta_x) \le 0 \ \text{ for all } (\zeta_t,\zeta_x) \in \partial^p \phi(t,x) \tag{4.1}$$

and condition

$$\phi(T,x) = l(x). \tag{4.2}$$

It was mentioned before that such functions exist. In particular, optimal value function (1.11) is continuous and satisfies (4.1)–(4.2).

For fixed compact set $D_0 \subset (-\infty,T] \times R^n$ we define compact sets D, D_1 (2.1) and function ϕ_α (2.4). Vector $(\zeta_t^\alpha, \zeta_x^\alpha)$ determined by (3.1) depends upon (t,x).

Let $u(t,x,p)$ be an arbitrary function such that

$$u(t,x,p) \in \operatorname{Arg}\min_{u\in P} \{\max_{v\in Q} [\langle p, f(x,u,v)\rangle - L(x,u,v)]\} \tag{4.3}$$

Then feedback control, corresponding parameter α and a set of initial conditions D_0, is defined as follows

$$u_f^\alpha(t,x) = u(t,x,\zeta_x^\alpha). \tag{4.4}$$

Before stating the main result of this section we introduce the following notation.

Because of continuity of ϕ on D_1 it is bounded on $D_1 : k_1 \ge \phi \ge k_2$. Constant C_2 is given by (3.3), constant C_1 is defined as in (2.2), (2.3). Functions

$$\begin{aligned} \gamma_1(\alpha) &= C_1C_2\alpha + \omega_l(C_1C_2\alpha) + \omega_\phi(C_2\alpha) \\ \gamma_2(\alpha) &= C_1(C_2\alpha + 2\omega_\phi(C_2\alpha), \end{aligned} \tag{4.5}$$

are defined by means of modulus of continuity ω_ϕ, ω_l of functions ϕ and l on D_1,

$$\gamma_3(\alpha,\Delta) = C_1^2C_2\Delta/\alpha + C_1^2\Delta + (1+C_1^2)\Delta/2\alpha^2. \tag{4.6}$$

Theorem 4.1. *Let continuous function $\phi(t,x)$ satisfies (4.1)–(4.2). Then for any compact set D_0 and $\alpha \in (0,1/C_2)$*

$$\Lambda_u(u_f^\alpha)(t_0,x_0) \le \phi(t_0,x_0) + (T-t_0)\gamma_1(\alpha) + \gamma_2(\alpha) \tag{4.7}$$

for all $(t_0,x_0) \in D_0$.

Note that for any $\epsilon \ge 0$ there exists such α_ϵ that for all $\alpha \in (0,\alpha_\epsilon)$ and all $(t_0,x_0) \in D_0$

$$(T-t_0)\gamma_1(\alpha) + \gamma_2(\alpha) < \epsilon.$$

It is clear that if ϕ and l are Lipschitz functions we may assume that $\alpha_\epsilon = O(\epsilon)$. We obtain from (4.7) that for any such α feedback control u_f^α (4.4) is universal and provides (2.8).

Proof. Let us fix $\alpha \in (0, C_2^{-1})$ and define a set

$$F = \{(T,x) : (T,x) \in D\} \cup F_\alpha,$$

where F_α is given by (3.9).

To prove Theorem it is sufficient to show that ϕ_α (2.4) is a Lyapunov function for feedback control u_f^α (4.4), set D_0, constants $\gamma_1(\alpha)$, $\gamma_2(\alpha)$, function $\gamma_3(\alpha,\Delta)$ and set F. Then relation (4.7) will follow from Proposition 2.1 and its Corollary.

At first we prove that ϕ_α, F satisfy **D1** in Definition 2.1.

Let $u(\cdot)$, $v(\cdot)$ be arbitrary control and disturbance, $x(\cdot)$ be the corresponding solution of (1.1) with initial condition $x(t)=x$.

Lemma 4.1. *Let $(t,x)\in F$ then*

$$l(x(T)) \le \phi_\alpha(t,x) + \int_t^T L(x(s),u(s),v(s))ds + \gamma_2(\alpha). \tag{4.8}$$

It is clear that if $(t,x)\in F_\alpha$ then

$$l(x) = \phi(T,x) \le \phi_\alpha(t,x).$$

Because of (2.2) and (3.10)

$$|x(T)-x| \le C_1(T-t), \quad (T-t)\le C_2\alpha,$$

which implies that

$$l(x(T)) \le \phi_\alpha(t,x) + \omega_l(C_1C_2\alpha).$$

By using boundness of $|L|$ (2.2) we obtain

$$l(x(T)) - \int_t^T L(x(s),u(s),v(s))ds \le \phi_\alpha(t,x) + C_1C_2\alpha + \omega_l(C_1C_2\alpha),$$

but it means (4.9).

If $t=T$ then $x(T)=x$ and

$$\phi_\alpha(T,x(T)) \ge \phi(\tau_\alpha,y_\alpha) \ge l(x(T)) - \omega_l(C_2\alpha),$$

which means (4.9) again. Lemma is proved.

Now we verify that ϕ_α satisfies **D2** in Definition 2.1.

In view of Lemma 3.4 for all $\alpha\in(0,1/C_2)$ and $(t,x)\in D\backslash F_\alpha$ (see (3.9))

$$(\zeta_t^\alpha,\zeta_x^\alpha)\in\partial^p\phi(\tau_\alpha,y_\alpha).$$

Lemma 4.2. *Let $(t,x)\in D\backslash F_\alpha$, then*

$$\zeta_t^\alpha + H^+(x,\zeta_x^\alpha) \le \gamma_1(\alpha). \tag{4.9}$$

To prove (4.9) we use (4.1) and previous inclusion to obtain

$$\zeta_t^\alpha + H^+(y_\alpha,\zeta_x^\alpha) \le 0.$$

Since $H^+(x,p)$ satisfies (2.3), it follows that

$$\zeta_t^\alpha + H^+(x,\zeta_x^\alpha) \le C_1|x-y_\alpha|(1+|\zeta_x^\alpha|).$$

We use Lemma 3.2 and 3.3 to derive

$$\zeta_t^\alpha + H^+(x,\zeta_x^\alpha) \le C_1C_2\alpha + C_1\omega_\phi(C_2\alpha),$$

which means (4.9).

Now we fix $(t,x) \in D\backslash F_\alpha$, arbitrary $\Delta > 0$, measurable disturbance $v(\cdot)$. Then there exists a unique solution $x(\cdot)$ of the equation

$$\dot{x}(s) = f(x(s), u_f^\alpha(s), v(s)), \quad x(t) = x, \ s \in [t, t_\Delta),$$

where $t_\Delta = t + \Delta$, u_f^α is defined by (4.4). It is easy to see that this construction represents the element of the implementation scheme for feedback control u_f^α (4.4).

Below we use the following notation

$$f = \frac{1}{\Delta}\int_t^{t_\Delta} f(x(s), u_f^\alpha(t,x), v(s))ds, \quad L = \frac{1}{\Delta}\int_t^{t_\Delta} L(x(s), u_f^\alpha(t,x), v(s))ds.$$

Because of the boundness of $|f|, |L|$ and Lipschitzness of f, L in x (2.2), we have the representation

$$f = \frac{1}{\Delta}\int_t^{t_\Delta} f(x, u_f^\alpha(t,x), v(s))ds + b, \quad L = \frac{1}{\Delta}\int_t^{t_\Delta} L(x, u_f^\alpha(t,x), v(s))ds + c, \tag{4.10}$$

where

$$|b| \le C_1^2\Delta, \qquad |c| \le C_1^2\Delta. \tag{4.11}$$

Lemma 4.3. *Let $(t,x) \in D\backslash F_\alpha$ then*

$$\phi_\alpha(t_\Delta, x(t_\Delta)) \le \phi_\alpha(t,x) + \int_t^{t_\Delta} L(x(s), uu_f^\alpha(t,x), v(s))ds + \Delta(\gamma_1(\alpha) + \gamma_3(\alpha,\Delta)) \tag{4.12}$$

We use the representation

$$x(t_\Delta) = x + \Delta f$$

and Lemma 3.5 to obtain the following inequality:

$$\phi_\alpha(t_\Delta, x(t_\Delta)) \le \phi_\alpha(t,x) + \Delta(\zeta_t^\alpha + \langle \zeta_x^\alpha, f\rangle) + \Delta^2(1+|f|^2)/2\alpha^2.$$

Then

$$\phi_\alpha(t_\Delta, x(t_\Delta)) - \int_t^{t_\Delta} L(x(s), u_f^\alpha(t,x), v(s))ds \le$$
$$\phi_\alpha(t,x) + \Delta(\zeta_t^\alpha + \langle \zeta_x^\alpha, f\rangle - L) + \Delta^2(1+C_1^2)/2\alpha^2.$$

By using (4.10), (4.11), the definition of $u_f^\alpha(t,x)$ and (4.4), we get

$$\begin{aligned} & \zeta_t^\alpha + \langle \zeta_x^\alpha, f\rangle - L \\ \le & \int_t^{t_\Delta} [\zeta_t^\alpha + \langle \zeta_x^\alpha, f(x, u_f^\alpha(t,x), v(s))\rangle - L(x, u_f^\alpha(t,x), v(s))]ds + |\zeta_x^\alpha||b|\Delta + |c|\Delta \\ \le & \int_t^{t_\Delta} [\zeta_t^\alpha + H^+(x, \zeta_x^\alpha)]ds + \frac{C_1^2 C_2 \Delta^2}{\alpha} + C_1^2\Delta^2. \end{aligned}$$

The last term is less then

$$\gamma_1(\alpha)\Delta + C_1^2 C_2 \Delta^2/\alpha + C_1^2\Delta^2$$

in view of (4.9).

Comparing three last relations we obtain (4.12).

Thus, function ϕ_α is Lyapunov function. We use Proposition 2.1 and its Corollary for this function and relation (3.6) to obtain (4.7). Theorem is proved.

5 Universal feedback control for lower semicontinuous functions satisfying Hamilton-Jacobi inequality.

Let lower semicontinuous function $\phi : (-\infty, T] \times R^n \to (-\infty, +\infty]$ satisfy Hamilton-Jacobi inequality (4.1) and the condition (4.2). In this Section universal feedback control u_f providing the estimate (2.8) for compact set D_0 of initial conditions (t_0, x_0) is constructed. The main difference in this construction from the case of continuous function ϕ lays in the fact that function ϕ_α connot now play a role of Lyapunov function η from Proposition 2.1. To explain this fact we note that key Lemma 4.3 stating that ϕ_α has property **D2** of Lyapunov function was based on Lemma 4.2 in which proof the continuity of ϕ was used.

In order to overcome this difficulty, a new virtual control system

$$\dot{x}' = g_\alpha(x', u, w, v) \tag{5.1}$$

is considered, where x' is virtual state, u is control, v is disturbance, $\alpha > 0$ is some fixed parameter,

$$g_\alpha(x, u, w, v) = f(x, u, v) + \alpha w(v).$$

Function $w(v)$ is some continuous function of v.

Virtual system (5.1) satisfies the condition which is analogus to (4.9) and some of its solutions satisfy the relation analogus to (4.12). Then it is sufficient to apply a feedback control which provides tracking of this virtual trajectory by the trajectory of the original system (1.1).

This approach has as its predecessor the method of control with guide [12].

Now we introduce some notation used in this Section.

Let us fix a compact set D_0 of initial points (t_0, x_0). Without loss of generality we may assume that

$$D_0 \subset [0, T] \times R^n$$

which means that for all (t_0, x_0) we have a lower bound

$$t_0 \geq 0. \tag{5.2}$$

Compact sets D and D_1 are determined by D_0 as in (2.1). Constant C_1 is defined by (2.2), k_2 denotes a lower bound for ϕ on D_1.

To clarify the exposition we start the construction of universal feedback control under the following assumption

for some constant $k_1' \geq k_2$

$$\phi(t, x) \leq k_1' \quad \textit{on } D_0. \tag{5.3}$$

Let $k_1 = k_1' + C_1 T$ and constant C_2 be defined by (3.3), $C_3 = C_2 \exp(C_1 T)$.

It was told that we can not use directly function ϕ_α as a Lyapunov function in the case of lower semicontinuous function ϕ. In this Section the role of Lyapunov function is played by function

$$\psi_\alpha(t, x) = \min_{x' \in R_\alpha(t,x)} \phi_\alpha(t, x'), \tag{5.4}$$

where

$$R_\alpha(t, x) = \{x' : |x' - x| \leq \alpha \int_0^t e^{C_1} s C_1 C_2 ds, \}, \tag{5.5}$$

function

$$\phi_\alpha(t, x') = \min_{(\tau, y) \in D_1} \{\phi(\tau, y) + \frac{1}{2\alpha^2} \|(t - \tau, x' - y)\|^2\} \tag{5.6}$$

was defined and used in previous Sections.

We use the following notation: for any given (t,x) the point x' denotes the point of $R_\alpha(t,x)$ at which the minimum in (5.8) is attained; (τ',y') denotes a point at D_1 at which the minimum at (5.9) is attained for this x'. Thus, everywhere points x',τ',y' are met they defined by (t,x) as it was mentioned before. Analogously

$$\zeta'_t = \frac{t-\tau'}{\alpha^2}, \qquad \zeta'_x = \frac{x'-y'}{\alpha^2} \tag{5.7}$$

depend upon (t,x)

Let us define sets

$$\begin{aligned} M_\alpha &= \{(t,x)\in D : \psi_\alpha(t,x) \le k'_1 + C_1 t\}, \\ F_\alpha &= \{(t,x)\in D : \tau' = T\}. \\ F &= \{(T,x) : (T,x)\in D\} \cup F_\alpha. \end{aligned} \tag{5.8}$$

It is easy to see from (5.2) that $D_0 \subset M_\alpha$. We demonstrate that for the implementation of the feedback control defined below pair $(t, x_\pi(t))$ will stay on M_α. This implies in accordance with Remark 2.1 to the Proposition 2.1 that we need to verify the property **D2** for function ψ_α only for points from M_α.

Functions

$$\begin{aligned} \chi_1(\alpha) &= \omega_l((C_1C_2 + C_2 + C_3)\alpha) + \omega_{l,\phi}((C_2+C_3)\alpha) + C_1C_2\alpha, \\ \chi_2(\alpha) &= (C_1C_2 + C_1C_3)\alpha, \\ \chi_3(\alpha,\Delta) &= C_1^2(1+C_2)\alpha + C_1^2(1+C_2)\Delta/\alpha + [1+C_1^2)(1+C_2)^2]\Delta/2\alpha^2 \end{aligned} \tag{5.9}$$

are used in the statement and proof of the next Theorem 5.1. Here

$$\omega_{l.\phi}(\Delta) = \sup\{l(x) - \phi(\tau,y) : |T-\tau| \le \Delta, |x-y| \le \Delta, (T,x)\in D_1, (\tau,y)\in D_1\}.$$

Because of condition (4.2), continuity of l and lower semicontinuity of ϕ

$$\lim_{\Delta\downarrow 0} \omega_{l,\phi}(\Delta) = 0.$$

It implies that

$$\lim_{\alpha\downarrow 0} \chi_1(\alpha) = 0.$$

The universal feedback control

$$u^\alpha_f := u^\alpha(t,x)$$

corresponding to a parameter α and a compact set of initial condition D_0 is defined as follows:

$$u^\alpha(t,x) = u(t,x',\zeta'_x) \tag{5.10}$$

where $u(t,x,p)$ is given by (4.3), y',ζ'_x correspond to (t,x) as it was pointed out before.

Theorem 5.1 *Let lower semicontinuous function $\phi(t,x)$ satisfy Hamilton-Jacobi inequality (4.1)-(4.2) and be bounded on a compact set $D_0 \subset [0,T]\times R^n$ (5.3). Then for all $\alpha \in (0,(C_1+c_3)^{-1})$ small enough*

$$\Lambda_u(u^\alpha_f)(t_0,x_0) \le \phi(t_0,x_0) + \chi_1(\alpha) + (T-t_0)\chi_2(\alpha) \tag{5.11}$$

for all $(t_0,x_0)\in D_0$.

Remark that since $\chi_1(\alpha), \chi_2(\alpha)$ are converging to 0 when $\alpha \downarrow 0$ the inequality (5.11) implies that for any $\epsilon > 0$ there exist $\alpha_\epsilon > 0$ such that for all $\alpha \in (0, \alpha_\epsilon)$

$$\chi_1(\alpha) + T\chi_2(\alpha) < \epsilon,$$

which implies (2.8) for universal feedback control u_f^α.

Proof . Let parameter α be as in the statement of Theorem.We may assume that $\alpha \leq 1$.

We need to verify that ψ_α (5.4) is a Lyapunov function for feedback control u_f^α (5.10), set D_0, constants $\chi_1(\alpha), \chi_2(\alpha)$, function $\chi_3(\alpha, \Delta)$ and set F (5.8).

At first we show that ψ_α, F satisfy **D1** in Definition 2.1.

Let $u(\cdot), v(\cdot)$ be an arbitrary control and a disturbance, $x(\cdot)$ be the corresponding solution of (1.1) with the initial condition $x(t) = x$.

Lemma 5.1 *Let $(t, x) \in F$, then*

$$l(x(T)) - \int_t^T L(x(s), u(s), v(s))ds \leq \psi_\alpha(t, x) + \chi_1(\alpha) \tag{5.12}$$

Because of the definition (5.8) of F we need to consider two cases: $(t, x) \in F_\alpha$, and $t = T$.

Let $(t, x) \in F_\alpha$ then

$$\psi_\alpha(t, x) = \phi_\alpha(t, x') \geq \phi(T, y') = l(y').$$

We have

$$|x(T) - x| \leq C_1(T - t) \quad \text{(from (2.2))} ,$$

$$|T - t| \leq C_2\alpha, |y' - x| \leq C_2\alpha, \quad \text{(from (5.8) and Lemma 3.2)}$$

$$|x - x'| \leq C_3\alpha, \qquad \text{(from (5.5))}.$$

It follows from the above relations that

$$|x(T) - y'| \leq (C_1C_2 + C_2 + C_3)\alpha,$$

which implies

$$\psi_\alpha(t, x) \geq l(x(T)) - \omega_l((C_1C_2 + C_2 + C_3)\alpha).$$

Taking into account boundness of $|L|$ on D by C_1, we obtain (5.12).

Let $t = T$ then $x(T) = x$ and

$$\psi_\alpha(T, t) = \phi_\alpha(T, x') \geq \phi(\tau', y'),$$

where τ', y', x' correspond to (T, x).

It follows from Lemma 3.2 and (5.5), that

$$\|(T - \tau', y' - x')\| \leq C_2\alpha, \quad |x' - x| \leq C_3\alpha.$$

We obtain from these relations that

$$\phi(\tau', y') \geq l(x) - \omega_{l,\phi}(\|T - \tau', x - y'\|) \geq l(x) - \omega_{l,\phi}((C_2 + C_3)\alpha),$$

which implies (5.12). Lemma is proved.

Now we verify that ψ_α satisfies **D2** in Definition 2.1 on $M_\alpha \backslash F_\alpha$.

Let $(t,x) \in M_\alpha$, then we have from Lemma 3.2 that

$$\|t-\tau', x'-y')\| \leq C_2\alpha. \tag{5.13}$$

Since $x' \in R_\alpha(t,x)$ (5.5), we have:

$$|x'-x| \leq C_3\alpha.$$

It follows from the choice of α that

$$\|(t-\tau', x-y') < 1.$$

So (τ', y') is an interior point of D_1 if (t,x) does not belong to F_α. In view of Lemma 3.1 it proves the following

Lemma 5.2 *For any* $(t,x) \in M_\alpha \backslash F_\alpha$

$$(\zeta'_t, \zeta'_x) \in \partial^p \phi(\tau', y').$$

This Lemma is used in the proof of the fact that function ψ_α has the property **D2** in the Definition 2.1 of Lyapunov function which is an assertion of the following Lemma 5.3.

Let us fix a point $(t,x) \in M_\alpha \backslash F_\alpha$, an arbitrary $\Delta > 0$, a measurable disturbance $v(\cdot)$, and define a constant control $u(\tau) \equiv u'$ on $[t, t_\Delta]$, where $t_\Delta = t + \Delta$, and

$$u' = u(t, y', \zeta'_x), \tag{5.14}$$

where $u(t,x,p)$ is defined by (4.3).

Then there exists a unique solution $x(\cdot)$ of (1.1) with the control $u(\cdot)$ the disturbance $v(\cdot)$ and the initial condition $x(t) = x$. It is obvious that the control $u(\cdot)$ on $[t, t_\Delta]$ is implemented by discontinuous feedback control (5.10).

Lemma 5.3 *Let* $(t,x) \in M_\alpha \backslash F_\alpha$. *Then* $(t_\Delta, x(t_\Delta)) \in M_\alpha$ *and*

$$\psi_\alpha(t_\Delta, x(t_\Delta)) \leq \psi_\alpha(t,x) + \int_t^{t_\Delta} L(x(s), u(s), v(s))ds + \Delta(\chi_2(\alpha) + \chi_3(\alpha, \Delta)) \tag{5.15}$$

The proof of this Lemma is given in several steps.

Step 1. It follows from Lemma 5.2, Hamilton-Jacobi inequality (4.1) and (4.3) that

$$\zeta'_t + \langle \zeta'_x, f(y', u', v)\rangle - L(y', u', v) \leq 0. \tag{5.16}$$

Let us define a virtual control system (5.1) by determining function $w(v)$ as follows:

$$w(v) = 1/\alpha(f(y', u', v) - f(x', u', v)).$$

Then from (5.16) we obtain the following inequality:

$$\zeta'_t + \langle \zeta'_x, g_\alpha(x', u', v) - L(x', u', v)\rangle \leq C_1 C_2 \alpha, \tag{5.17}$$

which is analogous to the relation (4.9).

It follows from (5.13) that

$$|w(v)| \leq C_1 C_2.$$

Particularly, it implies that function g_α has the same properties as function f: $|g_\alpha|$ is bounded on D_1 by the constant $C_1(1+C_2)$, g_α is Lipschitz in z with the constant C_1.

Step 2. This fact in combination with the inequality (5.17) means that we can apply Lemma 4.3 to the solution $z(\cdot)$ of the virtual control system (5.1), the corresponding control $u(\cdot) \equiv u'$, an arbitrary disturbance $v(\cdot)$ and the initial condition $z(t) = x'$. Then taking into account properties of g_α, we obtain that

$$\phi_\alpha(t_\Delta, z(t_\Delta)) \le \phi_\alpha(t, x') + \int_t^{t_\Delta} L(z(s), u', v(s))ds + \Delta(C_1C_2\alpha + \chi_3(\alpha, \Delta)). \tag{5.18}$$

Here function χ_3 is obtained by substituting C_1 in $\gamma_3(\alpha, \Delta)$ by $C_1(C_2+1)$, where C_1 appeared as an upper bound for $|f|$.

Step 3. Now we show that the trajectory $x(\cdot)$ in the statement of Lemma is tracking the trajectory $z(\cdot)$. For any $t' \in [t, t_\Delta]$ we have the following:

$$|x(t') - z(t') \le |x - x'| + \int_t^{t'} [|f(x(s), u', v(s)) - f(z(s), u', v(s))| + C_1C_2]ds \le$$

$$\le |x - x'| + \int_t^{t'} [C_1|x(s) - z(s)| + C_1C_2]ds.$$

It implies that

$$|x(t') - z(t')| \le e^{C_1(t'-t)}|x - x'| + \int_t^{t'} e^{C_1(t'-s)}C_1C_2\alpha ds.$$

Since $x' \in R_\alpha(t, x)$ (5.5) it follows

$$z(t') \in R_\alpha(t', x(t')) \tag{5.19}$$

for all $t' \in [t, t_\Delta]$.

Thus, we obtain from the above relation and (5.5) that

$$|x(t') - z(t')| \le C_3\alpha,$$

and

$$|L(x(s), u', v(s)) - L(z(s), u', v(s))| \le C_1C_3\alpha. \tag{5.20}$$

Because of the definition of x' and (5.19) for $t' = t_\Delta$

$$\psi_\alpha(t, x) = \phi_\alpha(t, x'), \quad \psi_\alpha(t_\Delta, x(t_\Delta)) \ge \phi_\alpha(t_\Delta, z(t_\Delta)).$$

Using these relations, (5.20) and the inequality (5.18) we obtain (5.15).

It follows from (5.15) that

$$\psi_\alpha(t_\Delta, x(t_\Delta)) \le \psi_\alpha(t, x) + C_1\Delta \le k_1' + C_1t_\Delta,$$

which implies that $(t_\Delta, x(t_\Delta))$ stays on M_α. So we have from Lemma 5.1 and 5.3 that function ψ_α is Lyapunov function for the feedback control u_f^α (5.10).

Thus the assertion of the Theorem and the estimate (5.11) follows from Proposition 2.1, its Corollary, (3.6) and definition of ψ_α.

References

[1] L.D. Berkowitz. Characterization of the values of differntial games *Appl.Math.Optim.* **17** (1988), 177–183.

[2] L.D.Berkowitz. Optimal feedback controls. SIAM J. Control Optim. **27** (1989), 991–1007.

[3] A. Blaquiere, L. G. Leitmann, *Jeux quantitatifs*, Gauthier-Villars, Paris, 1969.

[4] F. H. Clarke. *Optimization and Nonsmooth Analysis*. Wiley-Interscience, New York, 1983. Third edition as Vol. 5 of Classics in Applied Mathematics, S.I.A.M., Philadelphia, 1990.

[5] F. H. Clarke. *Methods of Dynamic and Nonsmooth Optimization*. Volume 57 of *CBMS-NSF Regional Conference Series in Applied Mathematics*, S.I.A.M., Philadelphia, 1989.

[6] M. Crandall, P.-L. Lions. Viscosity solutions of Hamilton-Jacobi equations *Trans.Amer.Math.Soc.*. **277** (1983), 1–42.

[7] Differential equations with discontinuous right-hand side. *Amer. Math. Soc. Transl.* Ser. 2, 42:199–231, 1964.

[8] A. Friedman, *Differential games*, Wiley-Interscience, 1971.

[9] G. G. Garnysheva, A. I. Subbotin, *Strategies of minimax aiming in the direction of quasigradient*, Prikl. Matematika i Mekhanika **58** (1994) (in Russian).

[10] N.N. Krasovskii. Differential games.Approximation and formal models. *Math. USSR Sbornik*, **35**, (1979), 795 –822.

[11] N.N. Krasovskii. *Control of Dynamical Systems* . Nauka, Moscow, 1985 [in Russian].

[12] N.N.Krasovskii and A.I.Subbotin. *Positional differential game*. Nauka, Moscow, 1974 [in Russian]. French translation *Jeux differentiels*. Editions Mir, Moscou, 1979. Revised English translation *Game-Theoretical Control Problems* Springer-Verlag, New York, 1988.

[13] P. Loewen. *Optimal Control via Nonsmooth Analysis. CRM Proceedings and Lecture Notes*, Amer. Math. Soc. Providence, 1993.

[14] A.I. Subbotin. A generalization of the basic equation of the theory of differential games. *Soviet Math.Dokl.* **22** (1980), 358-362.

[15] A. I. Subbotin, *Generalized solutions of first order PDEs: The dynamic optimization perspective*, Birkhauser, Boston, 1994.