Intrinsic nonlinear models of shells for Saint Venant-Kirchhoff materials*

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1 Introduction

The first complete linear theory, based on Kirchhoff's [1850] earlier works on plates, was given by Love [1888]. The modern theory (linear and nonlinear) comes from the works by Naghdi and Koiter in the sixties. In such models the displacement variables are functions defined on the mean surface and the resulting equations are differential equations on a smooth 2-dimensional submanifold of the three-dimensional Euclidean space. This is done by introducing local coordinates and local covariant and contravariant bases for the tangent plane and equations are written in terms of covariant and contravariant derivatives. In this framework expressions of constitutive laws become heavy and it is not always easy to separate purely mechanical approximations from approximations of the characteristics of the geometry of the mean surface.

Research in Shell Theory has currently been stimulated by questions arising from a number of applications in control and design of large space structure, flexible robots, composite materials, etc, where intrinsic and mathematically more tractable models would be much prefered.

Recently, Delfour and Zolsio (cf. [10] and [11]) introduced a new way to deal with the differential geometry, to express tangential differential operators, and to do the differential calculus on submanifolds of the n-dimensional Euclidean space. This was successfully used in the Natural Theory and the Love-Kirchhoff Theory (cf. for instance Germain [14], Dautray and Lions [7]) of static and dynamical linear thin/shallow shells. The new linear models are expressed in

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terms of intrinsic tangential differential operators such as the tangential gradient, divergence, strain tensor, and they fully preserve the rigid displacements. In is also shown in [12] that the classical linear models of Naghdi and Koiter can be rewritten in terms of these intrinsic tangential differential operators.

This approach gives a direct access to the geometrical properties of a surface such as the normal, the principal curvatures, and their derivatives, and avoids local bases for the tangent spaces, Christoffel symbols, or an a priori parametrization of the surface. It centers around the use of the *oriented (signed or algebraic) distance function*, an important tool in *Shape Optimization*. It becomes easier to separate the geometry from the approximation of the displacement variables, and the resulting models are mathematically more tractable in both static and dynamical control, design and optimization problems. It is important to emphasize that this not just another model with a few extra terms, but a fundamental *intrinsic model* which fully preserves the geometry of the main surface and the rigid displacements. Mathematically the proof of existence, uniqueness and smoothness (cf. Theorem 2 in section 3.2) of solutions becomes easier and more in line with the modern theory of partial differential equations.

The main objective of this paper is to follow this approach to extend the Natural Theory of linear thin/shallow shells to a large class of nonlinear shells made up of a Saint Venant-Kirchhoff material. This yields an intrinsic second order nonlinear tangential differential system which is significantly different from more traditional models of thin/shallow shells. This is one of the main contributions of this paper. We prove that the nonlinear operator related to this system is differentiable. A direct calculation shows that the derivative of the nonlinear operator, i.e. its linear part, coincides with the linear operator in the Delfour-Zolsio model [11]. This suggests to use the inverse function theorem (cf. Hörmander [21]) to prove the existence of solutions. By using the Agmon, Douglis and Nirenberg's classical results [2] and the index of elliptic operators (cf. Geymonat [17]), we prove regularity of the weak solutions of the Delfour-Zolsio model [11]. It is a key result which implies that the derivative of the nonlinear operator is an isomorphism between the given function spaces. Then the existence of a solution of the nonlinear model follows from the inverse function theorem.

Few authors have studied questions of existence for nonlinear models of shells, largely because of the geometrically nonlinear behavior of shells. We mention, for example, Bernadou and Oden [4] for Koiter's models (one may find more references in this paper) and Figueiredo [16] for Donnell-Mushtari-Vlasov's models. The pseudo-monotone operators are the main tools in [4], and the implicit function theorem is used in [16].

Again the main advantage of the model described in this paper is that it is mathematically more tractable than currently available models which use local coordinate system and Christoffel symbols. It also offers definite technical advantages in dealing with theoretical issues since the geometry is easier to manage.

2 Preliminaries

We first recall the notation, definitions, and some technical results about the tangential differential calculus and the oriented distance functions. All results can be found in Delfour and Zolsio ([9] to [11]).

2.1 Notation

Let $\mathbb{R}^N$ be the N-dimensional Euclidean space for some integer $N \geq 1$. In practice N will be 3. The *set-theoretic boundary* of a set Ω in $\mathbb{R}^N$ will be denoted $\partial\Omega$. It is equal to $\overline{\Omega} \cap \overline{\complement\Omega}$, where $\complement\Omega = \{x \in \mathbb{R}^N \,:\, x \notin \Omega\}$ is the *complement* of Ω and $\overline{S}$ is the closure of a set S. A vector x in $\mathbb{R}^N$ will usually be assumed to be a column vector. Its transposed will be denoted ${}^*x = (x_1, \ldots, x_N)$. The inner product and norm will be denoted as

$$x \cdot y = \sum_{i=1}^{N} x_i \, y_i \quad |x| = \sqrt{x \cdot x}$$

for ${}^*x = (x_1, \ldots, x_N)$ and ${}^*y = (y_1, \ldots, y_N)$. The identity matrix in $\mathbb{R}^N$ will be written I. The transpose of a tensor (or an N by N matrix) M in $\mathcal{M}^N$ will be denoted ${}^*M$, that is $({}^*Mx) \cdot y = x \cdot (My)$. The same notation will be used for the transpose of linear operators. Note that for two vectors x and y

$${}^*x \, y = x \cdot y \quad \text{and} \quad (x \, {}^*y)_{ij} = x_i \, y_j.$$

We also define the double contraction of two $N \times N$ tensors A and B (or two matrixes $A, B \in \mathcal{M}^N$) as follows

$$A \cdot\cdot B = \sum_{i,j=1}^{N} A_{ij} B_{ij}.$$

We have the following straightward relationships

$$A \cdot\cdot BC = A^*C \cdot\cdot B = {}^*BA \cdot\cdot C,$$

and

$$A \cdot\cdot \ell^* e = Ae \cdot \ell.$$

Given an open subset V of $\mathbb{R}^N$ and an integer $k \geq 1$, $C^k(V)$ will be the space of k-times continuously differentiable functions on V; for $0 < \lambda \leq 1$, $C^{k,\lambda}(V)$ will be the space of k-times differentiable functions on V whose partial derivatives up to order k are λ-Hlderian continuous.

2.2 Oriented boundary distance function

Associate with Ω in $\mathbb{R}^3$ the *oriented boundary distance function* which is defined as

$$b_\Omega(x) = d_\Omega(x) - d_{\complement\Omega}(x), \tag{1}$$

where d_A is the usual distance function to a subset A of $\mathbb{R}^3$. This function captures all the geometrical properties of the boundary $\partial\Omega$. For $k \geq 2$ a domain Ω has a C^k (resp. $C^{1,1}$) boundary $\partial\Omega$ if and only if in each point $X \in \Omega$ there exists a bounded open neighbourhood $N(X)$ of X such that $b_\Omega \in C^k(N(X))$ (resp. $C^{1,1}(N(X))$). In the sequel the submanifold $\partial\Omega$ of $\mathbb{R}^3$ will be denoted by Γ

At each point X of $\partial\Omega$, its gradient $\nabla b_\Omega(X)$ coincides with the unitary exterior normal field n to Γ. The trace of $D^2b(X)$ is the *mean curvature* (up to the multiplying factor 1/2 which is used to make the mean curvature of the unit ball in $\mathbb{R}^3$ equal to one)

$$H(X) = \mathrm{tr}\,(D^2b(X)) = \triangle b(X),$$

and the trace of its matrix of cofactor $M(D^2b(X))$ is the *total or Gaussian curvature*

$$K(X) = \mathrm{tr}\,(M(D^2b(X)).$$

D^2b, H, and K belong to $C^{k-2}(\Gamma)$ (resp. $L^\infty(\Gamma)$)

Since the domain Ω is fixed thoughtout this paper, from now on the function b_Ω is denoted by b. For each $X \in \Gamma$, the *projection mapping* $p : N(X) \to \Gamma$ is obtained directly from the function b as

$$p(x) = x - b(x)\nabla b(x). \tag{2}$$

This definition is independent of the choice of $N(X)$ and X. Its Jacobian matrix is given by

$$Dp(x) = I - b(x)D^2b(x) - \nabla b(x)^*\nabla b(x). \tag{3}$$

For $x \in N(X)$, the *linear projector* onto the tangential plane $T_{p(x)}\Gamma$ is given by

$$P(x) = I - \nabla b(x)^*\nabla b(x). \tag{4}$$

2.3 Definition of the shell

A shell is characterized by its mean surface ω and its thickness h. The *mean surface* ω of the shell is a bounded (relatively) open domain in the submanifold Γ of $\mathbb{R}^3$. When $\omega = \Gamma$ (hence compact), the shell has no boundary. When $\omega \neq \Gamma$, the relative boundary of ω in Γ will be denoted by γ and assumed to be bounded and uniformly Lipschizian in Γ.

Assume that ω is bounded and Γ is C^2 (resp. $C^{1,1}$). There exist $h > 0$ and a bounded neighbourhood $N_h(\omega)$ of ω of the form $\Gamma_h = \{x \in \mathbb{R}^3 : |b(x)| \leq h\}$ such that $b \in C^2(\Gamma_h)$. For that h, the *shell* is defined as the set

$$S_h \stackrel{def}{=} \{x \in \mathbb{R}^3 : p(x) \in \omega \text{ and } |b(x)| < h\}. \tag{5}$$

By assumption on ω the set S_h is a bounded open domain in $\mathbb{R}^3$ with Lipschizian boundary. When $\omega \neq \Gamma$, S_h has a *lateral boundary*

$$\Sigma_h = \{x \in \mathbb{R}^3 : p(x) \in \gamma \text{ and } |b(x)| < h\}, \tag{6}$$

which is a surface normal to the mean surface ω.

In practice the mean surface Γ is given first and the underlying assumption is the existence of an appropriate domain Ω with the above properties. It is important to keep in mind that we use the distance function $b = b_\Omega$ and not the distance function to ω.

2.4 Flow of the gradient of b and local coordinates

For a C^2 (resp. $C^{1,1}$ submanifold, $\nabla b \in C^1(S_h)$ (resp. $C^{0,1}(S_h)$) and we can consider the *flow mapping* $T_z = T_z(\nabla b)$ defined by $T_z(X) = x(z)$, where $x(z)$ is the solution of

$$\begin{cases} \frac{dx}{dz}(z) = \nabla b(x(z)), & |z + b(X)| \leq h, \\ x(0) = X, & |b(X)| \leq h. \end{cases}$$

It can be shown that $b(x(z)) = b(X) + z$ and that the solution is of the form

$$T_z(X) = X + z\nabla b(X), \quad \forall z\, |z + b(X)| \leq h$$

and the elements of the Jacobian matrix $DT_z(X) = I + z\, D^2 b(X)$ belong to $C^0(S_h)$ (resp. $L^\infty(S_h)$). It is a homeomorphism from ω onto $\omega_z = \{x \in \mathbb{R}^3 : p(x) \in \omega \text{ and } b(x) = z\}$. In particular it induces a *curvilinear coordinates system* $(X, z) \in \omega \times (-h, h)$. The points on the level set ω_z are given by $\{X + z\nabla b(X) : X \in \omega\}$ and for each $(X, z) \in \omega \times (-h, h)$

$$\nabla b(T_z(X) = \nabla b(X + z\nabla b(X)) = \nabla b(X).$$

We have the following identities and properties on ω

$$\begin{aligned} &p \circ T_z = p,\, b \circ T_z = z, \quad DT_z = I + zD^2 b, \\ &\frac{\partial}{\partial z} DT_z = D^2 b \circ T_z DT_z = D^2 b, \quad \frac{\partial}{\partial z} \det DT_z = \nabla b \circ T_z \det DT_z, \\ &D^2 b = D^2 b \circ T_z[I + zD^2 b], D^2 b \circ T_z = [I - zD^2 b \circ T_z]D^2 b, \\ &[I + zD^2 b]^{-1} = I - zD^2 b \circ T_z, \end{aligned}$$

and

$$j(z)(X) \stackrel{def}{=} \det DT_z(X) = 1 + z\, H(X) + z^2\, K(X).$$

2.5 Tangential differential operators

For any scalar function $w : \omega \to \mathbb{R}$, denote by $\nabla_\Gamma w$ the *tangential gradient*

$$\nabla_\Gamma w = \nabla W|_\omega - \frac{\partial W}{\partial n} n \tag{7}$$

defined in terms of an extension W of w to S_h. It can be shown that this definition is independent of the choice of the extension W and that $\nabla_\Gamma w(X)$ is the projection of ∇W onto the tangential plane $T_X\Gamma$ to Γ in X. It is easy to check that

$$\nabla(w \circ p) = Dp(\nabla_\Gamma w) \circ p = [I - bD^2 b]\nabla_\Gamma w \circ p$$

and that $\nabla(w \circ p)_{|\omega} = \nabla_\Gamma w$. The last two identities will play a crutial role in the tangential differential calculus. The *tangential Jacobian matrix* of a vector $v : \omega \to \mathbb{R}^3$ is defined through its transposed

$$^*D_\Gamma v = (\nabla_\Gamma v_1, \nabla_\Gamma v_2, \nabla_\Gamma v_3) \tag{8}$$

in terms of the column tangential gradients. In particular

$$\begin{aligned} ^*D(v \circ p) &= (\nabla(v_1 \circ p), \nabla(v_2 \circ p), \nabla(v_3 \circ p)) \\ &= Dp(^*D_\Gamma v) \circ p = [I - bD^2 b](^*D_\Gamma v) \circ p, \end{aligned}$$

and

$$D(v \circ p) = (D_\Gamma v) \circ pDp = (D_\Gamma v) \circ p[I - bD^2 b]$$

and $D(v \circ p)_{|\omega} = D_\Gamma v$.

In the same way we define the *tangential divergence* as

$$\mathrm{div}_\Gamma v \stackrel{def}{=} \mathrm{tr}\, D_\Gamma(v) \ \text{ on } \ \omega \tag{9}$$

or equivalently in term of an extension V of v to a neighbourhood of Γ

$$\mathrm{div}_\Gamma v \stackrel{def}{=} \mathrm{div} V|\omega - DVn \cdot n.$$

It is easy to verify that

$$\mathrm{div}(v \circ p) = \mathrm{div}_\Gamma v \circ p - b\,\mathrm{tr}\,[D_\Gamma v \circ pD^2 b]$$

and that $\mathrm{div}(v \circ p)_{|\omega} = \mathrm{div}_\Gamma v$. Similarly the *tangential strain tensor* is defined as

$$\varepsilon_\Gamma(v) \stackrel{def}{=} \frac{1}{2}(D_\Gamma v + {}^*D_\Gamma v)$$

and $\varepsilon_\Gamma(v) = \varepsilon(v \circ p)_{|\omega}$, where $\varepsilon(V)$ is the usual strain tensor of a vector V in $\mathbb{R}^3$

$$\varepsilon(V) \stackrel{def}{=} \frac{1}{2}(D(V) + {}^*D(V)).$$

We shall also use the *tangential vectorial divergence* of a matrix or tensor A which is defined as

$$(\overrightarrow{\mathrm{div}_\Gamma}\ A)_i = \mathrm{div}_\Gamma A_{i,\cdot}$$

The technique extends to second order derivatives and general tensors. For instance for a scalar function $w : \omega \to \mathbb{R}$

$$\begin{aligned} D^2(v \circ p) &= D(\nabla(v \circ p)) \\ &= D(\nabla_\Gamma w \circ p) - bD(D^2 b \nabla_\Gamma w \circ p) - D^2 b \nabla_\Gamma w \circ p\, {}^*\nabla b, \end{aligned}$$

$$\begin{aligned} D^2(v \circ p) &= {}^*D(\nabla(v \circ p)) \\ &= {}^*D(\nabla_\Gamma w \circ p) - b\, {}^*D(D^2 b \nabla_\Gamma w \circ p) - \nabla b\, {}^*(D^2 b \nabla_\Gamma w \circ p). \end{aligned}$$

By taking the restriction to ω, we obtain the noncommutativity of the second order tangential derivatives which differ by a first order term

$$D_\Gamma(\nabla_\Gamma w) - D^2 b \nabla_\Gamma w\, {}^*\nabla b = D^2(v \circ p)_{|\omega} = {}^*D_\Gamma(\nabla_\Gamma w) - \nabla b\, {}^*(D^2 b \nabla_\Gamma w).$$

Therefore ${}^*[D_\Gamma(\nabla_\Gamma w)] = {}^*D_\Gamma(\nabla_\Gamma w)$ and it is natural to define the *matrix of second order tangential derivative* as

$$D^2_\Gamma(w) \stackrel{def}{=} D_\Gamma(\nabla_\Gamma w) \quad \Rightarrow \quad {}^*D^2_\Gamma(w) = {}^*D_\Gamma(\nabla_\Gamma w)$$

and

$$D^2_\Gamma(w) - {}^*D^2_\Gamma(w) = (D^2 b \nabla_\Gamma w)\, {}^*\nabla b - \nabla b\, {}^*(D^2 b \nabla_\Gamma w).$$

2.6 Integration by parts on ω

For simplicity we assume that γ is the finite union of C^1 closed curves in $\mathbb{R}^3$. First for a scalar function $f : \omega \to \mathbb{R}$ and a vector function $g : \omega \to \mathbb{R}^3$ of appropriate smoothness, for example, $f \in H^1(\omega)$ and $g \in H^1(\omega)^3$, we have the following result,

$$\int_\omega \nabla_\Gamma \cdot g + f \mathrm{div} g \, d\Gamma = \int_\omega H f g \cdot n \, d\Gamma + \int_\gamma f g \cdot \nu \, ds \tag{10}$$

where $\cdot$ denotes the inner product in $\mathbb{R}^3$, ν is the exterior unit normal to γ tangent to Γ and orthogonal to $n = \nabla b$, $H = \triangle b$ (cf. for instance [25]). We have the following vectorial version of formula (10).

Lemma 1. *For any smooth enough functions, we have $e : \omega \to \mathbb{R}^3$ and $G : \omega \to \mathbb{R}^{3\times 3}$,*

$$\int_\omega D_\Gamma e \cdot\cdot G + e \cdot \overrightarrow{div_\Gamma}\ G \, d\Gamma = \int_\omega H\, e \cdot G\, n \, d\Gamma + \int_\gamma e \cdot G\, \nu \, ds \tag{11}$$

and

$$\begin{aligned} &\int_\omega \varepsilon_\Gamma(e) \cdot\cdot G + e \cdot \overrightarrow{div_\Gamma}\ \frac{G + {}^*G}{2} \, d\Gamma \\ &= \int_\omega H \frac{G + {}^*G}{2}\, e \cdot n \, d\Gamma + \int_\gamma \frac{G + {}^*G}{2} e \cdot \nu \, ds. \end{aligned} \tag{12}$$

3 Linear theory

In [11] Delfour and Zolsio studied the Natural Theory and Love-Kirchhoff theory for thin/shallow shells preserving the rigid displacements. They obtain the linear models for the N-dimensional shells and arbitrary boundary conditions. Here we just recall the linear models of a three-dimensional thin/shallow shell with Dirichlet boundary conditions, i.e., the clamped boundary conditions.

3.1 Linear model of Delfour and Zolsio

The basic mechanical assumption in the natural theory of thin/shallow plates and shells is that the three-dimensional *displacement vector* is approximated by a first order development with respect to the local normal coordinate to the mean surface (cf. for instance Germain [14], Dautray and Lions [7]). Delfour and Zolsio [11] observed that this is equivalent to a first order development with respect to $b(x)$ which is the oriented normal distance to the submanifold Γ.

Assumption 1. At each point x of the shell S_h the *approximate displacement vector* $V(x)$ is of the form

$$V(x) = e \circ p(x) + b(x)\, \ell \circ p(x), \quad x \in S_h, \tag{13}$$

for vector-valued mappings e and ℓ from ω to $\mathbb{R}^3$

The following assumption is a mechanical one, and we will discuss it in the Love-Kirchhoff theory. In this paper, we mainly work on Assumption 1 unless we point out. But we will give results related with Assumption 1^t.

Assumption 1^t. *In addition to Assumption 1, assume that $\ell(X)$ is a tangential vector, that is $\ell(X)$ belongs to the tangent space $T_X\omega$ at X on ω or equivalently*

$$\ell(X) \cdot \nabla b(X) = 0 \qquad \text{on } \omega. \tag{14}$$

where $\cdot$ denotes the inner product in $\mathbb{R}^3$.

In curvilinear coordinates

$$V \circ T_z(x) = e(x) + z\, \ell(x), \quad x \in \omega,\ |z| < h. \tag{15}$$

With the help of the tangential calculus of section 2.5 the Jacobian matrix DV in S_h is given by

$$DV = [D_\Gamma e \circ p + b\, D_\Gamma \ell \circ p + \ell \circ p^* \nabla b][I - bD^2 b].$$

and the *strain tensor* $\varepsilon(V)$ by

$$\begin{aligned} 2\varepsilon(V) &= D(V) +^* D(V) \\ &= [D_\Gamma e \circ p + bD_\Gamma \ell \circ p + \ell \circ p^* \nabla b]\,[I - bD^2 b] \\ &\quad + [I - bD^2 b]^*[D_\Gamma e \circ p + bD_\Gamma \ell \circ p + \ell \circ p^* \nabla b]. \end{aligned}$$

In the (X, z) coordinates system the above expression becomes

$$DV \circ T_z = [D_\Gamma e + z D_\Gamma \ell + \ell^* \nabla b][I + z D^2 b]^{-1},$$

and

$$\begin{aligned} 2\varepsilon(V) \circ T_z &= [D_\Gamma e + z\, D_\Gamma \ell + \ell^* \nabla b][I + z\, D^2 b]^{-1} \\ &\quad + [I + z\, D^2 b]^{-1*}[D_\Gamma e + z\, D_\Gamma \ell + \ell^* \nabla b]. \end{aligned}$$

If there exists β, $0 \leq \beta < 1$, such that for all z, $|z| \leq h$, and $x \in \omega$ $\|z\, D^2 b(x)\| \leq \beta$, $[I + z\, D^2 b]^{-1}$ and $\varepsilon(V) \circ T_z$ can be written as an infinite sum and

$$\varepsilon(V) \circ T_z = \sum_{i=0}^{\infty} \varepsilon^i(e, \ell)\, z^i$$

where for 0, 1, 2 and $i > 2$

$$\begin{aligned} 2\varepsilon^0(e, \ell) &= D_\Gamma e + \ell^* \nabla b + {}^* D_\Gamma e + \nabla b^* \ell, \\ 2\varepsilon^1(e, \ell) &= D_\Gamma \ell - D_\Gamma e\, D^2 b + {}^* D_\Gamma \ell - D^2 b^* D_\Gamma e, \\ 2\varepsilon^2(e, \ell) &= (D_\Gamma \ell - D_\Gamma e\, D^2 b)(-D^2 b) + (-D^2 b)({}^* D_\Gamma \ell - D^2 b^* D_\Gamma e) \\ 2\varepsilon^i(e, \ell) &= (D_\Gamma \ell - D_\Gamma e\, D^2 b)\, (-D^2 b)^{i-1} + (-D^2 b)^{i-1}\, ({}^* D_\Gamma \ell - D^2 b^* D_\Gamma e). \end{aligned}$$

The tensor ε^0 is related to the *membrane stresses*, ε^1 to the *bending stresses* and ε^2 to the *shearing stresses*. From the Theorem 3.1 in [11], we know that $\varepsilon(V) \circ T_z = 0$ is equivalent to $\varepsilon^0 = \varepsilon^1 = \varepsilon^2 = 0$. So $\varepsilon^0 + \varepsilon^1 z + \varepsilon^2 z^2$ is a good approximation of $\varepsilon(V)$ near the origin which preserves the rigid displacements. Following this point, the following assumption is given by by Delfour and Zolsio [11].

Assumption 2. There exists β, $0 \leq \beta < 1$, such that

$$\forall z,\ |z| \leq h,\ \text{and}\ \forall x \in \omega, \quad \|z\, D^2 b(x)\| \leq \beta \tag{16}$$

and the *approximate strain tensor* is chosen as

$$\tilde{\varepsilon}(V) \circ T_z \stackrel{def}{=} \varepsilon^0 + \varepsilon^1 z + \varepsilon^2 z^2. \tag{17}$$

This is a mathematical assumption which says that the dimensionless quantity $\|zD^2 b\|$ is small compared to one or equivalently that the shell is either thin (h small) or shallow ($\|D^2 b\|$ small) or both.

The associated natural function spaces are

$$\begin{aligned} \mathcal{H}(\omega) &= L^2(\omega)^3 \times L^2(\omega)^3, \\ \mathcal{V}(\omega) &= \{(e, \ell) \in \mathcal{H}(\omega)\ :\ \varepsilon^i \in L^2(\omega)^{3\times 3},\ 0 \leq i \leq 2\} \end{aligned}$$

and from a generalized Korn's inequality (cf. Theorem 7.2 in [11])

$$\mathcal{V}(\omega) = H^1(\omega)^3 \times H^1(\omega)^3,$$

where

$$H^1(\omega) = \{w \in L^2(\omega) \,:\, \nabla_\Gamma w \in L^2(\omega)^3\}$$

Under the following *rheological law* for isotropic materials

$$\sigma = \lambda \,\mathrm{tr}\, \tilde{\varepsilon}(V) I + 2\mu\, \tilde{\varepsilon}(V),$$

where $\lambda \geq 0$ and $\mu > 0$ are the Lam's coefficients. If we assume that the mean surface ω of the shell has a boundary γ and Dirichlet boundary conditions we get the following variational equation which characterizes the stationary point of the sum of the strain energy and the work of the external forces and torques: there exists $(e, \ell) \in \mathcal{V}_0(\omega)$ such that for all $(\bar{e}, \bar{\ell}) \in \mathcal{V}_0(\omega)$

$$< A(e,\ell), (\bar{e}, \bar{\ell}) >_{\mathcal{V}} + < \mathcal{B}(f,m)(\bar{e}, \bar{\ell}) >_{\mathcal{H}} = 0 \tag{18}$$

where f and m belong to $L^2(\omega)^3$ with $m \cdot n = 0$ in ω,

$$\mathcal{V}_0(\omega) = H_0^1(\omega)^3 \times H_0^1(\omega)^3,$$

$$< A(e,\ell), (\bar{e}, \bar{\ell}) >_{\mathcal{V}} = \sum_{m=0}^{4} \int_\omega \alpha_m a_m^L((e,\ell), (\bar{e}, \bar{\ell}))\, d\Gamma,$$

$$< \mathcal{B}(f,m), (\bar{e}, \bar{\ell}) >_{\mathcal{H}} = \int_\omega \alpha_0 [f \cdot \bar{e} + m \cdot \bar{\ell}] + \alpha_1 f \cdot \bar{\ell}\, d\Gamma,$$

$$\alpha_m(X) = \int_{-h}^{h} j(z)(X)\, z^m\, dz, \quad X \in \omega,\, 0 \leq m \leq 4$$

$$\alpha_0 = 2h + \frac{2}{3}h^3 K, \alpha_1 = \frac{2}{3}h^3 H, \alpha_2 = \frac{2}{3}h^3 + \frac{2}{5}h^5 K, \alpha_3 = \frac{2}{5}h^5 H, \alpha_4 = \frac{2}{5}h^5 + \frac{2}{7}h^7 K.$$

The bilinear symmetrical forms appearing in the definition of A are

$$\begin{aligned}
a_0^L((e,\ell), (\bar{e}, \bar{\ell})) &= 2\mu \varepsilon^0 \cdot\cdot \bar{\varepsilon^0} + \lambda \mathrm{tr}\, \varepsilon^0 \mathrm{tr}\, \bar{\varepsilon^0}, \\
a_1^L((e,\ell), (\bar{e}, \bar{\ell})) &= 2\mu [\varepsilon^0 \cdot\cdot \bar{\varepsilon^1} + \bar{\varepsilon^0} \cdot\cdot \varepsilon^1] + \lambda [tr \varepsilon^0 \mathrm{tr}\, \bar{\varepsilon^1} + \mathrm{tr}\, \varepsilon^1 \mathrm{tr}\, \bar{\varepsilon^0}], \\
a_2^L((e,\ell), (\bar{e}, \bar{\ell})) &= 2\mu [\varepsilon^0 \cdot\cdot \bar{\varepsilon^2} + \bar{\varepsilon^0} \cdot\cdot \varepsilon^2 + \varepsilon^1 \cdot\cdot \bar{\varepsilon^1}] \\
&\quad + \lambda [\mathrm{tr}\, \varepsilon^0 \mathrm{tr}\, \bar{\varepsilon^2} + \mathrm{tr}\, \varepsilon^1 \mathrm{tr}\, \bar{\varepsilon^1} + \mathrm{tr}\, \varepsilon^2 \mathrm{tr}\, \bar{\varepsilon^0}], \\
a_3^L((e,\ell), (\bar{e}, \bar{\ell})) &= 2\mu [\varepsilon^1 \cdot\cdot \bar{\varepsilon^1} + \bar{\varepsilon^1} \cdot\cdot \varepsilon^1] + \lambda [\mathrm{tr}\, \varepsilon^1 \mathrm{tr}\, \bar{\varepsilon^1} + \mathrm{tr}\, \varepsilon^1 \mathrm{tr}\, \bar{\varepsilon^1}], \\
a_4^L((e,\ell), (\bar{e}, \bar{\ell})) &= 2\mu \varepsilon^2 \cdot\cdot \bar{\varepsilon^2} + \lambda \mathrm{tr}\, \varepsilon^2 \mathrm{tr}\, \bar{\varepsilon^2},
\end{aligned}$$

where $\varepsilon^i = \varepsilon^i(e,\ell)$ and $\bar{\varepsilon^i} = \varepsilon^i(\bar{e}, \bar{\ell}), 0 \leq i \leq 2$. For $0 \leq i \leq 2$, let

$$A^i(e,\ell) = 2\mu\, E^i(e,\ell) + \lambda\, \mathrm{tr}\, E^i(e,\ell)\, I$$

$$E^i(e,\ell) = \alpha_i \varepsilon^0(e,\ell) + \alpha_{i+1} \varepsilon^1(e,\ell) + \alpha_{i+2} \varepsilon^2(e,\ell) = \sum_{j=0}^{2} \alpha_{i+j}\, \varepsilon^j(e,\ell).$$

Therefore

$$< A(e,\ell), (\bar{e}, \bar{\ell}) >_{\mathcal{V}} = \sum_{i=0}^{2} \int_\omega A^i \cdot\cdot \bar{\varepsilon}^i\, d\Gamma = \int_\omega L_1 \cdot\cdot D_\Gamma \bar{e} + L_2 \cdot\cdot D_\Gamma \bar{\ell} + L_3 \cdot \bar{\ell}\, d\Gamma$$

where

$$L_1 = A^0 - A^1 D^2 b + A^2 (D^2 b)^2 \quad L_2 = A^1 - A^2 D^2 b \quad L_3 = A^0 \nabla b.$$

Then by using the integration by parts formula, we get

$$\begin{aligned} &< A(e,\ell), (\bar{e},\bar{\ell}) >_{\mathcal{V}} \\ &= \int_\omega \{ H L_1 \nabla b - \overrightarrow{\operatorname{div}_\Gamma}\, L_1 \} \cdot \bar{e} + \{ H L_2 \nabla b + \overrightarrow{\operatorname{div}_\Gamma}\, L_2 + L_3 \} \cdot \bar{\ell}\, d\Gamma \\ &\quad + \int_\gamma L_1 \nu \cdot \bar{e} + L_2 \nu \cdot \bar{\ell}\, ds \end{aligned} \tag{19}$$

and this leads to the strong form of (18)

$$\begin{cases} -\overrightarrow{\operatorname{div}_\Gamma}\, L_1 + H L_1 \nabla b = -\alpha_0 f & \text{on } \omega \\ -\overrightarrow{\operatorname{div}_\Gamma}\, L_2 + H L_2 \nabla b + L_3 = -\alpha_0 m - \alpha_1 f & \text{on } \omega \end{cases} \tag{20}$$

$$e = \ell = 0 \quad \text{on } \gamma. \tag{21}$$

It is convenient to rewrite the system (20) and (21) in the form: to find vector fields $(e,\ell) : \omega \to \mathbb{R}^3 \times \mathbb{R}^3$ that verify

$$\begin{cases} \mathcal{L}(e,\ell) = \vec{G} & \text{in } \omega \\ (e,\ell) = 0 & \text{on } \gamma \end{cases} \tag{22}$$

where

$$\mathcal{L}(e,\ell) \stackrel{def}{=} (-\overrightarrow{\operatorname{div}_\Gamma}\, L_1 + H L_1 \nabla b, -\overrightarrow{\operatorname{div}_\Gamma}\, L_2 + H L_2 \nabla b + L_3)$$

and

$$\vec{G} \stackrel{def}{=} (-\alpha_0 f, -\alpha_0 m - \alpha_1 f).$$

The following theorem is Theorem 4.3 in [11].

Theorem 1. *There exists $\bar{h} > 0$, such that for all h, $0 < h < \bar{h}$, there exists a unique weak solution $(e,\ell) \in \mathcal{V}_0(\omega)$ to the boundary problem* (20) *and* (21)*.*

Remark 1. *The condition $\ell \cdot \nabla b = 0$ in Assumption 1^t is a mechanical one. It is equivalent to the condition of tangential normal strain $\varepsilon n \cdot n = 0$. We may add this condition, and get the same result in the corrsponding spaces*

$$\begin{aligned} \mathcal{H}(\omega)^t &= \{(e,\ell) \in L^2(\omega)^3 \times L^2(\omega)^3 \ : \ \ell \cdot n = 0 \ \textit{in } \omega\} \\ \mathcal{V}(\omega)^t &= \{(e,l) \in H^1(\omega)^3 \times H^1(\omega)^3 \ : \ \ell \cdot n = 0 \ \textit{on } \omega\} \\ \mathcal{V}_0(\omega)^t &= \{(e,\ell) \in H_0^1(\omega)^3 \times H_0^1(\omega)^3 \ : \ \ell \cdot n = 0 \ \textit{on } \omega\}, \end{aligned}$$

since the proof of the generalized Korn's inequality [11] does not use the fact that ℓ is tangent. In fact, if we use $\ell - \ell \cdot \nabla b \nabla b$ as a test function in (19)*, the*

following linear equations of the shell with the homogeneous Dirichlet boundary condition are given

$$\begin{cases} -\overrightarrow{div}_\Gamma\, L_1 + HL_1\nabla b = -\alpha_0 f \qquad on\ \omega \\ -\overrightarrow{div}_\Gamma\, L_2 + \overrightarrow{div}_\Gamma\, L_2\cdot\nabla b\,\nabla b + HL_2\nabla b - HL_2\nabla b\cdot\nabla b\,\nabla b \\ \qquad +L_3 - L_3\cdot\nabla b\,\nabla b = -\alpha_0 m - \alpha_1(f - f\cdot\nabla b\,\nabla b) \qquad on\ \omega \end{cases}$$

$$e = \ell = 0 \qquad on\ \gamma.$$

The operators related to the above system are defined as following.

$$\begin{aligned} \mathcal{L}^t(e,l) \overset{def}{=} \ & (-\overrightarrow{div}_\Gamma\, L_1 + HL_1\nabla b, \\ & -\overrightarrow{div}_\Gamma\, L_2 + \overrightarrow{div}_\Gamma\, L_2\cdot\nabla b\,\nabla b + HL_2\nabla b \\ & \qquad -HL_2\nabla b\cdot\nabla b\,\nabla b + L_3 - L_3\cdot\nabla b\,\nabla b), \end{aligned}$$

and

$$\vec{G}^t \overset{def}{=} (-\alpha_0 f, -\alpha_0 m - \alpha_1(f - f\cdot\nabla b\,\nabla b)).$$

Moreover, by adding some appropriate conditions on the stress or strain tensor, we can also discuss the Love-Kirchhoff theory, and Naghdi's and Koiter's models. We also mention that all the results in this paper for the nonlinear theory are true under the condition $\ell\cdot\nabla b = 0$.

3.2 Smoothness of solutions

We now prove regularity property of the weak solutions of (20) and (21). The key tools here are Agmon, Douglis and Nirenberg's classical results [2] about elliptic differential systems, and the index of elliptic operators by Geymonat [17]. There are few results about regularity property of the weak solutions for linear models of shells. We mention that, for example, H^2 regularity property of the weak solutions for linear Donnell-Mushtari-Vlasov's models is given by Figueiredo [16] by using Guseva's results [20].

It will be useful to introduce the following subspaces for an arbitrary $p \geq 1$

$$\mathcal{W}^p(\omega) = \{(e,\ell) \in W^{2,p}(\omega)^3 \times W^{2,p}(\omega)^3 \,:\, (e,\ell) \in \mathcal{V}_0(\omega)\}$$
$$\mathcal{L}^p(\omega) = L^p(\omega)^3 \times L^p(\omega)^3.$$

Theorem 2. *Let Ω be a domain in $\mathbb{R}^3$ with a boundary Γ which is a $C^{2,1}$ submanifold of $\mathbb{R}^3$, let ω be a C^2 domain in the submanifold Γ, and $f, m \in L^p(\omega)^3$, $p \geq 2$. Then the weak solution $(e,l) \in \mathcal{V}_0(\omega)$ of equations (20) belongs to the space $\mathcal{W}^p(\omega)$, and satisfies (20) in $L^p(\omega)^3 \times L^p(\omega)^3$.*

Proof. (i) First of all, since Γ is a $C^{2,1}$ submanifold of $\mathbb{R}^3$, b, ∇b, and $D^2 b$ respectively belong to $C^{2,1}(S_h)$, $C^{1,1}(S_h)$, and $C^{0,1}(S_h)$. Moreover since the coefficients of $D^2 b$ are uniformly Lipschitz continuous in S_h, their partial derivatives

exist almost everywhere and belong to $L^\infty(S_h)$ and $L^\infty(\omega)$. So the coefficients in equation (18) are at least $C^{0,1}(\omega)$, and those of equations (20)-(21) are at least $L^\infty(\omega)$, but the coefficients of the second order derivatives are $C^{0,1}$. From Delfour and Zolsio's Theorem 4.3 and Lemma 4.2 in [12], the bilinear form associated with the linear shell equations is strongly coercive, and then it is strongly elliptic from the result in Morrey [24] (cf. also Fichera [15]). Therefore since, in addition, ω is a C^2 domain in the submanifold Γ from classical results on linear elliptic differential systems (cf. Agmon, Douglis and Nirenberg [2] and Guseva [20])

$$(f,m) \in L^2(\omega)^3 \times L^2(\omega)^3 \Rightarrow (e,\ell) \in [H^2(\omega) \cap H_0^1(\omega)]^3 \times [H^2(\omega) \cap H_0^1(\omega)]^3.$$

Hence the results holds for $p = 2$.
(ii) Next we use the result in Geymonat [17] to complete the proof of the theorem. First of all, we are to verify the conditions in Geymonat ([17], Theorem 3.5). Condition (I), strong ellipticity of the system, is satisfied from the first part of the proof. The assumption (II), proper ellipticity, is verified from the results in Morrey [24]. The hypothesis (III), the complementing boundary condition, is satisfied from the results in Agmon, Douglis and Nirenberg [2] since the boundary conditions are of Dirichlet type. The smoothness of the boundary γ (V; α), the smoothness of the coefficients of the equations (V; α) and the smoothness conditions on the coefficients of the boundary operators (VI; α) are satisfied from the assumptions of the theorem. It follows from Theorem 3.5 in Geymonat [17] that the mapping $\mathcal{L} : \mathcal{W}^p(\omega) \to \mathcal{L}^p(\omega)$ has an index ind$\mathcal{L}$ which is independent of $p \in (1,\infty)$. We recall that

$$\text{ind}\mathcal{L} = \dim \text{Ker}\mathcal{L} - \dim \text{Coker}\mathcal{L},$$

where $\dim \text{Coker}\mathcal{L}$ is the quotient space of the space $\mathcal{L}^p(\omega)$ by the space Im$\mathcal{L}$. Here, we know by the first part that ind$\mathcal{L} = 0$ for $p = 2$ since $\mathcal{L}$ is a bijection.
(iii) Since $\mathcal{W}^p(\omega) \hookrightarrow \mathcal{V}_0(\omega)$ for $p \geq 2$, the mapping $\mathcal{L} : \mathcal{W}^p(\omega) \to \mathcal{L}^p(\omega)$ is injective for these value of p because the weak solution is unique in the space $\mathcal{V}_0(\omega)$ if $(m,f) \in \mathcal{L}^p(\omega)$. Hence dim Ker$\mathcal{L} = 0$.
On the other hand, since ind $\mathcal{L} = 0$, $\dim \text{Coker}\mathcal{L} = 0$, therefore the mapping $\mathcal{L}$ is surjective in this case. Hence the regularity result holds for $p \geq 2$. Moreover, the weak solution $(e,\ell) \in [W^{2,p}(\omega)^3 \times W^{2,p}(\omega)^3] \cap \mathcal{V}_0(\omega)$ satisfies the variational equation

$$< A(e,\ell), (\bar{e},\bar{\ell}) >_{\mathcal{V}} + < \mathcal{B}(f,m), (\bar{e},\bar{\ell}) >_{\mathcal{H}} = 0.$$

for all $(\bar{e},\bar{\ell}) \in \mathcal{V}_0(\omega)$. Hence we can apply the integration by parts formula on ω. This gives

$$\int_\omega \{HL_1\nabla b - \overrightarrow{\text{div}}_\Gamma\, L_1\} \cdot \bar{e} + \{HL_2\nabla b - \overrightarrow{\text{div}}_\Gamma\, L_2 + L_3\} \cdot \bar{\ell}\, d\Gamma$$
$$= -\int_\Gamma \alpha_0(f \cdot \bar{e} + m \cdot \bar{\ell}) + \alpha_1 f \cdot \bar{\ell}\, d\Gamma$$

and the conclusion follows. □

If $(m, f) \in \mathcal{L}^p(\omega)$, the mapping

$$(m, f) \longrightarrow \vec{G} = (-\alpha_0 f, -\alpha_0 m - \alpha_1 f) \in \mathcal{L}^p(\omega)$$

is one-to-one. Therefore, $\vec{G} \in \mathcal{L}^p(\omega)$ from the definition of $\vec{G}$. By the last theorem, we get that the problem given in term of the operator $\mathcal{L}$

$$\mathcal{L}(e, \ell) = \vec{G}$$

has one and only one solution $(e, \ell) \in \mathcal{W}^p(\omega)$ for any $(m, f) \in \mathcal{L}^p(\omega)$.

3.3 Love-Kirchhoff Theory, Naghdi's and Koiter's models, and associated dynamical models

In [11] the Love-Kirchhoff Theory was obtained from the natural theory by observing that

$$\boxed{\tilde{\varepsilon}\, n = 0 \text{ in } S_h} \iff \boxed{\ell = D^2 b\, e - \nabla_\Gamma (e \cdot n) \text{ in } \omega} \tag{23}$$

are two equivalent formulations of the Love-Kirchhoff condition. Specifically in curvilinear coordinates

$$2\, [\tilde{\varepsilon}(e, \ell) \circ T_z]\, \nabla b = [I - z\, D^2 b + z^2\, (D^2 b)^2]\, [\ell + 2\, \varepsilon_\Gamma(e)\, \nabla b] \tag{24}$$

and

$$\ell + 2\, \varepsilon_\Gamma(e)\, \nabla b = \ell + \nabla_\Gamma (e \cdot n) - D^2 b\, e.$$

Under Assumption 2, $z\, \|D^2 b\| \ll 1$, we get the equivalence. Therefore the Love-Kirchhoff Theory is readily obtained from the Natural Theory by including the condition in the definition of the spaces of solutions

$$\mathcal{H}_{LK} = \{(e, \ell) \in \mathcal{H} \,:\, \ell + 2\, \varepsilon_\Gamma(e)\, \nabla b = 0 \text{ in } \omega\} \tag{25}$$

$$\mathcal{V}_{LK} = \{(e, \ell) \in \mathcal{V} \,:\, \ell + 2\, \varepsilon_\Gamma(e)\, \nabla b = 0 \text{ in } \omega\}. \tag{26}$$

In [12] detailed computations were made to express Naghdi's and Koiter's models in terms of intrinsic tangential differential operators. The basic difference between the two models is the previous Love-Kirchhhoff condition. The common additional assumption in both models is

$$\boxed{\sigma\, n \cdot n = 0 \text{ in } S_h.}$$

For isotropic materials with a rheological law of the form

$$\sigma = 2\, \mu\, \tilde{\varepsilon} + \lambda\, \mathrm{tr}\, \tilde{\varepsilon}\, I$$

this condition yields

$$\sigma\, n \cdot n = 2\, \mu\, \tilde{\varepsilon}\, n \cdot n + \lambda\, \mathrm{tr}\, \tilde{\varepsilon} = 0. \tag{27}$$

This last identity is often used to compute $\tilde{\varepsilon}\, n \cdot n$ in Naghdi's and Koiter's models. The fact that $\tilde{\varepsilon}\, n \cdot n$ is not zero in those linear models is referred to as an "occasional contradiction". However in the model of Delfour and Zolsio $\tilde{\varepsilon}\, n \cdot n$ is indeed equal to zero since the "geometry is not approximated". To see this go back to identity (24)

$$\begin{aligned} 2\,[\tilde{\varepsilon}(e,\ell)\circ T_z]\;\nabla b\cdot\nabla b &= [I - z\,D^2 b + z^2\,(D^2 b)^2]\,[\ell + 2\,\varepsilon_\Gamma(e)\,\nabla b]\cdot\nabla b \\ &= [\ell + 2\,\varepsilon_\Gamma(e)\,\nabla b]\cdot\nabla b = 2\,\varepsilon_\Gamma(e)\,\nabla b\cdot\nabla b = 0. \end{aligned}$$

Therefore under Assumptions 1 and 2

$$\boxed{\tilde{\varepsilon}(e,\ell)\, n\cdot n = 0 \text{ in } S_h.} \tag{28}$$

The direct consequence of this property is that for $\lambda \neq 0$

$$\boxed{\operatorname{tr}\tilde{\varepsilon}(e,\ell) = 0 \text{ in } S_h} \tag{29}$$

or equivalently

$$\operatorname{tr}\varepsilon^0 = \operatorname{tr}\varepsilon^1 = \operatorname{tr}\varepsilon^2 = 0 \text{ in } \omega \tag{30}$$

or also

$$\operatorname{div}_\Gamma e = 0, \quad \operatorname{div}_\Gamma \ell - \operatorname{tr}(D_\Gamma e\, D^2 b), \quad \operatorname{tr}(D_\Gamma \ell\, D^2 b - D_\Gamma e\,(D^2 b)^2) = 0 \text{ in } \omega.$$

So again the three conditions (30) can be included in the definition of the spaces of solutions and the theory of section 3.2 applies to Naghdi's case with the function spaces

$$\mathcal{H}_N = \{(e,\ell)\in\mathcal{H} \,:\, \operatorname{tr}\varepsilon^0 = \operatorname{tr}\varepsilon^1 = \operatorname{tr}\varepsilon^2 = 0 \text{ in } \omega\} \tag{31}$$

$$\mathcal{V}_N = \{(e,\ell)\in\mathcal{V} \,:\, \operatorname{tr}\varepsilon^0 = \operatorname{tr}\varepsilon^1 = \operatorname{tr}\varepsilon^2 = 0 \text{ in } \omega\}. \tag{32}$$

and to Koiter's case with

$$\mathcal{H}_K = \{(e,\ell)\in\mathcal{H} \,:\, \begin{array}{l} \operatorname{tr}\varepsilon^0 = \operatorname{tr}\varepsilon^1 = \operatorname{tr}\varepsilon^2 = 0 \\ \ell + 2\,\varepsilon_\Gamma(e)\,\nabla b = 0 \end{array} \text{ in } \omega\} \tag{33}$$

$$\mathcal{V}_K = \{(e,\ell)\in\mathcal{V} \,:\, \begin{array}{l} \operatorname{tr}\varepsilon^0 = \operatorname{tr}\varepsilon^1 = \operatorname{tr}\varepsilon^2 = 0 \\ \ell + 2\,\varepsilon_\Gamma(e)\,\nabla b = 0 \end{array} \text{ in } \omega\}. \tag{34}$$

Of course the actuals models slightly differ in their form, but they incorporate the same set of assumptions.

This approach made it possible in [11] to deal simultaneouly with the dynamical versions of the Natural and the Love-Kirchhoff models. From the above remarks the same approach applies to the Naghdi's and Koiter's cases by identifying the dual of the respective spaces $\mathcal{H}_N$ and $\mathcal{H}_K$ with themselves.

4 Nonlinear equations

For simplicity we assume that the shell S_h has constant thickness, but almost everything can be applied to shells $S_{\tilde{h}}$ with variable thickness under suitable smoothness of the function $\tilde{h}$. For details in this direction see [11].

As in the linear case Assumptions 1 and an extension of Assumptions 2 will be the basic assumptions for the Natural Theory of nonlinear thin/shallow shells.

4.1 Strain tensor

Under Assumption 1, consider the *Green Saint Venant strain tensor* $\varepsilon(V)$ over S_h associated with the displacement field $V(x)$ (cf. Ciarlet [5], Marsden and Hughes [23]),

$$\begin{aligned} 2\varepsilon(V) &= DV + {}^*DV + {}^*DV\,DV \\ &= [D_\Gamma e \circ p + bD_\Gamma \ell \circ p + \ell \circ p^* \nabla b][I - bD^2 b] \\ &\quad + [I - bD^2 b]^*[D_\Gamma e \circ p + bD_\Gamma \ell \circ p + \ell \circ p^* \nabla b] \\ &\quad + [I - bD^2 b]^*[D_\Gamma e \circ p + bD_\Gamma \ell \circ p + \ell \circ p^* \nabla b] \\ &\qquad [D_\Gamma \ell \circ p + bD_\Gamma \ell \circ p + \ell \circ p^* \nabla b][I - bD^2 b] \end{aligned}$$

in S_h, where in terms of curvilinear coordinates (X, z),

$$DV \circ T_z = [D_\Gamma e + zD_\Gamma \ell + \ell^* \nabla b][I + zD^2 b]^{-1}, \tag{35}$$

and

$$[I + zD^2 b]^{-1} = I - zD^2 b \circ T_z.$$

Under the thin/shallow Assumption 2

$$[I + zD^2 b]^{-1} = \sum_{i=0}^{\infty} (-D^2 b)^i z^i, \tag{36}$$

and we get

$$DV \circ T_z = \sum_{i=0}^{\infty} v^i(e, \ell)\, z^i$$

where

$$v^0(e, \ell) = D_\Gamma e + \ell^* \nabla b \text{ and for } i \geq 1,\ v^i(e, \ell) = [D_\Gamma \ell - D_\Gamma e\, D^2 b](-D^2 b)^{i-1}.$$

With this we can also write the strain tensor as an infinite sum in terms of $v^i = v^i(e, \ell)$

$$\begin{aligned} 2\varepsilon(V) \circ T_z &= \sum_{i=0}^{\infty} ({}^*v^i + v^i)\, z^i + \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} {}^*v^i\, v^j\, z^{i+j} \\ &= \sum_{i=0}^{\infty} [{}^*v^i + v^i + \sum_{j=0}^{i} {}^*v^j\, v^{i-j}]\, z^i = \sum_{i=0}^{\infty} 2\,[\varepsilon^i + \eta^i]\, z^i \end{aligned}$$

where $\varepsilon^i = \varepsilon^i(e,\ell)$ and $\eta^i = \eta^i(e,\ell)$ are given by

$$2\,\varepsilon^i(e,\ell) \stackrel{def}{=} v^i(e,\ell) + {}^*v^i(e,\ell) \text{ and } 2\,\eta^i(e,\ell) \stackrel{def}{=} \sum_{j=0}^{i} {}^*v^j(e,\ell)\, v^{i-j}(e,\ell).$$

In particular

$$\begin{aligned}
2\eta^0(e,\ell) &= {}^*v^0\, v^0 = {}^*[D_\Gamma e + \ell^* \nabla b]\,[D_\Gamma e + \ell^* \nabla b] \\
2\eta^1(e,\ell) &= {}^*v^0\, v^1 + {}^*v^1\, v^0 \\
&= {}^*[D_\Gamma e + \ell^* \nabla b]\,[D_\Gamma \ell - D_\Gamma e\, D^2 b] \\
&\quad + {}^*[D_\Gamma \ell - D_\Gamma e\, D^2 b]\,[D_\Gamma e + \ell^* \nabla b] \\
2\eta^2(e,\ell) &= {}^*v^0\, v^2 + {}^*v^1\, v^1 + +{}^*v^2\, v^0 \\
&= -\,{}^*[D_\Gamma e + \ell^* \nabla b]\,[D_\Gamma \ell - D_\Gamma e\, D^2 b]\, D^2 b \\
&\quad + {}^*[D_\Gamma \ell - D_\Gamma e\, D^2 b]\,[D_\Gamma \ell - D_\Gamma e\, D^2 b] \\
&\quad - D^2 b\, {}^*[D_\Gamma \ell - D_\Gamma e\, D^2 b]\,[D_\Gamma e + \ell^* \nabla b].
\end{aligned}$$

We get

$$2\varepsilon(V) \circ T_z = \varepsilon^0 + \eta^0 + (\varepsilon^1 + \eta^1) z + (\varepsilon^2 + \eta^2) z^2 + O(z^3). \tag{37}$$

From a result by Delfour and Zolsio ([11], Theorem 3.1), we know that, for the linear part of the strain tensor $\varepsilon(V)$, $\varepsilon^0 + \varepsilon^1 z + \varepsilon^2 z^2$ is a good approximation near the origin which preserves the rigid displacements. Following this point of view here we introduce the following approximation of the strain tensor.

Assumption 3. There exists β, $0 \le \beta < 1$, such that

$$\forall z,\ |z| \le h,\ \text{ and } x \in \omega, \quad \|z\, D^2 b(x)\| \le \beta \tag{38}$$

and the *approximate Green Saint Venant strain tensor* is chosen as

$$\tilde{\varepsilon}(V) \circ T_z = \varepsilon^0 + \eta^0 + (\varepsilon^1 + \eta^1) z + (\varepsilon^2 + \eta^2) z^2. \tag{39}$$

Again this is a mathematical assumption which says that the dimensionless quantity $\|z D^2 b(X)\|$ is small or equivalently that the shell is either thin (h small) or shallow ($\|D^2 b(X)\|$ small) or both.

4.2 Strain energy and work of the external forces

We now follow the standard approach and compute the total energy made up of the strain energy plus the work of the external forces. External forces mean forces and torques applied to the mean surface of the shell. The shell equation corresponds to a stationary point of the total energy functional.

Define the *strain energy* $\mathcal{P}$ and the *work of the external forces* $\mathcal{W}$ as follows:

$$\mathcal{P} = \frac{1}{2} \int_{\Omega_h} \sigma \cdot\cdot\, \tilde{\varepsilon}(V)\, dx \tag{40}$$

and

$$\mathcal{W} = \int_{S_h} [F \cdot V + M \cdot (\ell \circ p)]\, dx, \tag{41}$$

where σ is the stress tensor, F is the loading force and M is the moment applied to the shell. Assume that F and M are smooth enough so that their traces are well-defined on the surface ω, Furthermore assume that S_h is made up of a Saint Venant-Kirchhoff material which obeys the following rheological law (cf. Marsden and Hughes [23] and Ciarlet [5]),

$$\sigma = \lambda \operatorname{tr} \tilde{\varepsilon}(V) I + 2\mu\, \tilde{\varepsilon}(V), \tag{42}$$

where $\lambda \geq 0$ and $\mu > 0$ are the Lam coefficients. Hence

$$\begin{aligned} \sigma \circ T_z \ \cdot\cdot \tilde{\varepsilon}(V) \circ T_z &= [\lambda \operatorname{tr} \tilde{\varepsilon}(V) \circ T_z I + 2\mu\, \tilde{\varepsilon}(V) \circ T_z] \ \cdot\cdot \tilde{\varepsilon}(V) \circ T_z \\ &= \lambda |\operatorname{tr} \ \tilde{\varepsilon}(V) \circ T_z|^2 + 2\mu\, \|\tilde{\varepsilon}(V) \circ T_z\|^2. \end{aligned}$$

Using Federer's decomposition of the measure (cf. Evans and Gariepy [13]) along the level curves of the oriented distance function b for which $|\nabla b(x)| = 1$ in S_h.

$$\begin{aligned} \mathcal{P} &= \frac{1}{2} \int_{\Omega_h} \sigma \ \cdot\cdot \tilde{\varepsilon}(V)\, dx \\ &= \frac{1}{2} \int_{\omega} \int_{-h}^{h} dz\, d\Gamma\, [\, \lambda\, |\operatorname{tr} \tilde{\varepsilon}(V) \circ T_z|^2 + 2\mu\, \|\tilde{\varepsilon}(V) \circ T_z\|^2]\, j(z). \end{aligned}$$

The integrand with respect to $d\Gamma$ in the expression for $\mathcal{P}$ generates the following symmetrical form

$$\begin{aligned} & p((e,\ell),(\bar{e},\bar{\ell})) \\ &= \int_{-h}^{h} dz\, j(z)\ [2\mu\, \tilde{\varepsilon}(V) \circ T_z \cdot\cdot \tilde{\varepsilon}(\overline{V}) \circ T_z + \lambda \operatorname{tr} \tilde{\varepsilon}(V) \circ T_z \operatorname{tr} \tilde{\varepsilon}(\overline{V}) \circ T_z] \\ &= \sum_{i=0}^{2} \sum_{j=0}^{2} [2\mu\, (\varepsilon^j + \eta^j) \cdot\cdot (\bar{\varepsilon}^i + \bar{\eta}^i) + \lambda \operatorname{tr} (\varepsilon^j + \eta^j) \operatorname{tr} (\bar{\varepsilon}^i + \bar{\eta}^i)] \int_{-h}^{h} j(z)\, z^{i+j}\, dz \\ &= \sum_{i=0}^{2} \sum_{j=0}^{2} [2\mu\, (\varepsilon^j + \eta^j) \cdot\cdot (\bar{\varepsilon}^i + \bar{\eta}^i) + \lambda \operatorname{tr} (\varepsilon^j + \eta^j) \operatorname{tr} (\bar{\varepsilon}^i + \bar{\eta}^i)]\, \alpha_{i+j}. \end{aligned}$$

Then since $\operatorname{tr} \kappa = \kappa \cdot\cdot I$

$$p((e,\ell),(\bar{e},\bar{\ell})) = \sum_{i=0}^{2} B^i(e,\ell) \cdot\cdot [\varepsilon^i(\bar{e},\bar{\ell}) + \eta^i(\bar{e},\bar{\ell})]$$

where

$$\begin{aligned} B^i(e,\ell) &= 2\mu\, F^i(e,\ell) + \lambda \operatorname{tr} F^i(e,\ell)\, I \\ F^i(e,\ell) &= \sum_{j=0}^{2} \alpha_{i+j} (\varepsilon^j(e,\ell) + \eta^j(e,\ell)) \end{aligned}$$

are symmetrical tensors. By construction

$$B^i(e,\ell)\cdot\cdot\,[\varepsilon^i(\bar{e},\bar{\ell})+\eta^i(\bar{e},\bar{\ell})] = B^i(\bar{e},\bar{\ell})\cdot\cdot\,[\varepsilon^i(e,\ell)+\eta^i(e,\ell)]$$

and

$$\mathcal{P} = \frac{1}{2}\int_\omega \sum_{i=0}^{2} B^i(e,\ell)\cdot\cdot\,[\varepsilon^i(e,\ell)+\eta^i(e,\ell)]\,d\Gamma.$$

Assuming that the force F and the moment M are of the forms

$$F = f\circ p \text{ and } M = m\circ p \quad \text{ in } S_h, \tag{43}$$

where $f, m \in L^2(\omega)^3$, and $m\cdot\nabla b = 0$. Recall that

$$V\circ T_z = e + z\ell$$

and that from [11] the work $\mathcal{W}$ of the external forces and torques becomes

$$\begin{aligned}\mathcal{W} &= \int_{-h}^{h} dz \int_\omega d\Gamma\,[F\circ T_z\cdot V\circ T_z + M\circ T_z\cdot(\ell\circ p\circ T_z)]\,j(z)\\ &= \int_\omega [\alpha_0\,(f\cdot e + m\cdot\ell) + \alpha_1\, f\cdot\ell]\,d\Gamma. \end{aligned}\tag{44}$$

Then the total energy is

$$\begin{aligned}\mathcal{E} &= \mathcal{P}+\mathcal{W}\\ &= \frac{1}{2}\int_\omega \sum_{i=0}^{2} B^i(e,\ell)\cdot\cdot\,[\varepsilon^i(e,\ell)+\eta^i(e,\ell)]\,d\Gamma + \int_\omega [\alpha_0(f\cdot e + m\cdot\ell) + \alpha_1 f\cdot\ell]\,d\Gamma.\end{aligned}$$

4.3 Nonlinear equations of the shell

The *variational principle* may now be used to obtain the equations of the displacement. A straightforward calculation of the directional derivative $d\mathcal{P}(e,\ell;\bar{e},\bar{\ell})$ at (e,ℓ) in the direction $(\bar{e},\bar{\ell})$ gives

$$d\mathcal{P}(e,\ell;\bar{e},\bar{\ell}) = \int_\omega \sum_{i=0}^{2} B^i(e,\ell)\cdot\cdot\,[\varepsilon^i(\bar{e},\bar{\ell}) + d\eta^i(e,\ell;\bar{e},\bar{\ell})]\,d\Gamma. \tag{45}$$

where

$$\begin{aligned}2\,d\eta^i(e,\ell;\bar{e},\bar{\ell}) &= \sum_{j=0}^{i} {}^*v^j\,\bar{v}^{i-j} + {}^*\bar{v}^j\,v^{i-j}\\ 2\,\varepsilon^i(\bar{e},\bar{\ell}) &= \bar{v}^i + {}^*\bar{v}^i.\end{aligned}$$

Therefore

$$\sum_{i=0}^{2} B^i\cdot\cdot\sum_{j=0}^{i} {}^*v^j\,\bar{v}^{i-j} + {}^*\bar{v}^j\,v^{i-j} = 2\sum_{i=0}^{2}\sum_{j=i}^{2} v^{j-i}\,B^j\cdot\cdot\,\bar{v}^i \tag{46}$$

and

$$d\mathcal{P}(e,\ell;\bar{e},\bar{\ell}) = \sum_{i=0}^{2}\int_{\omega}[B^i + \sum_{j=i}^{2} v^{j-i}\, B^j]\cdot\cdot\,\bar{v}^i\, d\Gamma = \sum_{i=0}^{2}\int_{\omega} M^i\cdot\cdot\,\bar{v}^i\, d\Gamma,$$

where

$$M^i = B^i + \sum_{j=i}^{2} v^{j-i}\, B^j.$$

In view of

$$\begin{aligned}
\bar{v}^0 &= D_\Gamma\bar{e} + \bar{\ell}^*\nabla b\\
\bar{v}^1 &= D_\Gamma\bar{\ell} - D_\Gamma\bar{e}\, D^2 b\\
\bar{v}^2 &= -[D_\Gamma\bar{\ell} - D_\Gamma\bar{e}\, D^2 b]\, D^2 b
\end{aligned}$$

we can rearrange again to isolate $\bar{\ell}$, $D_\Gamma\bar{e}$, and $D_\Gamma\bar{\ell}$

$$\sum_{i=0}^{2} M^i\cdot\cdot\,\bar{v}^i = N_1\cdot\cdot\, D_\Gamma\bar{e} + N_2\cdot\cdot\, D_\Gamma\bar{\ell} + N_3\cdot\bar{\ell}$$

where

$$\begin{aligned}
N_1 &= M^0 - M^1\, D^2 b + M^2\,(D^2 b)^2 = \sum_{i=0}^{2} M^i\,(-D^2 b)^i\\
N_2 &= M^1 - M^2\, D^2 b = \sum_{i=1}^{2} M^i\,(-D^2 b)^{i-1}\\
N_3 &= M^0\nabla b.
\end{aligned}$$

Finally

$$\begin{aligned}
N_1 &= \sum_{i=0}^{2}[B^i + \sum_{j=i}^{2} v^{j-i}\, B^j]\,(-D^2 b)^i\\
N_2 &= \sum_{i=1}^{2}[B^i + \sum_{j=i}^{2} v^{j-i}\, B^j]\,(-D^2 b)^{i-1}\\
N_3 &= [B^0 + \sum_{j=0}^{2} v^j\, B^j]\,\nabla b
\end{aligned}$$

and then, we get

$$d\mathcal{P}(e,\ell;\bar{e},\bar{\ell}) \qquad = \int_{\omega}[N_1\cdot\cdot\, D_\Gamma\bar{e} + N_2\cdot\cdot\, D_\Gamma\bar{\ell} + N_3\cdot\bar{\ell}]\, d\Gamma. \tag{47}$$

By using integration by parts

$$\begin{aligned} d\mathcal{P}(e,\ell;\bar{e},\bar{\ell}) &= \int_\omega (HN_1\nabla b - \overrightarrow{\mathrm{div}}_\Gamma\, N_1)\cdot \bar{e}\, d\Gamma \\ &\quad + \int_\omega (HN_2\nabla b - \overrightarrow{\mathrm{div}}_\Gamma\, N_2 + N_3)\cdot \bar{\ell}\, d\Gamma \\ &\quad + \int_\gamma [N_1\cdot\nu\cdot\bar{e} + N_2\cdot\nu\cdot\bar{\ell}]\, ds. \end{aligned}$$

Moreover it is obvious that

$$\begin{aligned} d\mathcal{W}(e,\ell;\bar{e},\bar{\ell}) &= \mathcal{W}(\bar{e},\bar{\ell}) \\ &= \int_\omega [\alpha_0(f\cdot\bar{e} + m\cdot\bar{\ell}) + \alpha_1 f\cdot\bar{\ell}]\, d\Gamma. \end{aligned} \tag{48}$$

This yields the following variational equation

$$d\mathcal{P}(e,\ell;\bar{e},\bar{\ell}) + d\mathcal{W}(e,\ell;\bar{e},\bar{\ell}) = \int_\gamma g\cdot\bar{e} + q\cdot\bar{\ell}\, ds$$

for some appropriate vector functions $g:\gamma\to\mathbb{R}^3$ and $q:\gamma\to\mathbb{R}^3$. From this we get the following nonlinear boundary problem for the shells

$$\begin{cases} -\overrightarrow{\mathrm{div}}_\Gamma\, N_1 + HN_1\nabla b = -\alpha_0 f \quad \text{on } \omega \\ -\overrightarrow{\mathrm{div}}_\Gamma\, N_2 + HN_2\nabla b + N_3 = -\alpha_0 m - \alpha_1 f \quad \text{on } \omega \end{cases} \tag{49}$$

$$\begin{cases} N_1\nu = g \quad \text{on } \gamma_1 \\ N_2\nu = q \quad \text{on } \gamma_1 \\ \ell = e = 0 \quad \text{on } \gamma_2 \end{cases} \tag{50}$$

where $\gamma = \gamma_1\cup\gamma_2$ and $\overset{\circ}{\gamma}_1\cap\overset{\circ}{\gamma}_2 = \varnothing$. But here we shall only consider the clamped case, that is $\gamma_1 = \varnothing$.

Remark 2. *For the Assumption 1^t, the same exercise can be repeated and the strong form of the nonlinear equations will be same up to a few terms of the form $F\cdot\nabla b\nabla b$ that will appear in the system. In fact, we will use $\ell - \ell\cdot\nabla b\nabla b$ as a test function in the variational equation, we get the following nonlinear system*

$$\begin{cases} -\overrightarrow{div}_\Gamma\, N_1 + HN_1\nabla b = -\alpha_0 f \quad on\ \omega, \\ -\overrightarrow{div}_\Gamma\, N_2 + \overrightarrow{div}_\Gamma\, N_2\cdot\nabla b\,\nabla b + HN_2\nabla b \\ \qquad -HN_2\nabla b\cdot\nabla b\,\nabla b + N_3 - N_3\cdot\nabla b\,\nabla b \\ \qquad\quad = -\alpha_0 m - \alpha_1(f - f\cdot\nabla b\,\nabla b) \quad on\ \omega \end{cases} \tag{51}$$

$$\begin{cases} N_1\nu = g \quad on\ \gamma_1 \\ N_2\nu - (N_2\nu\cdot\nabla b)\nabla b = q \quad on\ \gamma_1 \\ \ell = e = 0 \quad on\ \gamma_2 \end{cases} \tag{52}$$

4.4 Linear model and linear part of the nonlinear model

We conclude this section by comparing the linear and the nonlinear equations. For convenience, we write the nonlinear equations (49) and (50) as follows: to find vector field $(e,\ell):\omega\to\mathbb{R}^3\times\mathbb{R}^3$ that satisfies

$$\begin{cases} \mathcal{N}(e,\ell)=\vec{G}, \\ (e,\ell)=0 \quad \text{on} \quad \gamma, \end{cases} \tag{53}$$

where

$$\mathcal{N}(e,\ell)\stackrel{def}{=}(-\overrightarrow{\operatorname{div}}_\Gamma\, N_1+HN_1\nabla b,-\overrightarrow{\operatorname{div}}_\Gamma\, N_2+HN_2\nabla b+N_3)$$

and

$$\vec{G}\stackrel{def}{=}(-\alpha_0 f,-\alpha_0 m-\alpha_1 f).$$

Before discussing the existence of solutions, we first give the following result, which relates the operators $\mathcal{N}$ and $\mathcal{L}$. Denote by $\mathcal{T}$ the grouping of "terms which are at least of second order". For simplicity we use this as a generic notation in different equations.

Theorem 3.

$$\mathcal{N}(e,\ell)=\mathcal{L}(e,\ell)+\mathcal{T}(e,\ell,D_\Gamma e,D_\Gamma\ell,\overrightarrow{div}_\Gamma\,(D_\Gamma e),\overrightarrow{div}_\Gamma\,(D_\Gamma\ell)). \tag{54}$$

Proof. From the previous definition, we have for $0\le i\le 2$

$$\begin{aligned} F^i &= \alpha_i\varepsilon^0+\alpha_{i+1}\varepsilon^1+\alpha_{i+2}\varepsilon^2+\mathcal{T}(e,\ell,D_\Gamma e,D_\Gamma\ell) \\ &= E^i+\mathcal{T}(e,\ell,D_\Gamma e,D_\Gamma\ell), \end{aligned}$$

therefore

$$\begin{aligned} B^i=2\mu\,F^i+\lambda\,\mathrm{tr}\,F^i I &= 2\mu\,E^i+\lambda\,tr E^i I+\mathcal{T}(e,\ell,D_\Gamma e,D_\Gamma\ell) \\ &= A^i+\mathcal{T}(e,\ell,D_\Gamma e,D_\Gamma\ell). \end{aligned}$$

But

$$\begin{aligned} N_1 &= B^0-B^1D^2b+B^2(D^2b)^2+\mathcal{T}(e,\ell,D_\Gamma e,D_\Gamma\ell) \\ &= A^0-A^1D^2b+A^2(D^2b)^2+\mathcal{T}(e,\ell,D_\Gamma e,D_\Gamma\ell) \\ &= L_1+\mathcal{T}(e,\ell,D_\Gamma e,D_\Gamma\ell), \end{aligned}$$

$$\begin{aligned} N_2 &= B^1-B^2D^2b+\mathcal{T}(e,\ell,D_\Gamma e,D_\Gamma\ell) \\ &= A^1-A^2D^2b+\mathcal{T}(e,\ell,D_\Gamma e,D_\Gamma\ell) \\ &= L_2+\mathcal{T}(e,\ell,D_\Gamma e,D_\Gamma\ell), \end{aligned}$$

$$\begin{aligned} N_3 &= B^0\nabla b+\mathcal{T}(e,\ell,D_\Gamma e,D_\Gamma\ell) \\ &= A^0\nabla b+\mathcal{T}(e,\ell,D_\Gamma e,D_\Gamma\ell) \\ &= L_3+\mathcal{T}(e,\ell,D_\Gamma e,D_\Gamma\ell). \end{aligned}$$

So from the definitions of $\mathcal{N}$ and $\mathcal{L}$, we get the result. □

5 Existence of solutions to the nonlinear equations

We first prove the differentiability of the nonlinear operator $\mathcal{N}$ associated with the nonlinear model. The key observation is that the Sobolev space $W^{1,p}(\omega)$ is a Banach algebra for $p > 2$ (see Adams [1]). As a consequence, the nonlinear operator $\mathcal{N}$ mapping the space $\mathcal{W}^p(\omega)$ into the space $\mathcal{L}^p(\omega)$ is well-defined and infinitely differentiable between these two spaces.

Theorem 4. *Assume that $p > 2$. Then the associated nonlinear operator*

$$(e, \ell) \mapsto \mathcal{N}(e, \ell) \,:\, \mathcal{W}^p(\omega) \to \mathcal{L}^p(\omega)$$

is well-defined and differentiable, and

$$d\mathcal{N}(0, 0; e, \ell) = \mathcal{L}(e, \ell) \tag{55}$$

in $\mathcal{W}^p(\omega)$.

Proof. From the definitions, we know that N_1, N_2 and N_3 are sums of continuous linear, bilinear, and trilinear mappings of $e, \ell, D_\Gamma e$, and $D_\Gamma \ell$. Since $p > 2$, the Sobolev space $W^{1,p}(\omega)$ is a Banach algebra. Therefore the nonlinear mappings

$$\begin{aligned}&(e, \ell) \mapsto N_i(e, \ell, D_\Gamma e, D_\Gamma \ell) :\\ &W^{2,p}(\omega)^3 \times W^{2,p}(\omega)^3 \to W^{1,p}(\omega)^{3\times 3}\end{aligned}$$

are well-defined for $i = 1, 2$. Similarly,

$$\begin{aligned}&(e, \ell) \mapsto N_3(e, \ell, D_\Gamma e, D_\Gamma \ell) :\\ &W^{2,p}(\omega)^3 \times W^{2,p}(\omega)^3 \to W^{1,p}(\omega)^3\end{aligned}$$

is well-defined. Since $\mathcal{W}^p(\omega) \subset W^{2,p}(\omega)^3 \times W^{2,p}(\omega)^3$, hence, from definition, the nonlinear mappings

$$\overrightarrow{\mathrm{div}}_\Gamma \, N_i \,:\, \mathcal{W}^p(\omega) \to L^{1,p}(\omega)^3$$

are well-defined for $i = 1, 2$, and

$$N_3 \,:\, \mathcal{W}^p(\omega) \to W^{1,\,p}(\omega)^3 \hookrightarrow L^{1,p}(\omega)^3$$

is well-defined. Therefore the nonlinear operator

$$(e, \ell) \mapsto \mathcal{N}(e, \ell) \,:\, \mathcal{W}^p(\omega) \to L^p(\omega)^3 \times L^p(\omega)^3$$

is well-defined. From the definition of the nonlinear operator $\mathcal{N}$, the nonlinear mapping

$$(e, \ell) \mapsto \mathcal{N}(e, \ell) \,:\, \mathcal{W}^p(\omega) \to \mathcal{L}^p(\omega)$$

is well-defined from the definition. Moreover the mapping

$$e \mapsto D_\Gamma e \,:\, W^{2,\,p}(\omega) \to W^{1,p}(\omega)^{3\times 3},$$
$$\ell \mapsto D_\Gamma \ell \,:\, W^{2,\,p}(\omega) \to W^{1,p}(\omega)^{3\times 3},$$

are smooth (C^∞), and the mapping

$$G \mapsto \overrightarrow{\mathrm{div}}_\Gamma\, G \,:\, W^{1,p}(\omega)^{3\times 3} \to L^p(\omega)^3$$

is also smooth (C^∞). Therefore by definition of $\mathcal{N}$, we get that $\mathcal{N}(e,\ell)$ is differentiable in $\mathcal{W}^p(\omega)$.
In order to compute the derivative, which is now known to exist, it is sufficient to compute terms which are linear with respect to $(e, \ell, D_\Gamma e, D_\Gamma \ell, \overrightarrow{\mathrm{div}}_\Gamma\, (D_\Gamma e), \overrightarrow{\mathrm{div}}_\Gamma\, (D_\Gamma \ell))$ in the difference $\mathcal{N}(e,\ell) - \mathcal{N}(0,0)$. In fact, we have for any $(e,\ell) \in \mathcal{W}^p(\omega)$,

$$\mathcal{N}(e,\ell) = \mathcal{L}(e,\ell) + \mathcal{T}(e,\ell, D_\Gamma e, D_\Gamma \ell, \overrightarrow{\mathrm{div}}_\Gamma\, (D_\Gamma e), \overrightarrow{\mathrm{div}}_\Gamma\, (D_\Gamma \ell)),$$

and

$$\mathcal{N}(0,0) = 0.$$

But $\mathcal{L} \,:\, \mathcal{W}^p(\omega) \to \mathcal{L}^p(\omega)$ is a bounded linear operator. This implies that

$$d\mathcal{N}(0,0;e,\ell) = \mathcal{L}(e,\ell),$$

from the definition of the derivative. □

We are now in a position to establish the existence of solutions to the non-linear equation. The objective is to apply the inverse function theorem (cf. Hörmander [21]) in neighborhoods of the origins in the spaces $\mathcal{L}^p(\omega)$ and $\mathcal{W}^p(\omega)$.

Theorem 5. *Let $\bar{h}$ be as in Theorem 1, and h, $0 < h < \bar{h}$ be fixed. Let Ω be a domain in $\mathbb{R}^3$ with a boundary Γ which is a $C^{2,1}$ submanifold of $\mathbb{R}^3$ and let ω be a C^2 domain in the submanifold Γ. Then for each $p > 2$, there exist two neighborhoods $\mathcal{L}_0^p$ and $\mathcal{W}_0^p$ of the origins in $\mathcal{L}^p(\omega)$ and $\mathcal{W}^p(\omega)$ respectively, such that, for each $(m,f) \in \mathcal{L}_0^p$, the boundary value problem*

$$\mathcal{N}(e,\ell) = \vec{G},$$

for

$$\vec{G} = (-\alpha_0 f, -\alpha_0 m - \alpha_1 f),$$

has exactly one solution (e,ℓ) in $\mathcal{W}_0^p$.

Proof. If $\vec{G} = \mathbf{0}$, then $m = 0, f = 0$. Therefore it is obvious that $(e,\ell) = (0,0)$ is a solution corresponding to $\vec{G} = \mathbf{0}$. From the last theorem, we know that

$$d\mathcal{N}(0,0;e,\ell) = \mathcal{L}(e,\ell).$$

Since ω is a C^2 domain in the submanifold Γ, the regularity result of Theorem 2 implies that the problem

$$\mathcal{L}(e,\ell) = \vec{G}$$

has one and only one solution (e,ℓ) in the space $\mathcal{W}^p(\omega)$ for each $\vec{G} \in \mathcal{L}^p(\omega)$, $p > 2$. Hence the linear bounded operator

$$\mathcal{N}'(0,0) = \mathcal{L} : \mathcal{W}^p(\omega) \to \mathcal{L}^p(\omega)$$

is bijective, where $\mathcal{N}'(0,0)$ denotes the linear mapping $(e,\ell) \mapsto d\mathcal{N}(0,0;e,\ell)$. Thus, its inverse is also continuous from the closed graph theorem. Therefore the operator $\mathcal{N}'(e,\ell)$ is an isomorphism between the spaces $\mathcal{W}^p(\omega)$ and $\mathcal{L}^p(\omega)$ at the origin. Then the inverse function theorem [21] can be used at the origins of the spaces $\mathcal{W}^p(\omega)$ and $\mathcal{L}^p(\omega)$. It implies that there exists a neighborhood $\mathcal{L}_1^p$ of the origin of $\mathcal{L}^p(\omega)$ and a neighborhood $\mathcal{W}_0^p$ of $\mathcal{W}^p(\omega)$, such that the problem

$$\mathcal{N}(e,\ell) = \vec{G}$$

has exactly one solution (e,ℓ) in $\mathcal{W}_0^p$ for each $\vec{G} \in \mathcal{L}_1^p$. Choosing a suitable neighborhood $\mathcal{L}_0^p$ of the origin of $\mathcal{L}^p(\omega)$ such that, for any $(m,f) \in \mathcal{L}_0^p$, $\vec{G} \in \mathcal{L}_1^p$. This completes the proof. □

Remark 3. *As shown in the above theorem, our result is for small load (m,f). But, it is well-known that, in the analysis of nonlinear shells, bifurcation occurs when the load is large (cf. Gould [19]). Based on this fact, our result is as strong as it can be.*

6 Love-Kirchhoff Theory, Naghdi's and Koiter's models

This section is for the nonlinear model the analogue of section 3.3 for the linear case. However the situation is much more complex. For simplicity, we use Assumption 1^t in this section. Recall that the approximate Green Saint Venant strain tensor is given by

$$2\tilde{\varepsilon}(V) \circ T_z = \sum_{i=0}^{2} \left[{}^*v^i + v^i + \sum_{j=0}^{i} {}^*v^j \, v^{i-j} \right] z^i. \tag{56}$$

Also

$$\begin{aligned} 2\tilde{\varepsilon}(V) \circ T_z \, \nabla b &= \sum_{i=0}^{2} [{}^*v^i \, \nabla b + v^i \, \nabla b + \sum_{j=0}^{i} {}^*v^j \, v^{i-j} \, \nabla b] \, z^i \\ &= \sum_{i=0}^{2} [{}^*v^i \, \nabla b + v^i \, \nabla b + {}^*v^i \, \ell] \, z^i \end{aligned}$$

and

$$\boxed{2\tilde{\varepsilon}(V)\circ T_z\,\nabla b=\ell+\sum_{i=0}^{2}{}^*v^i\,[\nabla b+\ell]\,z^i} \tag{57}$$

Therefore

$$2\tilde{\varepsilon}(V)\circ T_z\,\nabla b\cdot\nabla b=\ell\cdot\nabla b+\sum_{i=0}^{2}(\nabla b+\ell)\cdot(v^i\,\nabla b)\,z^i=(\nabla b+\ell)\cdot\ell=|\ell|^2$$

and

$$\boxed{2\tilde{\varepsilon}(V)\circ T_z\,\nabla b\cdot\nabla b=|\ell|^2} \tag{58}$$

Moreover

$$\begin{aligned}
{}^*v^0\,\nabla b&=[{}^*D_\Gamma e+\nabla b\,{}^*\ell]\,\nabla b={}^*D_\Gamma e\,\nabla b\\
{}^*v^1\,\nabla b&=[{}^*D_\Gamma\ell-D^2b\,{}^*D_\Gamma e]\,\nabla b=-D^2b\,[\ell+{}^*D_\Gamma e\,\nabla b]\\
{}^*v^2\,\nabla b&=-[D^2b]\,[{}^*D_\Gamma\ell-D^2b\,{}^*D_\Gamma e]\,\nabla b=(D^2b)^2\,[\ell+{}^*D_\Gamma e\,\nabla b]
\end{aligned}$$

and

$$2\tilde{\varepsilon}(V)\circ T_z\,\nabla b=[I-z\,D^2b+z^2(D^2b)^2]\,[\ell+{}^*D_\Gamma e\,\nabla b]+\sum_{i=0}^{2}{}^*v^i\ell.$$

But

$$\begin{aligned}
\sum_{i=0}^{2}{}^*v^i\ell&=[I-z\,D^2b+z^2(D^2b)^2]\,{}^*D_\Gamma e\,\ell+|\ell|^2\,\nabla b+z\,{}^*D_\Gamma\ell\,\ell-z^2\,D^2b\,{}^*D_\Gamma\ell\,\ell\\
&=[I-z\,D^2b+z^2(D^2b)^2]\,{}^*D_\Gamma e\,\ell+|\ell|^2\,\nabla b+z\,[I-z\,D^2b]\,{}^*D_\Gamma\ell\,\ell
\end{aligned}$$

and since $2\,D_\Gamma\ell\,\ell=\nabla_\Gamma|\ell|^2$

$$\begin{aligned}
2\tilde{\varepsilon}(V)\circ T_z\,\nabla b=&[I-z\,D^2b+z^2(D^2b)^2]\,[\ell+{}^*D_\Gamma e\,(\nabla b+\ell)]\\
&+|\ell|^2\,\nabla b+z\,[I-z\,D^2b]\,\frac{1}{2}\nabla_\Gamma|\ell|^2
\end{aligned}$$

The first observation is that

$$\tilde{\varepsilon}(V)\circ T_z\,\nabla b\cdot\nabla b=|\ell|^2. \tag{59}$$

The second observation is that the Love-Kirchhoff condition of the linear case $\ell+{}^*D_\Gamma e\,\nabla b=0$ no longer implies that $\tilde{\varepsilon}(V)\circ T_z\,\nabla b=0$.

As for the condition on $\sigma\,n\cdot n$ for Naghdi's and Koiter's models, we get

$$\sigma\,n\cdot n=2\,\mu\,\tilde{\varepsilon}(V)\,n\cdot n+\lambda\,\mathrm{tr}\,\tilde{\varepsilon}(V). \tag{60}$$

We have seen that $2\,\tilde{\varepsilon}(V)\circ T_z\, n\cdot n = |\ell|^2$. For the trace

$$
\begin{aligned}
2\,\mathrm{tr}\,\tilde{\varepsilon}(V)\circ T_z &= \sum_{i=0}^{2}\left[2\,\mathrm{tr}\,v^i + \sum_{j=0}^{i}\mathrm{tr}\,({}^*v^j\,v^{i-j})\right]\,z^i \\
&= \sum_{i=0}^{2}\left[2\,\mathrm{tr}\,v^i + \sum_{j=0}^{i}v^j\cdot\cdot v^{i-j})\right]\,z^i
\end{aligned}
$$

and finally

$$
\begin{aligned}
2\,\sigma\circ T_z\, n\cdot n =& \lambda\,[2\,\mathrm{tr}\,v^0 + v^0\cdot\cdot v^0] + 2\,\mu\,|l|^2 \\
&+ z\,\lambda\,[2\,\mathrm{tr}\,v^1 + 2\,v^0\cdot\cdot v^1] \\
&+ z^2\,\lambda\,[2\,\mathrm{tr}\,v^2 + 2\,v^0\cdot\cdot v^2 + v^1\cdot\cdot v^1]
\end{aligned}
$$

or more explicitly

$$
\begin{aligned}
\sigma\circ T_z\, n\cdot n =& \lambda\,(\,\mathrm{div}_\Gamma e + \frac{1}{2}(\|D_\Gamma e\|^2 + |\ell|^2)) + \mu\,|\ell|^2 \\
&+ \lambda\,z\,\left\{\mathrm{tr}\,(D_\Gamma\ell - D_\Gamma e\,D^2 b) + (D_\Gamma e + \ell\,{}^*\nabla b)\cdot\cdot(D_\Gamma\ell - D_\Gamma e\,D^2 b)\right\} \\
&+ \lambda\,z^2\,\{[\mathrm{tr}\,[(D_\Gamma\ell - D_\Gamma e\,D^2 b)\,D^2 b] \\
&\qquad + (D_\Gamma e + \ell\,{}^*\nabla b)\cdot\cdot[(D_\Gamma\ell - D_\Gamma e\,D^2 b)\,D^2 b] \\
&\qquad + \frac{1}{2}\|D_\Gamma\ell - D_\Gamma e\,D^2 b\|^2\}.
\end{aligned}
$$

When the above expressions are compared with thoses of the linear case it is clear that the same conditions cannot be used and that some mechanical intuition should be added to the above mathematical identities.

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