Approximate invariance and differential inclusions in Hilbert spaces

F. H. Clarke[*] Yu. S. Ledyaev[†‡]
M. L. Radulescu[§]

CRM-2316

September 1995

[*]Centre de recherches mathématiques, Université de Montréal, C. P.6128, succ. Centre-ville, Montréal QC H3C 3J7, clarke@CRM.UMontreal.CA

[†]Centre de recherches mathématiques, Université de Montréal, C. P.6128, succ. Centre-ville, Montréal QC H3C 3J7,ledyaev@CRM.UMontreal.CA

[‡]Permanent address: Steklov Institute of Mathematics, Moscow 117966, Russia.

[§]Département de mathématiques et de statistique, Université de Montréal, C. P.6128, succ. Centre-ville, Montréal QC H3C 3J7, radulesm@ere.umontreal.ca

Abstract

Consider a mapping F from a Hilbert space H to the subsets of H which is upper semicontinuous/Lipschitz, has nonconvex, noncompact values and satisfies to a linear growth condition. We give necessary and sufficient conditions for a subset S of H to be approximately weak/strong invariant with respect to approximate solutions of the differential inclusion $\dot{x}(t) \in F(x)$. The conditions are given in terms of the lower/upper Hamiltonians corresponding to F and involve nonsmooth analysis elements and techniques. The concept of approximate invariance generalizes the well known concept fo invariant and in turns relies on the notion of ε-trajectory corresponding to a differential inclusion.

Keywords. differential inclusion, approximate weak and strong invariance, ε-trajectory, lower and upper Hamiltonians, proximal normal cone, proximal aiming.

Résumé

Considérons une application F définie sur un espace de Hilbert H à valeurs dans les sous-ensembles de H qui est semi-continue supérieurement/Lipschitz avec des valeurs non convexes, non compactes et qui satisfait à une condition de croissance linéaire. Nous donnons des conditions nécessaires et suffisantes pour qu'un sous-ensemble S de H soit approximativement faible/fort invariant par rapport aux solutions approximatives de l'inclusion différentielle $\dot{x} \in F(x)$. Les conditions sont données en fonction des hamiltoniens inférieurs/supérieurs attachés à F et impliquent des éléments et des techniques d'analyse non lisse. Le concept d'invariance est a son tout basé sur la notion de ε-trajectoire correspondant à une inclusion différentielle. Il apparait naturellement en connection avec plusieurs issues en contrôle comme par exemple celui des solutions généralisées de l'équation Hamilton-Jacobi.

Mots clef. inclusion différentielle, invariance approximativement faible et forte, ε-trajectoire hamiltoniens inférieurs et supérieurs

1 Introduction

This paper is devoted to the infinitesimal characterization of the invariance properties of a closed subset S of a Hilbert space H with respect to exact or approximate solutions of the differential inclusion

$$\dot{x} \in F(x). \tag{1.1}$$

We make the following assumptions on the multivalued mapping $F\colon H \to 2^H \backslash \phi$.

Hypothesis (H)

(a) $x \mapsto F(x)$ is upper semicontinuous;

(b) for certain constant c and for all $x \in H$,

$$v \in F(x) \Longrightarrow \|v\| \leq c\big(1 + \|x\|\big). \tag{1.2}$$

An additional hypothesis on F which will be invoked later is that it be Lipschitz on bounded sets. As usual, an *exact solution* (or *trajectory*) for F is an absolutely continuous function $x\colon [a,b] \to H$ satisfy (1.1) for a.a. $t \in [a,b]$.

Various aspects of weak and strong invariance concepts were studied in numerous papers and books (see [3, 2, 4, 8, 10, 13, 14, 16, 17, 19]) and references therein). The large interest in the subject is due to its important applications in many areas of dynamic optimization, control and first order partial differential equations (see for example [4, 7, 11, 17, 20]). Let us recall the definitions of weakly and strongly invariant sets with respect to a differential inclusion. For a given multifunction F and a closed subset S of H, the pair (S, F) will be referred to as a *system*.

Definition 1.1. The system (S, F) is *weakly invariant* if for any $x_0 \in S$ there exists a trajectory $x(\cdot)$ for F satisfying $x(0) = x_0$ and $x(t) \in S$, $t \geq 0$.

Definition 1.2. The system (S, F) is strongly invariant if all trajectories for F, $x(\cdot)$ whose initial value $x(0)$ lies in S remains in $S\colon x(t) \in S$, $t \geq 0$.

Traditionally, criteria for weak and strong invariance of (S, F) were given in terms of the contingent (or Bouligan-cone, $T_S(x)$ [3, 2, 14, 21]).

For example, if $H = \mathbb{R}^n$, F has convex, compact values and satisfies hypothesis (H) the system (S, F) is weakly invariant iff

$$F(x) \cap T_S(x) \neq \varnothing, \quad \forall x \in S. \tag{1.3}$$

The apparently more general condition

$$F(x) \cap \operatorname{co} T_S(x) \neq \varnothing, \tag{1.4}$$

turns out to be equivalent to (1.2) (see [16]).

In the infinite dimensional case, weak invariance criterion in form (1.3) were obtained only under more restrictive assumptions on F (see for example [14]).

A different approach to the derivation of invariance criteria was initiated in [10] for the finite dimensional case. Weak and strong invariance criterions were formulated in terms of the lower and upper Hamiltonian corresponding to F;

$$
\begin{aligned}
h_F(x,p) &:= \inf\bigl\{\langle p,v\rangle : v\in F(x)\bigr\},\\
H_F(x,p) &:= \sup\bigl\{\langle p,v\rangle : v\in F(x)\bigr\}.
\end{aligned}
\tag{1.5}
$$

Let us state this results: the system (S,F) is *weakly invariant* iff

$$
h_F(x,\xi)\le 0,\quad \forall \xi\in N_S^P(x),\quad \forall x\in S; \tag{1.6}
$$

if F is locally Lipschitz, (S,F) is *strongly invariant* iff

$$
H_F(x,\xi)\le 0,\quad \forall \xi\in N_S^P(x),\quad \forall x\in S. \tag{1.7}
$$

$N_S^P(x)$ denotes the proximal normal cone to S at s a basic concept in nonsmooth analysis which have already proved its utility in nonsmooth and dynamic optimization theories [5].

The hamiltonian conditions (1.6), (1.7) reflect the close intrinsic connection between invariance properties of some systems (S,F) and generalized solutions of Hamilton-Jacobi equations. In particular, this connection provides a dynamic integral characterization [21] of viscosity solutions of first order partial differential equations [12]. It should be noted that A. I. Subbotin was the first to point out that the invariance is relevant to the generalized solutions of Hamilton-Jacobi equations [20,21].

As a second note, we remark that criterion (1.6) also explains, via the elementary inclusion $T_S(x)\subseteq \bigl(N_S^P(x)\bigr)^*$ (the polar of N_S^P), the equivalence of (1.3) and (1.4). The aim of the present paper is to generalize the above invariance results (framework of the paper [10]) to the case of an infinite dimensional Hilbert space without introducing new assumptions on F. Yet, Hypothesis (H) together with compactness and convexity assumptions on the values of F, does not ensure the existence of an exact solution for F (see the ? example due to J. Yorke in [22]). This is why we *shall deal* mainly with *approximate solutions* (or ε-trajectories) for F a concept which is not *completely unfamiliar* in the theory of differential equations [1,12].

Definition 1.3. An absolutely continuous function $x\colon [a,b]\to H$ is called an *ε-trajectory* for F if

$$
\dot{x}(t)\in F\bigl(x(t)+\varepsilon B\bigr),\ \text{a.e. on } [0,T],
$$

where B denotes the closed unit ball in H.

Correspondingly, the concept of invariance is replaced by the concept of approximate invariance. The *approximate weak invariance* of (S,F) means that for any $\varepsilon>0$, any compact interval $J\subseteq\mathbb{R}$ and any $x_0\in S$. There exists an ε-trajectory for F satisfying $x(0)=x_0$ and $x(t)\in S+\varepsilon B$ on J. The *approximate strong* invariance of (S,F) means that for any $\lambda>0$, $J\subseteq\mathbb{R}^n$ there exists $\varepsilon>0$ such that any ε-trajectory for F whose initial value lies in S remains in $S+\lambda B$ on J.

The main Theorems 3.1 and 4.1 give infinitesimal criteria for approximate weak and strong invariance in hamiltonian form (1.5), respectively. (1.6) while Theorems 3.2 and 4.2 establish the equivalence between hamiltonian and tangential type criteria.

We remark that this results are obtained under hypothesis (H) which does not include convexity and relative compactness assumptions on the values of F.

The plan of the paper is the following. In Section 2 we give some basic definitions and recall certain results from nonsmooth analysis. We also prove a new result on the approximation of δ-metric projections by proximal normals. Some elements of integration theory of vector-valued functions are mentioned here too. Sections 3 and 4 are devoted to approximate weak and strong invariance. The results therein are used in Section 5 in order to derive weak and strong invariance results under additional compactness assumptions on F. In this case we prove that (1.5) is a criterion for weak invariance which implies the well known tangential criterion (1.3) (see [13, 14]) and establishes that (1.3) is equivalent to (1.4).

2 Basic definitions and preliminary results

We begin this section by presenting some elements of nonsmooth analysis; we shall use [6], [11], [9] and [18] as general reference and guides to the literature.

As we already mentioned, H is a Hilbert space equiped with the inner product $\langle \cdot, \cdot \rangle$ and corresponding norm $\| \cdot \|$. If the sequence (v_n) converges weakly to v we write $v_n \rightharpoonup v$. Let S denote a closed subset of H. The distance of a point x to S is given by

$$d_S(x) := \inf\big\{ \|x - s\| \colon s \in S \big\}.$$

We recall the fact that the distance function $d_S \colon H \to \mathbb{R}$ is globally Lipschitz of rank 1. The (possibly empty) set of closest points to x in S is denoted

$$\operatorname{proj}_S(x) := \big\{ s \in S \colon \|x - s\| = d_S(x) \big\}.$$

The multifunction proj_S is referred to as the metric projection onto s. If $x \notin S$ and $s \in \operatorname{proj}(x)$ then we say that the vector $x - s$ is a *perpendicular* to S at s. The set of all nonnegative multiples of such perpendiculars is denoted by $N_S^P(s)$ and is referred to as the *proximal normal cone* to S at s. One can show that $\xi \in N_S^P(s)$ if and only if there exists $M(s) \geq 0$ such that the following *proximal normal inequality* holds:

$$\langle \xi, s' - s \rangle \leq M(s) \|s' - s\|^2 \quad \forall s' \in S. \tag{2.1}$$

If $x \in \operatorname{int} S$ or no perpendiculars to S' exist at s, then by convention we set $N_S^P(s) = \{0\}$.

Now let $f \colon H \to (-\infty, \infty]$ be a lower semicontinuous function and $x \in \operatorname{dom} f := \big\{ y \colon f(y) < \infty \big\}$. An element $\xi \in H$ is a *proximal subgradient* of f at x provided that for some $\sigma(x) \geq 0$ and for all y in some neighbourhood of x we have

$$f(y) - f(x) + \sigma(x) \|y - x\|^2 \geq \langle \xi, y - x \rangle. \tag{2.2}$$

We denote by $\partial_P f(x)$ the set of proximal subgradients of f at x. $\partial_P f(x)$ may be empty; however the domain of $\partial_P f(x)$, $\operatorname{dom} \partial_P f(x) := \big\{ y \colon \partial_P f(y) \neq \varnothing \big\}$ is dense in $\operatorname{dom} f$.

The connection between normals and subgradients is established via the *epigraph* of f, the set

$$\operatorname{epi} f := \big\{ (x, \alpha) \in H \times \mathbb{R}, f(x) \leq \alpha \big\},$$

namely

$$\xi \in \partial_P f(x) \Leftrightarrow (\xi, -1) \in N^P_{\operatorname{epi} f}\big(x, f(x)\big).$$

If we denote by ψ_S the *indicator function* of the set S then

$$\partial_P \psi_S(s) = N_S^P(s) \quad \forall s \in S'. \tag{2.3}$$

We shall use the following fact which could be immediately derived from the definitions.

Proposition 2.1. *Let $g\colon H \to \mathbb{R}$ be of class $\mathcal{C}^2$ on an open set Ω and $x_0 \in \Omega$ be a local minimum for the function $f + g$. Then $-g'(x_0) \in \partial_P f(x_0)$.*

The following results on proximal subdifferentiability of the distance function to a closed subset of a Hilbert space and the existence of closest points can be found in [4]. For related results in more general Banach space setting see the references in [4].

Proposition 2.2. (a) *Suppose $x \in S'$. Then*

$$N_S^P(x) = \big\{\lambda\xi\colon \lambda \geq 0, \xi \in \partial_P d_S(x)\big\}. \tag{2.4}$$

(b) *Suppose $x \notin S$ and $\partial_P d_S(x) \neq \varnothing$. Then $\operatorname{proj}_S(x)$ is a singleton $\{s\}$ and $\partial_P d_S(x)$ is the singleton $\xi := (x-s)/(d_S(x))$ i.e. $\partial_P d_S(x) = \{\xi\} \subseteq N_S^P(s)$.*

Since d_S has a nonempty proximal subdifferential on a dense subset of $H\backslash S$ it readily follows that $\operatorname{proj}_S(x) \neq \varnothing$ for all x in a dense subset of $H\backslash S$. In general, the dense set is a proper subset of $H\backslash S$ and it always coincides with $H\backslash S$ if H is finite dimensional. Let us introduce the following subset of S which is nonempty for any $x \in H$ and $\delta > 0$.

$$\operatorname{proj}_S^\delta(x) := \big\{s \in S\colon \|x-s\|^2 \leq d_S^2(x) + \delta^2\big\}. \tag{2.5}$$

The next result says that for any selector $s_\delta(x) \in \operatorname{proj}_S^\delta(x)$ the vector $x - s_\delta(x)$ can be approximated by some proximal normal vector $y_\delta(x) - \bar{s}_\delta(x)$ where $\bar{s}_\delta(x) \in S$ approximates $s_\delta(x)$.

Proposition 2.3. *Let $x \in H$, $\delta > 0$ and $s_\delta(x) \in \operatorname{proj}_S^\delta(x)$. Then there exists $y_\delta(x) \in H$ and $\bar{s}_\delta(x) \in S$ such that*

$$y_\delta(x) - \bar{s}_\delta(x) \in N_S^P(x), \tag{2.6}$$

$$\Big\|\big(y_\delta(x) - \bar{s}_\delta(x)\big) - \big(x - s_\delta(x)\big)\Big\| \leq 2\delta \ \textit{and} \tag{2.7}$$

$$\big\|s_\delta(x) - \bar{s}_\delta(x)\big\| \leq \delta. \tag{2.8}$$

Proof. If $x \in S$ we choose $y_\delta(x) := x$ and $\bar{s}_\delta(x) = s_\delta(x)$. Suppose $x \in H\backslash S$ and let us consider the function

$$g(s) := \|x-s\|^2 + \psi_S(s),$$

where ψ_S is the indicator function of S. By assumption

$$g\big(s_\delta(x)\big) \leq \inf_H g + \delta^2.$$

By the Borwein Preiss smooth variational principle [3] (see also [4] for the Hilbert space version) there exist $u \in H$ and $s_\delta(x) \in S$ such that

$$\big\|s_\delta(x) - \bar{s}_\delta(x)\big\| < \delta, \quad \big\|\bar{s}_\delta(x) - u\big\| < \delta, \quad g\big(s_\delta(x)\big) < \inf_H g + \delta^2 \tag{2.9}$$

and the function $\varphi(s) := g(s) + \|u-s\|^2$ has a unique minimum point at $\bar{s}_\delta(x)$.

Since $\|x-s\|^2 + \|u-s\|^2$ is of class $\mathcal{C}^2$ near s, by Proposition 2.1 and relation (2.3),

$$\big(x - \bar{s}_\delta(x)\big) + \big(u - \bar{s}_\delta(x)\big) \in \partial^p \psi_S\big(\bar{s}_\delta(x)\big) = N_S^P\big(\bar{s}_\delta(x)\big).$$

Finally we set $y_\delta := x + (u - \bar{s}_\delta)$. Then (2.6) holds and inequalities (2.7), (2.8) follow from (2.9). ∎

Another important concept of nonsmooth analysis is that of a tangent cone to the set S at $x \in S$. We shall introduce three types of tangent cones by means of the following directional derivatives: the *weak-Dini derivative*

$$D^w f(x;v) := \inf_{\{v_i\}} \liminf_{\substack{v_i \rightharpoonup v \\ t\downarrow 0}} \frac{f(x+tv_i)-f(x)}{t},$$

where the infimum is taken over all weakly convergent subsequences to v, the *Dini-derivative*

$$Df(x;v) := \liminf_{\substack{v'\to v \\ t\downarrow 0}} \frac{f(x+tv')-f(x)}{t},$$

and the *generalized derivative*

$$f^0(x;v) := \limsup_{\substack{y\to x \\ t\downarrow 0}} \frac{f(y+tv)-f(y)}{t}.$$

We remark that if f is Lipschitz near x then

$$Df(x;v) = \liminf_{t\downarrow 0} \frac{f(x+tv)-f(x)}{t}.$$

Let $x \in S$. We define the *weak-Dini tangent cone* (or the weak contingent cone) at x

$$T_s^w(x) := \big\{v \mid D^w d_S(x;v) = 0\big\},$$

the *Dini tangent cone* (or the contingent or Baeligand cone) at x

$$T_S(x) := \big\{v \mid Dd_S(x;v) = 0\big\},$$

and the *Clarke tangent cone* at x

$$T_s^c(x) := \big\{v \mid d_s^0(x;v) = 0\big\}.$$

One can easily prove that

$$\begin{aligned} T_S(x) = \big\{v \mid \exists (s_i) \subset S, s_i \to x \text{ and } t_i \downarrow 0 \text{ such that} \\ v = \lim(s_i - s)/t_i\big\} \text{ and} \\ T_S^w(x) = \big\{v \mid \exists (s_i) \subset S, s_i \to x \text{ and } t_i \downarrow 0 \text{ such that} \\ v = w - \lim(s_i - s)/t_i\big\}. \end{aligned} \tag{2.10}$$

Since $D^w f(x;v) \le Df(x;v) \le f^0(x;v)$ the following inclusions hold

$$T_S^c(x) \subset T_S(x) \subset T_S^w(x). \tag{2.11}$$

The next proposition gives the relation between the weak-Dini tangent cone and the proximal normal cone. First we recall that the *polar* of a set C is the convex cone

$$C^* := \big\{\xi \in H \mid \langle \xi, v\rangle \le 0 \quad \forall v \in C\big\}.$$

Proposition 2.4. *For any $x \in S$*

$$\operatorname{co} T_S^w(x) \subset \big(N_S^P(x)\big)^*.$$

where co *is the notation for the convex hull.*

Proof. Since $\big(N_S^P(x)\big)^*$ is convex, is enough to prove that

$$T_S^w(x) \subset \big(N_S^P(x)\big)^*.$$

Let $v \in T_S^w(x)$. Then by definition there exist a weakly convergent sequence $v_k \to v$ and a sequence $t_k \downarrow 0$ such that $d_S(x+t_kv_k)/t_k \to 0$. Let $\xi \in N_S^P(x)$. By (2.4) and the subgradient inequality (2.2) there exist $\lambda > 0$ and $\sigma > 0$ such that for all y near x

$$\big\langle \lambda\xi, y-x\big\rangle \le \sigma\|x-y\|^2 + d_S(y) - d_S(x).$$

Thus, for k large enough and $y = x + t_kv_k$

$$\langle \lambda\xi, v_k\rangle \le \sigma t_k\|v_k\|^2 + \frac{1}{t_k}d_S(x+t_kv_k),$$

and by passing to limit, $\langle \xi, v\rangle \le 0$ which completes the proof. □

One can also prove the following formula (see [11])

$$T_S^c(x) = \big\{\xi\colon \xi = w-\lim \xi_n, \xi_n \in N_S^P(x_n), (x_n) \subset S, x_n \to x\big\}^*. \tag{2.12}$$

In what concerns the lower and upper hamiltonians (which were defined in 1.5) we have the following properties.

Proposition 2.5. *For any x and p in H*

(a) $h_F(x,p) = h_{\overline{\mathrm{co}}F}(x,p)$

(b) $H_F(x,p) = H_{\overline{\mathrm{co}}F}(x,p)$

Proof. (a) Let $x, p \in H$. Clearly $h_{\overline{\mathrm{co}}F}(x,p) \le h_F(x,p)$. To prove the inverse inequality let $v_0 \in \overline{\mathrm{co}}F(x)$ be such that $\langle p, v_0\rangle = h_{\overline{\mathrm{co}}F}(x,p)$. Then $v_0 = \lim v_n$ where $v_n \in \operatorname{co} F(x)$. Since each v_n may be written as $v_n = \sum_{i=1}^N \alpha_i v_{n_i}$, $\sum_{i=1}^N \alpha_i = 1$, $v_{n_i} \in F(x)$ we readily obtain $\langle p, v_n\rangle \ge h_F(x,p)$ and by passing to limit the conclusion follows. The proof of (b) is analogous. □

We conclude this section by recalling a few useful results from the theory of integration of vector valued functions. They may be found together with other related topics and references in Diestel and Uhl [15].

We denote by $\mathcal{C}_H(J)$ the Banach space of all continuous functions from J to H equiped with the sup-norm, $\|\cdot\|_0$. All the integrals of vector valued functions are to be understood in the sense of the Bochner integral and are taken with respect to the Lebesgue measure on some interval $J \subseteq \mathbb{R}$. We recall the following results:

Theorem 2.1. *If f is Bochner integrable on $[a,b]$ with respect to the Lebesgue measure then*

$$\frac{1}{b-a}\int_a^b f(t)\,dt \in \overline{\mathrm{co}}\Big(f\big([a,b]\big)\Big).$$

Theorem 2.2. *Every absolutely continuous function $f\colon [a,b] \to H$ is differentiable almost everywhere. Then*

$$f(t) - f(a) = \int_a^t f'(t)\,dt \quad \forall t \in [a,b]. \tag{2.13}$$

We remark that every trajectory or ε-trajectory to the differential inclusion (1.1) is by definition an absolutely continuous function (see §1) and consequently it satisfies (2.13).

Finally, $L^2_H(J)$ stands for the Hilbert space of all (equivalence classes of) H-valued functions, Bochner integrable on J with respect to the Lebesgue-measure with $\int_J \big\|f(t)\big\|^2\,dt < \infty$. The norm $\|\cdot\|_2$ is defined by

$$\|f\|_2 := \left(\int_J \big\|f(t)\big\|^2\,dt\right)^{1/2}, \quad f \in L^2_H(J).$$

Corollary 2.1. *Let $x(\cdot)$ be a trajectory for a multifunction $G\colon H \to 2^H \backslash \varnothing$ on $[a,b]$. Then for a given $t \in [a,b)$ and for small enough positive h*

$$\frac{x(t+h)-x(t)}{h} \in \overline{\mathrm{co}} G\big(x\big([t,t+h]\big)\big).$$

Proof. Let $t \in [a,b)$. As a trajectory, $x(\cdot)$ is an absolutely continuous function so by Theorem 2.2

$$\frac{x(t+h)-x(t)}{h} = \frac{1}{h}\int_t^{t+h} \dot{x}(\tau)\,d\tau,$$

for h small enough. Let $f\colon [a,b] \to H$ be such that it coincides with $\dot{x}$ if the later does exist and takes a certain value $f(t) \in G\big(x(t)\big)$ otherwise. Then according to Theorem 2.1

$$\frac{1}{h}\int_t^{t+h} \dot{x}(\tau)\,d\tau = \frac{1}{h}\int_t^{t+h} f(\tau)\,d\tau \in \overline{\mathrm{co}}\big(f\big([t,t+h]\big)\big) \in \overline{\mathrm{co}} G\big(x\big([t,t+h]\big)\big),$$

as required. □

3 Approximate weak invariance

We begin by recalling the definition of approximate weak invariance which was given in the Introduction; the concept of ε-trajectory for a multifunction F was introduced in Definition 1.3; S is supposed to be a closed subset of H.

Definition 3.1. The system (S,F) is said to be *approximately weak invariant* if for any $\varepsilon > 0$, any interval $J = [t_0, T]$ and any $x_0 \in S$ there exists an ε-trajectory for F on J, $x(\cdot)$, such that $x(t_0) = x_0$ and

$$d_S\big(x(t)\big) \le \varepsilon \quad \forall t \in J.$$

Let us state the main result of this section, namely a necessary and sufficient condition for approximate weak invariance in terms of the lower Hamiltonian h_F (see (1.6)).

Theorem 3.1. *Under hypothesis* (H) *the system* (S,F) *is approximately weak invariant iff*

$$h_F(x,\xi) \le 0 \quad \forall \xi \in N^P_S(x), \quad \forall x \in S. \tag{3.1}$$

Remark 3.1. We may assume without loss of generality that $J = [0,T]$ and $\varepsilon \in (0,1)$. Then by means of the Gronwall Lemma we can deduce from the linear growth condition (**??**) that any ε-trajectory $x(\cdot)$ corresponding to (1.1) is bounded i.e.

$$\|x(t)\| \leq e^{cT}\big(2 + \|x_0\|\big) = K \quad \forall t \in [0,T]. \tag{3.2}$$

Hence

$$\|\dot{x}(t)\| \leq c(z + K) =: M \text{ for a.a } t \in [0,T] \tag{3.3}$$

and $x(\cdot)$ is globally Lipschitz of rank M.

Since we shall consider only ε-trajectories on $[0,T]$ starting from x_0, without loss of generality (1.2) may be replaced by

$$v \in F(x) \Rightarrow \|v\| \leq M, \tag{3.4}$$

where M depends upon T and $\|x_0\|$.

In the proof, an important role will be played by the concept of *Euler polygon* corresponding to a initial value problem. We recall it briefly. Consider the Cauchy problem on $[0,T]$

$$\begin{cases} \dot{x} = f(x) \\ x(0) = x_0 \end{cases} \tag{3.5}$$

where f is simply an *arbitrary* function from H to H.

Since in general (3.2) does not admit a classical solution we take a pragmatic approach based upon the well known numerical solution procedure which constructs piecewise linear "approximate solutions" for partitions of the time interval. Let $\pi = \{t_0, t_1, \ldots, t_N\}$ denote a partition of the interval $[0,T]$ where $t_0 = 0$ and $t_N = T$. We shall refer to

$$\mu(\pi) := \sup_{0 \leq i \leq N-1} (t_{i+1} - t_i)$$

as to the *mesh size* of the partition. On each interval $[t_i, t_{i+1}]$ we consider the linear solution of "slope" $f(x_i)$ where x_i is obtained recurrently. Thus we obtain a polygonal arc described by

$$\begin{cases} \dot{x}_\pi(t) = f(x_i) \quad \text{on } [t_i, t_{i+1}] \\ x(t_i) = x_i \end{cases} \tag{3.6}$$

for all $i \in \{0, 1, \ldots N-1\}$ which will be said to be an *Euler-polygon* for f on $[0,T]$.

Remark 3.2. (a) In our context, an important feature of Euler polygons is that for appropriate f they become ε-trajectories for F. Indeed let $\mu(\pi) \leq \varepsilon/2M$ and consider $f(x) \in F(x + \varepsilon/2B)$ $\forall x \in H$. Then by construction

$$\|x_\pi(t) - x_\pi(t_i)\| \leq M\mu(\pi) < \frac{\varepsilon}{2} \quad \forall t \in [t_i, t_{i+1}], \quad \forall i \in \{0, 1, \ldots N-1\}. \tag{3.7}$$

Hence

$$\dot{x}_\pi(t) \in F\bigg(x_\pi(t_i) + \frac{\varepsilon}{2}B\bigg) \subseteq F\big(x_\pi(t) + \varepsilon B\big), \text{ for a.a } t \in [0,T].$$

(b) If $H = \mathbb{R}^W$ inequalities (3.2) and (3.3) ensure the applicability of Arzela-Ascoli theorem, thus as $\mu(\pi) \to 0$ there exists a uniform limit of Euler-polygons namely the Euler solution of (3.5).

It was proven in [10] that (3.1) is a sufficient condition for the weak invariance of a finite dimensional system (S, F); the trajectory of F that remains in S was explicitely constructed as some Euler solution. It will be useful to recall the idea of the proof. Let $x_0 \in S$. We begin by associating to each $x \in \mathbb{R}^n$ a point $s(x) \in \operatorname{proj}_S(x)$. The vector $x - s(x)$ lies in $N_S^P\big(s(x)\big)$ by construction so in view of (3.1) there exists $f(x) \in F\big(s(x)\big)$ such that $\big\langle f(x), x - s(x)\big\rangle \leq 0$. The technique of choosing a velocity $f(x)$ ("aiming") in a opposite direction to a proximal $x - s(x)$ was labelled in [10] "proximal aiming". Now we invoke compactness of trajectories to show that the initial value problem (3.5) admits an Euler solution which satisfies $\dot{x} \in F\big(s(x)\big)$. One also proves, by the so called "Proximal aiming lemma", (see [4]) that the Euler solution lies in S. So $\dot{x} \in F\big(s(x)\big) = F(x)$ and $x(\cdot)$ is a trajectory as required.

The actual proof must deal with the possible emptyness of the sets $\operatorname{proj}_S(x)$ and with the lack of an Euler solution for (3.5). First we give the Hilbert version of the Proximal aiming lemma.

Lemma 3.1. *Let $x_0 \in H$ and $T > 0$. Suppose that $M := M(x_0, T)$ is a positive constant such that for any $\delta \in (0,1)$ and any selection s_δ of $\operatorname{proj}_S^\delta$ there exists $f_\delta \colon H \to H$ such that*

$$\big\|f_\delta(x)\big\| \leq M \quad \text{and} \tag{3.8}$$

$$\big\langle f_\delta(x), x - s_\delta(x)\big\rangle \leq 4M\delta \quad \forall x \in H. \tag{3.9}$$

Then for each $\varepsilon \in (0,1)$ there exists $\Delta \in (0, \varepsilon)$ and an Euler polygon on $[0, T]$ corresponding to the initial value problem

$$\begin{cases} \dot{x} = f_\Delta(x) \\ x(0) = x_0 \end{cases} \tag{3.10}$$

which satisfies

$$d_S\big(x(t)\big) \leq d_S(x_0) + \varepsilon \quad \forall t \in [0, T].$$

Proof. Let $\pi = \{t_0, t_1, \ldots, t_N\}$ be a partition of $[0, T]$ of mesh size

$$\mu(\pi) \leq \min\left(\frac{\varepsilon}{2M}, \frac{\varepsilon^2}{4T(M^2 + 8M + 1)}, \varepsilon\right). \tag{3.11}$$

Let us choose $\Delta > 0$ such that

$$\Delta \leq \min_{0 \leq i \leq N-1}(t_{i+1} - t_i) \tag{3.12}$$

and let $x(\cdot)$ be the Euler polygon corresponding to (3.11) and to the partition π. We use the following notations:

$$x_i := x(t_0), \quad s_i = s_\Delta(x_i) \text{ and } f_i := f_\Delta(x_i). \tag{3.13}$$

Then by construction

$$x(t) - x_i = \int_{t_i}^{t} f_i \, d\tau = (t - t_i) f_i \quad \forall t \in [t_i, t_{i+1}] \tag{3.14}$$

and from (3.9)

$$\big\|x(t) - x_i\big\| \le M(t - t_i) \quad \forall t \in [t_i, t_{i+1}]. \tag{3.15}$$

Due to (2.5), (3.15) and (3.16) we have

$$\begin{aligned} d_S^2(x_{i+1}) &\le \|x_{i+1} - s_i\|^2 \\ &= \|x_{i+1} - x_i\|^2 + \|x_i - \delta_i\|^2 + 2\langle x_{i+1} - x_i, x_i - \delta_i\rangle \\ &\le M^2(t_{i+1} - t_i)^2 + d_S^2(x_i) + \Delta^2 + 2\int_{t_i}^{t_{i+1}} \langle f_i, x_i - \delta_i\rangle \, d\tau. \end{aligned} \tag{3.16}$$

Now by (3.14) and (3.10)

$$\begin{aligned} d_S^2(x_{i+1}) - d_S^2(x_i) &\le \big(M^2\mu(\pi) + \mu(\pi) + 8M\mu(\pi)\big)(t_{i+1} - t_i) \\ &\le \varepsilon^2 \frac{(t_{i+1} - t_i)}{4T} \qquad \text{(in view of (3.12))}, \end{aligned}$$

and consequently

$$d_S(x_{i+1}) \le d_S(x_0) + \frac{\varepsilon}{2} \quad \forall i \in \{0, 1, 2, \ldots, N-1\}.$$

Finally, according to (3.16) and (3.12), $\big\|x(t) - x_i\big\| \le \varepsilon/2$ on $[t_i, t_{i+1}]$ and since d_S is Lipschitz of rank 1 the conclusion follows. □

Lemma 3.2. *Suppose that the functions f_δ in Lemma 3.1 are such that*

$$f_\delta(x) \in F(s_\delta(x) + \delta B) \quad \forall x \in H. \tag{3.17}$$

Let $x_0 \in S$. Then there exists an Euler polygon $x(\cdot)$ which satisfies $x(0) = x_0$,

$$\begin{gathered} d_S\big(x(t)\big) \le \varepsilon \quad \forall t \in [0, T] \text{ and} \\ \dot{x}(t) \in F\big(x(t) + \varepsilon B\big) \text{ a.e. in } [0, T]. \end{gathered}$$

(i.e. $x(\cdot)$ is an ε-trajectory for F on $[0, T]$ remaining in $S + \varepsilon B$).

Proof. Let us apply Lemma 3.1 with $\varepsilon/8$. Then there is a partition π of mesh size $\mu(\pi) \le \min(\varepsilon/16M, \varepsilon/8)$ (see (3.12)), $0 \le \Delta \le \mu(\pi) \le \varepsilon/8$ and an Euler polygon $x(\cdot)$ such that $d_S\big(x(t)\big) \le \varepsilon/8 \quad \forall t \in [0, T]$ and in particular $d_S(x_i) \le \varepsilon/8 \quad \forall i \in \{0, 1, 2, \ldots, N-1\}$ (see notations (3.14)).

By definition, $\|s_i - x_i\| \le d_S(x_i) + \Delta \le \varepsilon/4$, that is $s_i \in x_i + \varepsilon/4B$ and in view of (3.18)

$$f_i \in F(s_i + \Delta B) \subseteq F\left(x_i + \frac{\varepsilon}{2}B\right) \quad \forall i \in \{0, 1, 2, \ldots, N-1\}. \tag{3.18}$$

Finally, since $\big\|x(t) - x_i\| \le M\mu(\pi) \le \varepsilon/2$ and $\dot{x}(t) = f_i \quad \forall t \in (t_i, t_{i+1}), \forall i \in \{0, 1, 2, \ldots, N-1\}$ we obtain

$$\dot{x}(t) \in F\big(x(t) + \varepsilon B\big) \text{ for a.a } t \in [0, T]$$

as required. □

Proof of Theorem 3.1. Let us assume that (3.1) holds. We proceed to show that (S, F) is approximately weak invariant. To this end let $x_0 \in S$, $T > 0$ and $\varepsilon \in (0, 1)$ be given. We shall construct an ε-trajectory for F which satisfies $x(0) = x_0$ and $x(t) \in S + \varepsilon B$ a.e. in $[0, T]$. Our goal is to provide for each $\delta \in (0, 1)$ a function $f_\delta \colon H \to H$ which satisfies (3.9), (3.10) and (3.16). Then the conclusion follows immediately from Lemma 3.2. For each $x \in H$ let $s_\delta(x) \in \operatorname{proj}_S^\delta(x)$. According to Proposition 2.1 there exist $y_\delta(x) \in H$ and $\bar{s}_\delta(x) \in S$ such that

$$y_\delta(x) - \bar{s}_\delta(x) \in N_S^P\big(\bar{s}_\delta(x)\big),$$

$$\Big\|\big(y_\delta(x) - \bar{s}_\delta(x)\big) - \big(x - s_\delta(x)\big)\Big\| \le 2\delta \quad \text{and} \tag{3.19}$$

$$\big\|s_\delta(x) - \bar{s}_\delta(x)\big\| \le \delta. \tag{3.20}$$

By assumption we can choose an element $f_\delta(x) \in F\big(\bar{s}_\delta(x)\big)$ such that

$$\big\langle f_\delta(x), y_\delta(x) - \bar{s}_\delta(x)\big\rangle \le 2M\delta, \tag{3.21}$$

here, M is the constant defined in (3.3), (3.4). Thus

$$\begin{aligned}\big\langle f_\delta(x), x - s_\delta(x)\big\rangle &\le \big\langle f_\delta(x), y_\delta(x) - \bar{s}_\delta(x)\big\rangle \\ &\qquad + M\Big\|\big(y_\delta(x) - \bar{s}_\delta(x)\big) - \big(x - s_\delta(x)\big)\Big\| \\ &\le 4M\delta \qquad \text{by (3.19) and (3.21).}\end{aligned} \tag{3.22}$$

Now since $f_\delta(x) \in F\big(\bar{s}_\delta(x)\big)$ (3.20) implies

$$f_\delta(x) \in F\big(s_\delta(x) + \delta B\big).$$

Also $\big\|f_\delta(x)\big\| \le M$. Thus this relations together with (3.22) and (3.23) enable us to invoke Lemma 3.2 which complets the proof. The opposite implication stems from Theorem 3.2 below. □

For clarity of exposition we considered only the autonomous case $F(x)$. However the proximal aiming method equaly works if F has explicit t-dependence. In this case F is assumed to satisfy the following hypothesis:

(H')

(a) $F \colon \mathbb{R} \times H \to H$ is measurable in t and upper semicontinuous in x.

(b) For every $t \in \mathbb{R}$ and $x \in H$, $F(t, x)$ is a nonempty, subset of H.

(c) For a certain constant c, for all $t \in \mathbb{R}$ and $x \in H$

$$v \in F(t, x) \Rightarrow \|v\| < c\big(\|x\| + 1\big).$$

We define the ε-trajectory to the non-autonomous differential inclusion

$$\begin{cases} \dot{x} \in F(t, x) & \text{a.e. on } J \subset \mathbb{R} \\ x(0) = x_0 \end{cases} \tag{3.23}$$

as follows:

Definition 3.2. An *ε-trajectory* for (3.27) is an absolutely continuous function $x\colon J \to H$ which satisfies

$$\dot{x}(t) \in F\big(t, x(t) + \varepsilon B\big) \text{ a.e. on } J.$$

Definition 3.1 remains unchanged but we bear in mind that the ε-trajectory concept was redefined.

Now we are in a position to state the hamiltonian condition for approximately weak invariance of nonautonomous systems:

Theorem 3.2. *Suppose that F satisfies (H') and*

$$h_F(t,x,\xi) \le 0 \quad \forall \xi \in N_S^P(x), \quad \text{a.a.} t \in \mathbb{R}, \quad \forall x \in S, \tag{3.24}$$

where

$$h_F(t,x,\xi) := \min\big\{\langle \xi, v\rangle : v \in F(t,x)\big\}.$$

Then (S,F) is approximately weak invariant.

Proof. We shall point out the required modifications to the proof of Theorem 3.1. First we redefine the Euler polygon corresponding to a nonautonomous Cauchy problem

$$\begin{cases} \dot{x} = f(t,x) \\ x(t_0) = x_0 \end{cases}. \tag{3.25}$$

The function $f\colon [t_0, T] \times H \to H$ is assumed to be only measurable in t for every fixed $x \in H$. An Euler polygon corresponding to (3.28) and to a given partition $\pi = \{t_0, t_1, \ldots, t_N = T\}$ is defined recurrently by

$$x(t) = x(t_i) + \int_{t_i}^{t} f\big(\tau, x(t_i)\big)\, d\tau \quad \forall t \in [t_i, t_{i+1}], \quad \forall i \in \{0, 1, \ldots, N-1\}.$$

The conclusion of the nonautonomous version of Lemma 3.1 remains the same. We remark that f_i in (3.14) should be replaced by

$$f_i(t) = f_\Delta(t, x_i) \quad \forall t \in [t_i, t_{i+1}]. \tag{3.26}$$

□

Let us restate Lemma 3.2:

Lemma 3.3. *Suppose that the functions f_δ in Lemma 3.1 (nonautonomous case) are such that*

$$f_\delta(t,x) \in F(t, s_\delta(x) + \delta B) \quad \forall x \in H.$$

Let $x_0 \in S$. Then there exists an Euler polygon $x(\cdot)$ which satisfies $x(0) = x_0$, $d_S\big(x(t)\big) \le \varepsilon \quad \forall t \in [t_0, T]$ and

$$\dot{x}(t) \in F\big(t, x(t) + \varepsilon B\big) \quad \text{a.e. on } [t, T].$$

Proof. The proof is almost the same. We only change (3.19) into

$$f_i(t) \in F(t, s_i + \Delta B) \subseteq F\Big(t, x_i + \frac{\varepsilon}{2} B\Big) \quad \forall i \in \{0, 1, \ldots, N-1\},$$

where $f_i(t)$ was defined in (3.26) and the conclusion follows immediately. To complete the proof of the theorem the only remark we need is that by virtue of (3.1) and the t-measurability of $F(\cdot, x)$ we are able to provide a *t-measurable* selection

$$f_\delta(t,x) \in F\big(t, \bar{s}_\delta(x)\big),$$

such that

$$\big\langle f_\delta(t,x), y_\delta(x) - \bar{s}_\delta(x)\big\rangle \le 2\delta M.$$

□

The link between hamiltonian and tangential type criteria for approximate weak invariance is established via the following theorem.

Theorem 3.3. *Suppose that F satisfies hypothesis* (H) *then the following are equivalent:*

(i) $\overline{\mathrm{co}} F(x) \cap T_S^w(x) \neq \varnothing \quad \forall x \in S$;

(ii) $\overline{\mathrm{co}} F(x) \cap \mathrm{co}\, T_S^w(x) \neq \varnothing \quad \forall x \in S$;

(iii) $h(x,\xi) \le 0 \quad \forall \xi \in N_S^P(x), \quad \forall x \in S$;

(iv) (S,F) *is approximately weak invariant.*

Proof. Clearly (i) $\Longrightarrow$ (ii). To prove (ii) $\Longrightarrow$ (ii) let $x \in S$ be given. By Proposition 2.4 $\mathrm{co}\, T_S^w(x) \subset \big(N_S^P(x)\big)^*$ which together with (ii) implies that $h_{\overline{\mathrm{co}} F}(x,\xi) \le 0 \quad \forall \xi \in N_S^P(x)$. Now (ii) follows from Proposition 2.5. The implication (iii) $\Rightarrow$ (iv) was proven in Theorem 3.1 so there remains to show that (iv) $\to$ (i).

Let $x \in S$ and $\varepsilon_n \downarrow 0$. Since (S,F) is approximately weak invariant there exists a sequence of absolutely continuous functions $\big(x_n(\cdot)\big)$ such that $x_n(0) = x$,

$$\dot{x}_n(t) \in F\big(x_n(t) + \varepsilon_n B\big) \qquad \text{a.e. on } [0,T], \tag{3.27}$$

and

$$d_S\big(x_n(t)\big) \le \varepsilon_n \quad \forall t \in [0,T]. \tag{3.28}$$

According to (3.3), for $\varepsilon_n \in (0,1)$ $x_n(\cdot)$ is Lipschitz of rank M. Let us define a sequence (t_n) by

$$t_n := \sqrt{\varepsilon_n}. \tag{3.29}$$

Then for any sufficiently large n

$$x_n(t) \in B\big(x + (t_n M) \cdot B\big) \quad \forall t \in [0, t_n]. \tag{3.30}$$

Combining the above inclusion with (3.27) we obtain

$$\dot{x}_n(t) \in F\big(x + (\varepsilon_n + t_n M)B\big).$$

Let $\varepsilon > 0$. Since F is upper-semicontinuous then for n large enough

$$F\big(x + (\varepsilon_n + t_n M)B\big) \subseteq F(x) + \varepsilon B.$$

Now, the two last inclusions imply

$$\dot{x}_n(t) \subseteq F(x_0) + \varepsilon B, \quad \text{a.a.} t \in [0, t_n]. \tag{3.31}$$

We have the following inclusions

$$\begin{aligned} v_n := \frac{x_n(t_n) - x}{t_n} &= \frac{1}{t_n} \int_0^{t_n} \dot{x}_n(\tau)\, d\tau \\ &\in \overline{\text{co}}\, \dot{x}_n\big([0, t_n]\big) && \text{(in view of Theorem 2.1)} \\ &\subset \overline{\text{co}}\big(F(x) + \varepsilon B\big) && \text{(in view of 3.26).} \end{aligned}$$

Since v_n is bounded we may suppose without loss of generality that it weakly converges to some $v \in \overline{\text{co}} F(x) + \varepsilon B$ and so $v \in F(x)$ since $\varepsilon > 0$ was arbitrary choosen.

Now in view of (3.28) and (3.29)

$$\frac{1}{t_n} d_S(x + t_n v_n) = \frac{1}{t_n} d_S\big(x_n(t_n)\big) \leq \sqrt{\varepsilon_n}.$$

Hence $v \in T_S^w(x)$ too and the proof is completed. □

We end this section by briefly discussing some immediate properties of approximately weak invariant systems. At first, we note that in Definition 3.1 S is a arbitrary set (not necessary closed). Also, for the same multifunction F we shall talk about approximately weak invariant *sets* rather then systems.

Proposition 3.1. *Let $(S_\alpha)_{\alpha \in A}$ be a collection of approximately weak invariant sets. Then the set*

$$S = \bigcup_{\alpha \in A} S_\alpha$$

is approximately weak invariant too.

The proof is trivial since $d_S(x) \leq d_{S_\alpha}(x) \quad \forall x \in H, \quad \forall \alpha \in A.$

Proposition 3.2. *Let $S \subset H$ be approximately weak invariant. Then $\bar{S}$ is approximately weak invariant too.*

Proof. Let $\varepsilon > 0$, $T > 0$ and $x_0 \in \overline{S}$. Then there exists $z_0 \in S$ such that

$$\|z_0 - x_0\| \leq \frac{\varepsilon}{2}.$$

Since S is approximately weak invariant there exists a $\varepsilon/2$-trajectory for F on $[0, T]$, $z(\cdot)$ such that $z(0) = z_0$ and $d_S\big(z(t)\big) \leq \varepsilon/2 \quad \forall t \in [0, T]$. Let us define a function $x(\cdot)$ by

$$x(t) := z(t) + x_0 - z_0 \quad \forall t \in [0, T].$$

Obviously, $\|x(t) - z(t)\| = \|x_0 - z_0\| \leq \varepsilon/2$. Then $x(0) = x_0$ and $d_S\big(x(t)\big) \leq d_S\big(z(t)\big) + \big\|x(t) - z(t)\big\| \leq \varepsilon \quad \forall t \in [0, T]$. Also $\dot{x}(t) = \dot{z}(t) \in F\big(z(t) + (\varepsilon/2)B\big) \subset F\big(x(t) + \varepsilon B\big)$ which completes the proof. □

4 Approximate strong invariance

Definition 4.1. The system (S, F) is said to be *approximately strong invariant* if for any $\lambda > 0$, any interval $J = [t_0, T]$ and any $x_0 \in S$ there exists $\varepsilon > 0$ (depending on λ, j and x_0) such that any ε-trajectory $x(\cdot)$ of F starting from x_0 satisfies

$$d_S\big(x(t)\big) \leq X \quad \forall t \in [t_0, T].$$

Throughout this section F is supposed to satisfy hypothesis (H) and in addition to be Lipschitz an bounded sets: that is to every bounded set D there corresponds a constant L such that

$$F(x_2) \subseteq F(x_1) + L\|x_1 - x_2\|B \quad \forall x_1, x_2 \in D. \tag{4.1}$$

The following theorem gives a criterion for approximate weak invariance in terms of the upper Hamiltonian H_F (see (1.5)).

Theorem 4.1. *Let F be Lipschitz on bounded sets. Then under hypothesis* (H) *the system* (S, F) *is approximately strong invariant iff*

$$H_F(x, \xi) \leq 0 \quad \forall \xi \in N_S^P(x), \quad \forall x \in S. \tag{4.2}$$

Proof. Let us assume that (4.2) holds. Without loss of generality we may suppose that $\lambda \in (0, 1)$ and $J = [0, 1]$. Let $x_0 \in S$. We recall that there exist $K > 0$ depending on x_0 and T, such that if $\varepsilon \in (0, 1)$ any ε-trajectory for F on $[0, T]$ starting from x_0 lies in KB (see Remark 3.1). Let L be the Lipschitz rank of F corresponding to the set $(2K + \|x_0\| + 4)B$. Consider $\varepsilon > 0$ such that

$$\varepsilon \leq \varepsilon_0 := \min\left(\frac{\lambda}{2}, \frac{\lambda}{4}(e^{LT} - 1)\right). \tag{4.3}$$

Let $x(\cdot)$ be any ε–trajectory for F on $[0, T]$ starting from x_0. As we have already pointed out

$$\big\|x(t)\big\| \leq K \quad \forall t \in [0, T]. \tag{4.4}$$

We have to prove that $x(t) \in S + \lambda B \quad \forall t \in [0, T]$. To this end, let us define a multifunction $\widetilde{F}\colon [0, T] \times H \to H$ measurable in t and continuous in x such that

$$\widetilde{F}(t, x) = \dot{x}(t) + L\big(\|x - x(t)\| + \rho\big)B, \tag{4.5}$$

a.e. on $[0, T]$, $\forall x \in H$ where

$$\rho \in [\varepsilon, \varepsilon_0]. \tag{4.6}$$

Henceforth the steps of the proof are the following:

1. any ε-trajectory to $\widetilde{F}$ starting from x_0 is sufficiently close to $x(\cdot)$;
2. there exists an ε-trajectory $\widetilde{x}(\cdot)$ for $\widetilde{F}$ such that $\tilde{x}(0) = x_0$ and which remains sufficiently close to S too;
3. finally, 1 and 2 will imply that $x(t) \in S + \lambda B \quad \forall t \in [0, T]$.

Let us make rpecise this ideas. □

Claim 1. Let $\tilde{x}(\cdot)$ be an ε-trajectory for $\widetilde{F}$ on $[0,T]$ with $\tilde{x}(0) = x_0$. Then $\|\tilde{x}(t) - x(t)\| \le \lambda/2 \quad \forall t \in [0,T]$.

Proof of Claim 1. Clearly $\tilde{x}(0) - x(0) = 0$ and by the definition of $\widetilde{F}$

$$\dot{\tilde{x}}(t) - \dot{x}(t) \in L\big(\|\tilde{x}(t) - x(t)\| + \varepsilon + \rho\big)B \text{ a.e. on } [0,T].$$

Hence $\|\dot{\tilde{x}}(t) - x(t)\| \le L\|\tilde{x}(t) - x(t)\| + (\varepsilon+\rho)L$ and by Gronwall's Lemma, (4.3) and (4.6)

$$\|\tilde{x}(t) - x(t)\| \le (\varepsilon+\rho)(e^{LT} - 1) \le 2\varepsilon_0(e^{LT}-1) \le \frac{\lambda}{2} \quad \forall t \in [0,T].$$

□

We also note that (4.4) implies

$$\|\tilde{x}(t)\| \le K + 1 \quad \forall t \in [0,T]. \tag{4.7}$$

Claim 2. There exists an ε-trajectory for $\widetilde{F}$ on $[0,T]$ starting from x_0 such that $d_S\big(\tilde{x}(t)\big) \le \varepsilon \le \lambda/2 \quad \forall t \in [0,T]$.

Proof of Claim 2. We first remark that $\widetilde{F}$ satisfies hypothesis (H') and according to Theorem 3.3 a sufficient condition for $(S, \widetilde{F})$ to be approximately weak invariant is

$$h_{\widetilde{F}}(t,s,\xi) \le 0 \quad \forall \xi \in N_S^P(s), \quad \text{a.a.} t \in \mathbb{R} \text{ and } \forall s \in S. \tag{4.8}$$

However we don't need the system $(S,\widetilde{F})$ to be approximately weak invariant but to prove the existence of an ε-trajectory $\tilde{x}(\cdot)$, remaining in $S + \lambda B$ for *fixed* x_0 and T. In addition, Claim 1 gives an apriori bound for $\tilde{x}(\cdot)$. Thus condition (4.8) should be verified *only* for $t \in [0,T]$ and $x \in \big(2(K+2) + \|x_0\|\big)B$. Indeed, we recall that in Theorem 3.1 (and similarly in Theorem 3.3), (4.8) was invoked only for points $\bar{s}_\delta(\tilde{x}_0) \in S$ satisfying $\|\bar{s}_\delta(\tilde{x}_i) - s_\delta(\tilde{x}_i)\| \le 1$ where $\|s_\delta(\tilde{x}_i) - \tilde{x}_i\| \le d_S(\tilde{x}_i) + 1$ and $\tilde{x}_i \in \big\{\tilde{x}(t) \colon t \in [0,T]\big\}$. Consequently

$$\begin{aligned}
\|\bar{s}_\delta(\tilde{x}_i)\| &\le \|\tilde{x}_i\| + d_S(\tilde{x}_i) + 2 \le \|\tilde{x}_i\| + \|\tilde{x}_i - x_0\| + 2 \\
&\le 2\|\tilde{x}_i\| + \|x_0\| + 2 \\
&\le 2(K+2) + \|x_0\| \qquad\qquad \text{(in view of (4.7))}.
\end{aligned}$$

Now in accordance with (4.5)

$$\begin{aligned}
h_{\widetilde{F}}(t,x,\xi) &= \min_{u \in B} \big\langle \xi, \dot{x}(t) + L\big(\|x - x(t)\| + \rho\big)u \big\rangle \\
&= \big\langle \xi, \dot{x}(t) \big\rangle - L\|\xi\|\big(\|x - x(t)\| + \rho\big) \\
&\le \max\langle \xi, v\rangle - L\|\xi\|\big(\|x - x(t)\| + \rho\big) \text{ for } v \in F\big(x(t) + \varepsilon B\big), \\
&\qquad \forall \xi \in N_S^P(x), \quad \text{a.a.} t \in [0,T], \quad \forall x \in H.
\end{aligned}$$

Since $x \in \big(2K + \|x_0\| + 4\big)B$, $x(t) \in KB$ an F is Lipschitz of rank L on $\big(2K + \|x_0\| + 4\big)B$,

$$F\big(x(t) + \varepsilon B\big) \subseteq F(x) + L\big(\|x - x(t)\| + \varepsilon\big)B.$$

Hence we obtain

$$h_{\widetilde{F}}(t,x,\xi) \le \max_{v \in F(x)} \langle \xi, v\rangle + L\|\xi\|(\varepsilon - \rho) \le H_F(x,\xi) \le 0,$$

$\forall \xi \in N_S^P(x)$, a.e.$t \in [0,T]$ and $\forall x \in \big(2K + \|x_0\| + 4\big)B$ which completes the proof of the claim. □

Finally combining the two claims we obtain $x(t) \in S + \lambda B \quad \forall t \in [0,T]$ as required.

The sufficiency is part of Theorem 4.2 below which subsumes several criteria for approximate weak invariance to prove it we need the following density result.

Proposition 4.1. *Assume that F satisfies hypothesis* (H) *and that for a given $\varepsilon \in (0,1)$ the map $x \to F(x+\varepsilon B)$ is upper semicontinuous. Let $x(\cdot)$ be an ε-trajectory for $\overline{\text{co}}F$ on $[0,T]$. Then for any $\gamma > 0$ there exists a $(2\varepsilon+\gamma)$-trajectory for F, $z(\cdot)$ such that $z(0) = x(0)$ and*

$$\big\|x(t) - z(t)\big\| < \gamma \quad \forall t \in [0,T].$$

Proof. By assumption

$$\dot{x}(t) \in (\overline{\text{co}}F)\big(x(t) + \varepsilon B\big) \text{ for a.a.} t \in [0,T]. \tag{4.9}$$

Let us define the multifunction $\widetilde{F}$ by

$$\widetilde{F}(x) = \{1\} \times F(x + 2\varepsilon B), \tag{4.10}$$

and the set Ω by

$$\Omega := \big\{(\tau, x(\tau)) \colon \tau \in [0,T]\big\}.$$

Let $t \in [0,T]$ then

$$\{1\} \times \overline{\text{co}}\big((\overline{\text{co}}F)\big(x(t) + 2\varepsilon B\big)\big) \cap T^w_\Omega\big(t, x(t)\big) \neq \varnothing. \tag{4.11}$$

Indeed, let $t \in [0,T)$. Because $x(\cdot)$ is Lipschitz of rank M (see Remark 3.1), for some (h_n), $h_n \downarrow 0$ the sequence $\big(x(t+h_n) - x(t)\big)/h_n$ is bounded so we may suppose without loss of generality that

$$\left(1, \frac{x(t+h_n) - x(t)}{h_n}\right) \to v.$$

In view of (   ), $v \in T^w_\Omega\big(t, x(t)\big)$. Also, for n large enough

$$x(t+h) \in x(t) + \varepsilon B \quad \forall h \in [0, h_n]. \tag{4.12}$$

From (4.9) and Corollary 2.1 it follows that

$$\begin{aligned}\frac{x(t+h_n) - x(t)}{h_n} &\in \overline{\text{co}}\big((\overline{\text{co}}F)\big(x\big([t, t_{h_n}]\big) + \varepsilon B\big) \\ &\subseteq \overline{\text{co}}\big((\overline{\text{co}}F)\big(x(t) + 2\varepsilon B\big)\big) \qquad\qquad \text{(by (4.12))}.\end{aligned}$$

If $t = T$, we consider the sequence $\big(x(t_n - h_n) - x(t)\big)/h_n$ to obtain the same conclusion. This (4.11) is proven. Now, we claim that

$$\overline{\text{co}}\big((\overline{\text{co}}F)\big(x(t) + 2\varepsilon B\big)\big) = \overline{\text{co}}\big(F\big(x(t) + 2\varepsilon B\big)\big). \tag{4.13}$$

Clearly $(\overline{\text{co}}F)\big(x(t) + 2\varepsilon B\big) \subset \overline{\text{co}}\big(F\big(x(t) + 2\varepsilon B\big)\big)$ so the inclusion $\subseteq$ holds. Also $F\big(x(t) + 2\varepsilon B\big) \subseteq (\overline{\text{co}}F)\big(x(t) + 2\varepsilon B\big)$ so the other inclusion holds also true. Now, in view of (4.10), (4.11) et (4.13)

$$\overline{\text{co}}\widetilde{F}(x) \cap T^w_\Omega(x) \neq \varnothing \quad \forall x \in \Omega,$$

$\widetilde{F}$ satisfies hypothesis (H), so according to Theorem 3.2 $(\Omega, \widetilde{F})$ is approximately weak invariant.

This means that for any $\gamma > 0$ there exists a γ-trajectory for $\widetilde{F}, t \mapsto \big(t, z(t)\big)$ such that $z(0) = x(0)$ and

$$d_\Omega\big(t, z(t)\big) \leq \frac{\gamma}{2\max(1, M)} \quad \forall t \in [0,T].$$

By the definition of $\widetilde{F}$, $z(\cdot)$ is a $(2\varepsilon+\gamma)$-trajectory for F on $[0,T]$. It also readily follows that $\big\|x(t) - z(t)\big\| \leq \gamma \quad \forall t \in [0,T]$ as required. ☐

Corollary 4.1. *If* $(s, \overline{\text{co}}F)$ *is approximately weak invariant then* (S, F) *is as well.*

Proof. Let $\varepsilon > 0$, $x_0 \in S$, $T > 0$ and $x(\cdot)$ be an $\varepsilon/3$-trajectory for $\overline{\text{co}}F$, such that $x(0) = x_0$ and

$$d_S\big(x(t)\big) \le \frac{\varepsilon}{3} \quad \forall t \in [0, T]. \tag{4.14}$$

Then Proposition 4.1 is true in particular for $\gamma = \varepsilon/3$. Thus there exists an ε-trajectory for $F, z(\cdot)$, such that $z(0) = x(0)$ and

$$\big\|x(t) - z(t)\big\| < \frac{\varepsilon}{3} \quad \forall t \in [0, T]. \tag{4.15}$$

Hence, since $d_S(\cdot)$ is Lipschitz of rank 1, the conclusion follows from (4.14) and (4.15). □

Theorem 4.2. *Suppose that* F *satisfies hypothesis* (H) *and is Lipschitz on bounded sets. Then the following are equivalent*:

(i) $F(x) \subseteq T_S^c \quad \forall x \in S$;

(ii) $\overline{\text{co}}F(x) \subseteq T_S(x) \quad \forall x \in S$;

(iii) $F(x) \subseteq T_S(x) \quad \forall x \in S$;

(iv) $F(x) \subseteq \text{co}\, T_S(x) \quad \forall x \in S$;

(v) $F(x) \subseteq T_S^w(x) \quad \forall x \in S$;

(vi) $F(x) \subseteq \text{co}\, T_S^w(x) \quad \forall x \in S$;

(vii) $H_F(x, \xi) \le 0 \quad \forall \xi \in N_S^P(x), \forall x \in S$;

(viii) (S, F) *is approximately strong invariant.*

Proof. First we shall omit (v) from the chain of implications clearly (i) $\Longrightarrow$ (ii) from (2.13) and the fact that T_S^c is closed and convex (see [4]). The implications (ii) $\Longrightarrow$ (iii) $\Longrightarrow$ (iv) $\Longrightarrow$ (vi) are obvious; (vi) $\Longrightarrow$ (vii) is a consequence of Proposition 2.4 and (vii) $\Longrightarrow$ (viii) is the if-part of Theorem 4.1. We proceed to prove (viii) $\Longrightarrow$ (ii) and (vii) $\Longrightarrow$ (ii). □

Let $x_0 \in S$ and $v_0 \in \overline{\text{co}}F(x_0)$ be given. According to Michel's selection theorem, there exists a continuous selection $f(x) \in \overline{\text{co}}F(x)$. Moreover, f may be choosen such that $f(x_0) = v_0$ (see for example [14] §2). Obviously f satisfies hypothesis (H) and as in Remark 3.1, if J is a certain interval $[0, T]$, we may replace the linear growth condition (1.2) by $\big\|f(x)\big\| \le M$. Then for each $\varepsilon \in (0, 1)$ there exists an Euler-polygon for f on $[0, T]$, $x_\varepsilon(\cdot)$ such that $x_\varepsilon(0) = x_0$ and

$$\dot{x}_\varepsilon(t) \in f\big(x_\varepsilon(t) + \varepsilon B\big) \subseteq \overline{\text{co}}F\big(x_\varepsilon(t) + \varepsilon B\big), \quad \text{a.e. on } [0, T], \tag{4.16}$$

(see Remark 3.2). This means that for each $\varepsilon \in (0, 1)$ there exists an ε-trajectory for $\overline{\text{co}}F$ on $[0, T]$, $x_\varepsilon(\cdot)$. Now let $\lambda_n \downarrow 0$. By assumption (S, F) is approximately strong invariant, so for each n sufficiently large there exists $\varepsilon_n \in (0, \lambda_n/2)$ such that any ε_n-trajectory for F remains in $S+(\lambda_n/2)B$. Since F is Lipschitz on bounded sets, $x \mapsto F(x + \varepsilon B)$ also is, so in view of Proposition 4.1 there exists an ε_n-trajectory for F, $z_n(\cdot)$, such that

$$\big\|x_n(t) - z_n(t)\big\| \le \frac{\varepsilon_n}{2} \quad \forall t \in [0, T], \tag{4.17}$$

where x_n is the $\varepsilon_n/2$-trajectory to $\overline{\text{co}}F$ provided above. From $d_S\big(z_n(t)\big) \leq \lambda_n/2$ and (4.10) it follows that

$$d_S\big(x_n(t)\big) \leq \lambda_n \quad \forall t \in [0,T], \text{ for sufficiently large } n. \tag{4.18}$$

Let $\varepsilon > 0$ and define a sequence (t_n) by $t_n := \sqrt{\lambda_n}$.

We recall that $x_n(t) \in f\big(x_n(t) + (\varepsilon_n/2)B\big)$. Now we proceed as in the proof (iv) $\implies$ (i) in Theorem 3.2, with f instead of F to obtain

$$v_n := \frac{x_n(t_n) - x_0}{t_n} \in f(x_0) + \varepsilon B \text{ for } n \text{ large enough.}$$

Since (v_n) is bounded we may suppose that v_n converges weakly to some $f(x_0)+\varepsilon B$. This means that v_n strongly converges to v_0 because $\varepsilon > 0$ was arbitrary choosen and $f(x_0) = v_0$. Now, in view of (4.10)

$$\frac{1}{t_n} d_S(x_0 + t_n v_n) = \frac{1}{t_n} d_S\big(x_n(t_n)\big) \leq \sqrt{\lambda_n}$$

since $t_n = \sqrt{\lambda_n}$. Thus $v_0 \in T_S(x_0)$ which completes the proof.

Now let us prove that (vi) $\implies$ (i). To this end we recall formula (2.11)

$$T_S^c(x) = \big\{\xi \colon \xi = w - \lim \xi_n, \xi_n \in N_S^P(x_n), (x_n) \subset S, x_n \to x\big\}^*.$$

Let $x \in S$, $v \in F(x)$ and $\xi_n \to \xi$ where $\xi_n \in N_S^P(x_n)$, $(x_n) \subset S$ and $x_i \to x$. We want to prove that $\langle v, \xi \rangle \leq 0$.

Let L be the Lipschitz constant corresponding to some neighborhood of x. Then for n big enough

$$v \in F(x) \subseteq F(x_n) + L\|x - x_n\| B.$$

Thus there exists $v_n \in F(x_n)$ such that $v \in v_n + L\|x - x_n\|B$ which implies that

$$\langle v, \xi_n \rangle \leq \langle v_n, \xi_n \rangle + L\|x - x_n\| \, \|\xi_n\|. \tag{4.19}$$

By assumption $H_F(x, \xi) \leq 0$, so $\langle v_n, \xi_n \rangle \leq 0$ and

$$\langle v, \xi_n \rangle \leq L\|x - x_n\| \, \|\xi_n\|.$$

The conclusion follows by passing to limit in the above inequality.

Finally, the equivalence with (v) may be obtained via (i) $\implies$ (v) $\implies$ (viii) $\implies$ (i).

Corollary 4.2. *If $(S, \overline{\text{co}}F)$ is approximately weak invariant then (S, F) is as well.*

Proof. Let $\varepsilon > 0$, $x_0 \in S$, $T > 0$ and $x(\cdot)$ be an $\varepsilon/3$-trajectory for $\overline{\text{co}}F$, such that $x(0) = x_0$ and

$$d_S\big(x(t)\big) \leq \frac{\varepsilon}{3} \quad \forall t \in [0,T]. \tag{4.20}$$

The Proposition 4.1 is true in particular for $\gamma = \varepsilon/3$. Thus there exists an ε-trajectory for $F, z(\cdot)$, such that $z(0) = x(0)$ and

$$\big\|x(t) - z(t)\big\| < \frac{\varepsilon}{3} \quad \forall t \in [0,T]. \tag{4.21}$$

Hence, since $d_S(\cdot)$ is Lipschitz of rank 1, the conclusion follows from (4.20) and (4.21). $\square$

5 Weak and strong invariance under compactness conditions

In this section we intend to obtain exact invariance results from the approximate ones by a limiting process.

An important device in deriving exact solutions for the differential inclusion (1.1) is *compactness of trajectories* ([2], [21]) which, in our setting could be formulated as follows:

Theorem 5.1. *Suppose that F satisfies* (H) *and has closed, convex values for each $n \in \mathbb{N}$, let $\varepsilon_n > 0$ and $x_n(\cdot)$ be an ε_n-trajectory for F on J with initial value x_0. Suppose also that if $\varepsilon_n \downarrow 0$, x_n converges uniformly to some $x \in \mathcal{C}_H(J)$ i.e.*

$$\|x_n(\cdot) - x(\cdot)\|_0 \to 0 \text{ as } n \to \infty. \tag{5.1}$$

Then $x(\cdot)$ is a trajectory for F on J.

We note that since we explicitely assume (5.1) (which also implies the weak convergence in the reflexive Banach space $L^2_H(J)$) the proof of the theorem is the same as in the finite dimensional case. Obviously, the convergence in (5.1) is obtained via Ascoli-Arzela theorem which in infinite-dimensions needs extra-assumptions to hold true. Namely, for a bounded and equicontinuous family $\{x_n \in \mathcal{C}_H(J), n \geq 1\}$ to be relatively compact it is also necessary that the set

$$\{x_n(t) \mid n \geq 1\} \text{ is relatively compact } \forall t \in J. \tag{5.2}$$

To insure (5.2) we will assume that ? (see for example [14]).

(CH) There exists $k > 0$ such that for any bounded set E, $E \subset H$

$$\alpha\big(F(E)\big) \leq k\alpha(E),$$

where α is the *Kuratwoski measure* of *noncompactness* which is a map from the bounded subsets of H to $\mathbb{R}_+$ defined by

$$\alpha(D) := \inf\{d > 0 \mid D \text{ admits a finite cover by sets of diameter } \leq d\}.$$

We remark that if F satisfies (CH), it necessary has relatively compact values since $\alpha\big(\{x\}\big) = 0$. Using the properties of α and some technical results in ([14], §9) one can immediately derive the

Proposition 5.1. *Suppose that F satisfies* (H) *and* (CH)*.*

For each $n \geq 1$ let $\varepsilon_n \in (0,1)$ and $x_n(\cdot)$ be an ε_n-trajectory for F on J with initial value x_0. Then if $\varepsilon_n \downarrow$), the set $\{x_n(t) \mid n \geq 1\}$ is relatively compact and consequently, on a subsequence

$$\|x_n(\cdot) - x(\cdot)\|_0 \to 0 \text{ as } n \to infty.$$

The next theorem establishes the equivalence between the hamiltonian criterion (iii), the classical tangential criterion (i) and the apparently weaker one (ii).

Theorem 5.2. *Let F be a closed convex valued multifunction satisfying* (H) *and* (CH)*. Then the following are equivalent:*

(i) $F(x) \cap T_S(x) \neq \varnothing$;

(ii) $F(x) \cap \operatorname{co} T_S(x) \neq \varnothing$;

(iii) $h(x, \xi) \leq 0 \quad \forall \xi \in N_S^P(x), \forall x \in S$;

(iv) (S, F) *is weak invariant.*

Proof. Obviously (i) $\Longrightarrow$ (ii) and (ii) $\Longrightarrow$ (iii) because by (2.13) and Proposition 2.2 $\operatorname{co} T_S(x) \subset \left(N_S^P(x)\right)^*$. Now according to Theorem 3.1 (iii) is a sufficient condition for (S, F) to be approximately weak invariant. Then for any $x_0 \in S$, and $n \in \mathbb{N}$. There exists $x_n \in \mathcal{C}_H(J)$ $1/n$-trajectory for F on $[0, T]$ such that $x_n(0) = x_0$ and $x_n(t) \in S + 1/nB$ for all $t \in [0, T]$. Now by Proposition 5.1 we may suppose that $x_n \longrightarrow x \in \mathcal{C}_H(J)$. Hence (iv) is a consequence of Theorem 5.1. It remains to prove that (iv) $\Longrightarrow$ (i). Let $x_0 \in S$ and $\varepsilon_n \downarrow$). Then for each n there exists a trajectory for F on $[0, \varepsilon_n]$, $x_n(\cdot)$ such that $x_n(0) = x_0$ and $x_n(t) \in S \quad \forall t \in [0, \varepsilon_n]$. Since each x_n is Lipschitz of rank M, and F is upper-semicontinuous, for each $\varepsilon_n > 0$ there exists $t_n \in (0, \varepsilon_n)$ such that

$$F\big(x_n(t)\big) \subseteq F(x_0) + \varepsilon_n B \quad \forall t \in [0, t_n].$$

Then, according to Theorem 2.3 and the above inclusion

$$\begin{aligned} v_n := \frac{x_n(t_n) - x_0}{t_n} &= \frac{1}{t_n} \int_0^{t_n} \dot{x}_n(t)\, dt \in \overline{\operatorname{co}} \dot{x}_n\big([0, t_n]\big) \\ &\subseteq F(x_0) + \varepsilon_n B. \end{aligned}$$

Now, since $F(x_0)$ is compact (due to (CH)), (v_n) converges to some $v \in F(x_0)$. On the other hand, according to the definition of the contingent cone $v \in T_S(x_0)$ and we obtained the required conclusion. $\square$

Let us turn to the strong invariance problem. The next result subsumes several types of criteria for strong invariance. We note the presence of the apparently weaker criteria (given in terms of the weak contingent cone) (iv) and (v).

Theorem 5.3. *Let F be Lipschitz on bounded sets.*

Suppose also that F satisfies hypothesis (H) and has convex compact values. Then the following are equivalent.

(i) $F(x) \subseteq T_S^c(x) \quad \forall x \in S$;

(ii) $F(x) \subseteq T_S(x) \quad \forall x \in S$;

(iii) $F(x) \subseteq \operatorname{co} T_S(x) \quad \forall x \in S$;

(iv) $F(x) \subseteq T_S^w(x) \quad \forall x \in S$;

(v) $F(x) \subseteq \operatorname{co} T_S^w(x) \quad \forall x \in S$;

(vi) $H(\xi, x) \leq 0 \quad \forall \xi \in N_S^P(x), \forall x \in S$;

(vii) (S, F) is strongly invariant.

Proof. For the time being we omit (iv) from the chain of implications; (i) $\Longrightarrow$ (ii) $\Longrightarrow$ (iii) $\Longrightarrow$ (v) are obvious; (v) $\Longrightarrow$ (vi) follows from Proposition 2.2; to prove (vi) $\Longrightarrow$ (vii) let $x_0 \in S$, $T > 0$ and $x(\cdot)$ be a trajectory for F on $[0, T]$ with $x(0) = x_0$. Then clearly for any $\varepsilon > 0$, $x(\cdot)$ is an ε-trajectory for F. Since F is Lipschitz we can invoke Theorem 4.1 which says that (vi) is a sufficient condition for (S, F) to be approximately strong invariant. Thus $x(t) \in S + \lambda B \quad \forall t \in [0, T]$ and $\forall \lambda > 0$. Consequently $x(t) \in S$, $\forall t \in [0, T]$ is required.

Now it remains to prove (vii) $\Longrightarrow$ (ii) because (vi) $\Longrightarrow$ (i), which is the implication (vii) $\Longrightarrow$ (i) in Theorem 4.2, closes the chain. To this end let $x_0 \in S$ and $v_0 \in F(x_0)$ be given. According to Michel's selection theorem there exists a continuous selection $f(x) \in F(x)$ such that $f(x_0) = v_0$. Let us consider some neighbourhood of x_0, $\Omega := x_0 + \delta B$, $\delta > 0$, and a function $\widetilde{F}$ defined for example by

$$\widetilde{F}(x) := \begin{cases} f(x) \text{ if } x \in \operatorname{int} B \\ \overline{\operatorname{co}}\big(f(x), 0\big) \text{ if } x \notin \operatorname{int} B \end{cases}$$

where T is some positive constant. By assumption (S, F) is strongly invariant, so $d_S\big(x(t)\big) = 0 \quad \forall t \in [0, T]$. Now we proceed as in (viii) $\Longrightarrow$ (ii) in Theorem 4.2 to obtain

$$v_0 = f(x_0) \in T_S(x_0).$$

Finally the equivalence with (iv) follows via (i) $\Longrightarrow$ (iv) $\Longrightarrow$ (vi) $\Longrightarrow$ (i) and the proof of the Theorem is completed. □

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