Shape analysis via distance functions*

M. C. Delfour† ‡ J.-P. Zolésio§

CRM-2299

sAugust 1995

*Submitted to J. Functional Analysis.

†The research of the first author has been supported in part by National Sciences and Engineering Research Council of Canada operating grant A-8730 and by a FCAR grant from the Ministère de l'Éducation du Québec.

‡Centre de recherches mathématiques, Université de Montréal, C. P.6128, succ. Centre-ville, Montréal QC H3C 3J7. delfour@@crm.UMontreal.ca

§CNRS Institut Non Linéaire de Nice, 1361 route des lucioles, 06904 Sophia Antipolis Cedex, France.zolesio@@cma.cma.fr

Abstract

We expand and sharpen earlier results on the *oriented boundary* (resp. *signed* or *algebraic*) distance function b_Ω of a domain Ω in the N-dimensional Euclidean space $\mathbb{R}^N$. This function is particularily interesting in Shape Analysis where it provides a global access to the fine geometric properties of the boundary of the domain.

The set of points of singularity of the gradient of b_Ω can be divided into its *skeleton* and its *set of cracks*. The interesting range of domains are those for which b_Ω lies between $W^{1,p}$ and $C^{1,1}$, that is from no assumption on the domain Ω to domains with a $C^{1,1}$ boundary. In this spirit we consider a number of asumptions on the matrix D^2b of second order derivatives of b_Ω. It contains information on the skeleton and the curvatures since, in the smooth case, the eigenvalues of this matrix are 0 and the $N-1$ principal curvatures of the boundary.

By using ideas from Geometric Measure Theory, we introduced sets with *Bounded Global Curvature*. They are characterized by the fact that D^2b is a matrix of Radon measures. This large family contains convex and locally semiconvex sets. In this paper we present new local conditions which essentially yield the same type of compactness results as the global ones. Finally an application is given to the continuity of the solution of the generic homogeneous Dirichlet problem.

Résumé

On développe et raffine les résultats précédents sur la *fonction distance orientée* (resp. *algébrique* ou *signée*) b_Ω d'un domaine Ω de l'espace euclidien $\mathbb{R}^N$ de dimension N. Cette fonction présente un intérêt particulier en analyse de forme car elle donne accès aux propriétés géométriques fines de la frontière du domaine.

L'ensemble des points singuliers du gradient de b_Ω peut être divisé entre le *squelette* et l'*ensemble des fissures*. La plage intéressante de domaines correspond à ceux pour lesquels b_Ω se situe entre $W^{1,p}$ et $C^{1,1}$, c'est-à-dire, d'aucune hypothèse sur le domaine à un domaine dont la frontière est de classe $C^{1,1}$.

Dans cette optique on considère un certain nombre d'hypothèses sur la matrice D^2b des dérivées secondes de b_Ω. Cette dernière contient des renseignements sur le squelette et les courbures puisque dans le cas régulier ses valeurs propres sont les $N-1$ courbures principales de la frontière et zéro.

En reprenant des idées de la *Théorie de la mesure géométrique*, on introduit les domaines à *Courbure globale bornée*. Ils sont caractérisés par le fait que D^2b est une matrice de mesures de Radon. Cette grande famille contient les ensembles convexes et locallement semiconvexes. Dans cet article on présente de nouvelles conditions locales plus faibles qui donnent essentiellement les mêmes résultats de compacité qu'avec les globales. Finalement une application est donnée à la continuité de la solution du problème générique de Dirichlet homogène.

1 Introduction

The connection between geometric properties of sets and the properties of certain set-parametrized functions is important in problems where the geometry is a key variable in the modelling, optimization or control. In our previous paper ([13]) we discussed several set-parametrized functions which can be used in Shape Analysis and Optimization. One of them, the *oriented boundary* (resp. *signed* or *algebraic*) distance function b_Ω of a subset Ω of the N-dimensional Euclidean space $\mathbb{R}^N$, is particularly interesting since it readily provides a global approach to fine geometric properties of sets.

Among the many interesting properties of b_Ω is the fact that for $C^{1,1}$ or smoother domains the regularity of the boundary is completely characterized by the regularity of the function b_Ω in a neighbourhood of the boundary $\partial\Omega$. Since b_Ω and its derivatives contain all the information on the set Ω and its boundary, the question of characterizing domains which are less smooth naturally arises.

For domains with a C^2 boundary the oriented boundary distance function has already received a lot of attention in the context of equations of mean curvature type (cf. [18]). We have also developped a complete intrinsic tangential differential calculus on C^2 $(N-1)$-dimensional submanifolds of $\mathbb{R}^N$ without using Christoffel symbols. This has led to an original application to the *theory of shells* (cf. [14], [15], [16]). But there seems to be deeper implications of the study of the oriented boundary distance function in *Differential Geometry* and some key results can be reformulated in that framework.

In general the singularities of the gradient of b_Ω can be classified in two categories: the *skeleton* and the *set of cracks*. This last notion would have potentially interesting applications in image processing and in the theory of shading. Several topologies on subsets of $\mathbb{R}^N$ can be considered as the degree of differentiability of b_Ω is increased. The region which remains to carefully explore lies between $W^{1,p}$ and $C^{1,1}$, that is from no asumption on the domain Ω to domains with a $C^{1,1}$ boundary. This means that we are looking at the existence, the properties and the characterization of the matrix D^2b of second order derivatives of b_Ω. We know that in the smooth case the eigenvalues of this matrix are 0 and the $N-1$ principal curvatures of the corresponding level curve through the point at which $D^2b(x)$ is evaluated.

By adapting techniques and ideas from Geometric Measure Theory, new families of subsets were introduced in [13]: sets with *Bounded Global Curvature*. They are characterized by the fact that D^2b is a matrix of Radon measures whose norm contains the $(N-1)$-Hausdorff measure on the set of singularities of the gradient of the function. This is quite a large family which contains convex and locally semiconvex sets. In the convex case the diagonal elements are non-negative Radon measures. For that family of sets compactness theorems were obtained (cf. [13]). In this paper we present new sets of weaker local conditions which essentially yield the same type of compactness results, but do not involve the singularities of the gradient far away from the boundary. This parallels the results in the smooth case where all the information on the smoothness of the boundary is contained in a small neighbourhood around it.

In §2 we recall some basic definitions and results on topologies generated by distance functions. In §3 we sharpen a number of results from [13] for Hölderian domains and show the connection of the Hessian matrix with the second and third fundamental forms. We give a general classification of the singularities of the gradient and introduce families of subsets with *bounded local curvature* along with illustrative examples. We explore domains for which b_Ω belongs to $W^{2,p}$, $1 \leq p < \infty$ in a neighbourhood of the boundary. For such domains the boundary has zero Lebesgue measure. Finally we discuss convex domains and the notion of locally semiconvex domains which naturally include C^2 domains. This is made possible by the fact that the convexity of a set is completely characterized by the convexity of its oriented distance function. In particular all convex sets have bounded local curvature. §4 contains the new compactness theorem for sets with bounded local curvature. Finally as an application we give in §5 a theorem on the continuity of the solution of the generic homogeneous Dirichlet problem in the class of sets for which b_Ω is uniformly locally $W^{2,p}$ and discuss its relationship with other continuity results.

2 Topologies generated by distance functions

In this section we recall a number of recent results which can be found in [13]. They make use of various types of distance functions. Recall that the distance function associated with a non-empty subset A of the N-dimensional Euclidean space $\mathbb{R}^N$ ($N \geq 1$, a finite integer) is defined as

$$d_A(x) = \min_{y \in A} |y - x|, \quad \forall x \in \mathbb{R}^N.$$

We consider subsets Ω of a bounded open *hold-all* D in $\mathbb{R}^N$ with a Lipschitzian boundary. Our results can be extended to unbounded sets by locally restricting to bounded open subsets and using local spaces. We introduce the following families of distance functions

$$C_d^c(D) = \{d_{\complement\Omega} : \Omega \text{ open in } D, \complement\Omega \neq \varnothing\} = \{d_{\complement\Omega} : \Omega \text{ an open subset of } D\} \tag{2.1}$$

since $D \neq \mathbb{R}^N$ and

$$C_b(D) = \{b_\Omega : \Omega \subset \overline{D}, \partial\Omega \neq \varnothing\}, \tag{2.2}$$

where $d_{\complement\Omega}$ is the *complementary distance function* associated with the set $\complement\Omega = \{x \in \mathbb{R}^N : x \notin \Omega\}$ and b_Ω is the *oriented boundary distance function* defined as

$$b_\Omega(x) = d_\Omega(x) - d_{\complement\Omega}(x). \tag{2.3}$$

For an arbitrary subset Ω of D there is not necessarily an open representative in the equivalence class

$$[\Omega]_b = \{\Omega' : b_{\Omega'} = b_\Omega\} = \{\Omega' : \overline{\Omega'} = \overline{\Omega}, \partial\Omega' = \partial\Omega\}.$$

However if it does it is unique and equal to int Ω.

The above families are compact in the space $C^0(\overline{D})$ of continuous functions in $\overline{D}$ and induce complete metric topologies on the associated equivalence classes of sets

$$\rho_{H^c}(\Omega_2, \Omega_1) = \|d_{\complement\Omega_2} - d_{\complement\Omega_1}\|_{C^0(\overline{D})} \tag{2.4}$$

$$\rho_b(\Omega_2, \Omega_1) = \|b_{\Omega_2} - b_{\Omega_1}\|_{C^0(\overline{D})}. \tag{2.5}$$

In the first case we shall speak of the *uniform (or Hausdorff) complementary topology* and in the second case of the *uniform oriented distance topology*.

Since the functions $d_{\complement\Omega}$ and b_Ω are uniformly Lipschitz of constant equal to one, we always have the following pointwise estimates

$$|\nabla d_{\complement\Omega}(x)| \leq 1, \quad \text{and} \quad |\nabla b_\Omega(x)| \leq 1, \quad \text{a.e. in } D.$$

So $d_{\complement\Omega}$ and b_Ω are elements of $W^{1,p}(D)$ for all p, $1 \leq p \leq \infty$. The sets $C_d^c(D)$ and $C_b(D)$ are closed in $W^{1,p}(D)$ for all p, $1 \leq p < \infty$. Therefore they also induce complete metric topologies on the associated equivalence classes of sets

$$\rho_{H^c,p}(\Omega_2, \Omega_1) = \|d_{\complement\Omega_2} - d_{\complement\Omega_1}\|_{W^{1,p}(D)} \tag{2.6}$$

$$\rho_{b,p}(\Omega_2, \Omega_1) = \|b_{\Omega_2} - b_{\Omega_1}\|_{W^{1,p}(D)}. \tag{2.7}$$

In the first case we shall speak of the $W^{1,p}$*-complementary topology* and in the second case of the $W^{1,p}$*-oriented distance topology.*

The characteristic functions of the interior int Ω, closure $\overline{\Omega}$ and boundary $\partial\Omega$ and the complement $\complement\Omega$ of Ω are directly related to the gradient of b_Ω and its positive and negative parts. For almost all $x \in \mathbb{R}^N$,

$$\chi_{\partial\Omega}(x) = 1 - |\nabla b_\Omega(x)| \tag{2.8}$$
$$\chi_{\overline{\Omega}}(x) = 1 - |\nabla d_\Omega(x)| = 1 - |\nabla b_\Omega^+(x)| \tag{2.9}$$
$$\chi_{\overline{\complement\Omega}}(x) = 1 - |\nabla d_{\complement\Omega}(x)| = 1 - |\nabla b_\Omega^-(x)| \tag{2.10}$$
$$\chi_{\text{int }\Omega}(x) = |\nabla d_{\complement\Omega}(x)| = |\nabla b_\Omega^-(x)|, \tag{2.11}$$

where

$$b_\Omega^+(x) = \max\{b_\Omega(x), 0\} = d_\Omega(x) \tag{2.12}$$
$$b_\Omega^-(x) = \max\{-b_\Omega(x), 0\} = d_{\complement\Omega}(x). \tag{2.13}$$

It turns out that the maps

$$b_\Omega \mapsto \chi_{\partial\Omega} : W^{1,p}(D) \to L^p(D) \tag{2.14}$$
$$b_\Omega \mapsto \chi_{\overline{\Omega}} : W^{1,p}(D) \to L^p(D) \tag{2.15}$$
$$b_\Omega \mapsto \chi_{\overline{\complement\Omega}} : W^{1,p}(D) \to L^p(D) \tag{2.16}$$
$$d_{\complement\Omega} \mapsto \chi_{\text{int }\Omega} : W^{1,p}(D) \to L^p(D) \tag{2.17}$$

are all uniformly Lipschitz continuous for all p, $1 \le p < \infty$.

In view of the above properties the set

$$C_b^0(D) = \{b_\Omega : \Omega \subset \overline{D},\ \partial\Omega \neq \varnothing \text{ and } \mathrm{m}(\partial\Omega) = 0\} \tag{2.18}$$

is closed in $W^{1,p}(D)$ for all p, $1 \le p < \infty$ (m denotes the N-dimensional Lebesgue measure).

Since b_Ω is Lipschitz, its gradient always exists almost everywhere in $\mathbb{R}^N$. Moreover it has been shown in [13] that whenever it exists

$$|\nabla b_\Omega| = \begin{cases} 1 & x \notin \partial\Omega \\ 0 & x \in \partial\Omega. \end{cases}$$

Thus the set

$$\partial^*\Omega = \{x \in \partial\Omega : \nabla b_\Omega(x)\exists \text{ and } \nabla b_\Omega(x) \neq 0\}.$$

has zero Lebesgue measure. It is to be compared with the *reduced boundary* in the theory of Caccioppoli sets.

3 Smoothness of boundary, curvatures, skeletons, convexity

In addition to providing new topologies on families of subsets of an open hold-all D, the oriented distance function contains all the geometric properties of the set and its boundary.

The smoothness of the boundary is directly related to the smoothness of the distance function in a neighbourhood of the boundary. We give here a slightly improved version over [13] to include the Hölderian case.

Proposition 3.1. *Let Ω be an open subset of $\mathbb{R}^N$ such that $\partial\Omega \neq \varnothing$ and let $X \in \partial\Omega$.*

(i) Let $k \geq 1$ and $0 \leq \lambda \leq 1$. If there exists a neighbourhood $U(X)$ of X such that $b_\Omega \in C^{k,\lambda}(U(X))$, then $\partial\Omega \cap U(X)$ belongs to an $(N-1)$-submanifold of class $C^{k,\lambda}$ in $\mathbb{R}^N$.

(ii) Let $k \geq 2$ and $0 \leq \lambda \leq 1$, or $(k,\lambda) = (1,1)$. If there exists a neighbourhood $U(X)$ of X such that $\partial\Omega \cap U(X)$ belongs to an $(N-1)$-submanifold of class $C^{k,\lambda}$ in $\mathbb{R}^N$, then there exists another neighbourhood $U'(X)$ of X such that $b_\Omega \in C^{k,\lambda}(U'(X))$.

Proof. Part (i) follows from Theorem 5.5 in [13]. The same is true of part (ii) for $k \geq 2$. The case $k = 1$ and $\lambda = 1$ follows from Theorem 5.7(ii) in [13]. □

The converse of part (i) is generally not true for boundaries which are $C^{1,1-\varepsilon}$, $0 < \varepsilon < 1$. Counterexamples can be constructed which show that ∇b_Ω has points of discontinuity in any neighbourhood of $\partial\Omega$. Furthermore, always in the smooth case, the eigenvalues of the Hessian matrix on each level set of b_Ω are the $N-1$ principal curvatures of that level set and 0 since $\nabla b(x)$ is an eigenvector (cf. [18]).

It is also instructive to make the connection between the *second fundamental form* associated with the submanifold $\partial\Omega$ of $\mathbb{R}^N$ and the matrix $D^2 b$ (cf. for instance [2] for definitions and notation). For $N = 3$ consider a bounded open subset A of $\mathbb{R}^2$ and a C^2-mapping

$$(\xi_1, \xi_2) \mapsto \vec{\phi}(\xi_1, \xi_2) : \overline{A} \subset \mathbb{R}^2 \to \Gamma = \vec{\phi}(\overline{A}) \subset \partial\Omega \subset \mathbb{R}^3$$

with the usual assumptions. Define the tangent vectors $(\vec{a}_1, \vec{a}_2)$ and the unit normal vector $\vec{a}_3$

$$\vec{a}_\alpha = \frac{\partial \vec{\phi}}{\partial \xi_\alpha}, \quad \alpha = 1, 2, \quad \vec{a}_3 = \frac{\vec{a}_1 \times \vec{a}_2}{|\vec{a}_1 \times \vec{a}_2|}.$$

The normal vector coincides with our exterior unit normal to the boundary $\partial\Omega$ and since the normal to $\partial\Omega$ can be defined either by Ω or its complement, we can choose Ω such that

$$\vec{a}_3 = -\nabla b_\Omega.$$

Then the elements of the second fundamental form are defined as

$$b_{\alpha\beta} = b_{\beta\alpha} \stackrel{\text{def}}{=} -\vec{a}_\alpha \cdot \frac{\partial}{\partial \beta} \vec{a}_3 = \vec{a}_\alpha \cdot \frac{\partial}{\partial \beta} \nabla b_\Omega = \vec{a}_\alpha \cdot D^2 b_\Omega \, \vec{a}_\beta.$$

In view of the fact that $D^2 b \, \nabla b = 0$, for all $(\xi_1, \xi_2, \xi_3) \in \mathbb{R}^3$

$$D^2 b_\Omega \, \vec{\xi} = \xi_1 D^2 b_\Omega \, \vec{a}_1 + \xi_2 D^2 b_\Omega \, \vec{a}_2 + \xi_3 D^2 b_\Omega \, \vec{a}_3 = \xi_1 D^2 b_\Omega \, \vec{a}_1 + \xi_2 D^2 b_\Omega \, \vec{a}_2,$$

where $\vec{\xi} = \xi_1 \, \vec{a}_1 + \xi_2 \, \vec{a}_2 + \xi_3 \, \vec{a}_3$ is an arbitrary vector in $\mathbb{R}^3$. In particular

$$\forall \zeta, \, \xi \in \mathbb{R}^3, \qquad b_{\alpha\beta} \, \xi_\alpha \, \zeta_\beta = D^2 b_\Omega \, \vec{\xi} \cdot \vec{\zeta}$$

and the second fundamental form coincides with the bilinear form generated by $D^2 b_\Omega$. It can also be shown that the *third fundamental form* coincides with the bilinear form generated by $(D^2 b)^2$.

Outside a neighbourhood of the boundary the singularities of the vector field ∇b_Ω^2 coincide with the points x where the projection $p_{\partial\Omega}(x)$ on $\partial\Omega$ is not unique since

$$p_{\partial\Omega}(x) = x - b_\Omega(x) \, \nabla b_\Omega(x) = x - \frac{1}{2} \nabla b_\Omega^2(x) \qquad \text{a.e. in } \mathbb{R}^N.$$

In general for an arbitrary set Ω the singularities can be classified as follows

$$\mathrm{Sk}(\Omega) \stackrel{\text{def}}{=} \{x \in \mathbb{R}^N \,:\, \nabla b_\Omega^2(x) \nexists\} \tag{3.1}$$

$$\mathrm{C}(\Omega) \stackrel{\text{def}}{=} \{x \in \mathbb{R}^N \,:\, \nabla b_\Omega^2(x)\exists,\, \nabla b_\Omega(x) \nexists\}. \tag{3.2}$$

The set $\mathrm{Sk}(\Omega)$ is the *total skeleton* of Ω. The second set is necessarily a subset of the boundary $\partial\Omega$. In summary

$$\mathrm{Sing}\,\nabla b_\Omega \stackrel{\text{def}}{=} \{x \in \mathbb{R}^N \,:\, \nabla b_\Omega(x) \nexists\} = \mathrm{Sk}(\Omega) \cup \mathrm{C}(\Omega)$$
$$\mathrm{Sk}(\Omega) \cap \partial\Omega = \varnothing, \quad \mathrm{C}(\Omega) \subset \partial\Omega,$$

where $\mathrm{m}(\mathrm{Sing}\,\nabla b_\Omega) = \mathrm{m}(\mathrm{Sk}(\Omega)) = \mathrm{m}(\mathrm{C}(\Omega)) = 0$. Similarly

$$\mathrm{Sing}\,\nabla d_{\complement\Omega} \stackrel{\text{def}}{=} \{x \in \mathbb{R}^N \,:\, \nabla d_{\complement\Omega}(x) \nexists\} = \mathrm{Sk}_{\text{int}}(\Omega) \cup \mathrm{C}_{\text{int}}(\Omega)$$

where

$$\mathrm{Sk}_{\text{int}}(\Omega) \stackrel{\text{def}}{=} \{x \in \mathbb{R}^N \,:\, \nabla d_{\complement\Omega}^2(x) \nexists\} \subset \mathrm{int}\,\Omega$$
$$\mathrm{C}_{\text{int}}(\Omega) \stackrel{\text{def}}{=} \{x \in \mathbb{R}^N \,:\, \nabla d_{\complement\Omega}^2(x)\exists,\, \nabla d_{\complement\Omega}(x) \nexists\} \subset \partial\Omega$$

and $\mathrm{m}(\mathrm{Sing}_{\text{int}}\,\nabla b_\Omega) = \mathrm{m}(\mathrm{Sk}_{\text{int}}(\Omega)) = \mathrm{m}(\mathrm{C}_{\text{int}}(\Omega)) = 0$. Also

$$\mathrm{Sing}\,\nabla d_\Omega \stackrel{\text{def}}{=} \{x \in \mathbb{R}^N \,:\, \nabla d_\Omega(x) \nexists\} = \mathrm{Sk}_{\text{ext}}(\Omega) \cup \mathrm{C}_{\text{ext}}(\Omega)$$

where

$$\mathrm{Sk}_{\text{ext}}(\Omega) \stackrel{\text{def}}{=} \{x \in \mathbb{R}^N \,:\, \nabla d_\Omega^2(x) \nexists\} \subset \mathrm{int}\,\complement\Omega$$
$$\mathrm{C}_{\text{ext}}(\Omega) \stackrel{\text{def}}{=} \{x \in \mathbb{R}^N \,:\, \nabla d_\Omega^2(x)\exists,\, \nabla d_\Omega(x) \nexists\} \subset \partial\Omega,$$

and $\mathrm{m}(\mathrm{Sing}_{\text{ext}}\,\nabla b_\Omega) = \mathrm{m}(\mathrm{Sk}_{\text{ext}}(\Omega)) = \mathrm{m}(\mathrm{C}_{\text{ext}}(\Omega)) = 0$. Clearly $\mathrm{Sk}(\Omega)$ is the union of the external and internal skeletons in $\mathbb{R}^N \backslash \partial\Omega$

$$\mathrm{Sk}(\Omega) = \mathrm{Sk}_{\text{int}}(\Omega) \cup \mathrm{Sk}_{\text{ext}}(\Omega).$$

Notice that the set of cracks $C(\Omega)$ is a subset of the boundary $\partial\Omega$ with zero N-dimensional Lebesgue measure while the boundary itself may have non zero measure (cf. [13] for an example). Hence it would be quite natural to ask under what circumstances $C(\Omega)$ and $\mathrm{Sk}(\Omega)$ have finite $(N-1)$-dimensional Hausdorff measures. The notion of skeleton is widespread in the litterature on pattern recognition, robotics, mining engineering, and computational geometry where the term *offsets* is often used (cf. [3, 23, 21, 22, 25, 19], etc...).

In [13] we focussed our attention on subsets Ω of a fixed *hold-all* D verifying the following properties:

$$\nabla b_\Omega \text{ (resp. } \nabla d_\Omega,\, \nabla d_{\complement\Omega}) \in BV(D)^N \tag{3.3}$$

that is to say that the elements of the Hessian matrix of the function b_Ω (resp. d_Ω, $d_{\complement\Omega}$) are bounded measures on D

$$D^2 b_\Omega \text{ (resp. } D^2 d_\Omega,\, D^2 d_{\complement\Omega}) \in M^1(D)^{N^2}. \tag{3.4}$$

The notation $BV(D)$ stands for the space of functions in $L^1(D)$ for which ∇f belongs to $M^1(D)^N$ and $M^1(D) = \mathcal{D}^0(D)'$ is the space of bounded measures on D.

Definition 3.1. Let Ω be a subset of D.

(i) If $\partial\Omega \neq \varnothing$, we say that Ω has *Bounded Global Curvature* in D if there exists a constant $c > 0$ such that

$$\|D^2 b_\Omega\|_{M^1(D)} \leq c. \tag{3.5}$$

(ii) If $\Omega \neq \varnothing$ (resp. $\complement\Omega \neq \varnothing$), we say that Ω has *Bounded Global Exterior (resp. Interior) Curvature* in D if there exists a constant $c > 0$ such that

$$\|D^2 d_\Omega\|_{M^1(D)} \leq c \text{ (resp. } \|D^2 d_{\complement\Omega}\|_{M^1(D)} \leq c) \,, \tag{3.6}$$

where it is understood that the norm is in $M^1(D)^{N\times N}$.

We now relax the global conditions on D to local conditions in a neighbourhood of $\partial\Omega$. For this purpose we introduce the notion of *h-tubular* neighbourhoods.

Definition 3.2. Let Ω be a subset (resp. a bounded subset) of $\mathbb{R}^N$ such that $\partial\Omega \neq \varnothing$.

(i) Given $h > 0$, the *h-tubular neighbourhood* of the boundary $\partial\Omega$ is the open set

$$U_h(\partial\Omega) = \{x \in D \,:\, |b_\Omega(x)| < h\}. \tag{3.7}$$

(ii) We say that the set Ω has *Bounded Local Curvature* if there exists $h = h(\Omega) > 0$ such that

$$\nabla b_\Omega \in BV_{loc}(U_h(\partial\Omega))^N \quad \text{(resp. } BV(U_h(\partial\Omega))^N). \tag{3.8}$$

Definition 3.3. Let Ω be an open subset of $\mathbb{R}^N$ such that $\complement\Omega \neq \varnothing$.

(i) Given $h > 0$, the *h-tubular neighbourhood* of the complement $\complement\Omega$ is the open set

$$U_h(\complement\Omega) = \{x \in D \,:\, d_{\complement\Omega}(x) < h\}. \tag{3.9}$$

(ii) We say that the set Ω has *Bounded Interior Local Curvature* if there exists $h = h(\Omega) > 0$ such that

$$\nabla d_{\complement\Omega} \in BV_{loc}(U_h(\complement\Omega))^N \quad \text{(resp. } BV(U_h(\complement\Omega))^N). \tag{3.10}$$

Definition 3.4. Let Ω be a subset of $\mathbb{R}^N$ such that $\Omega \neq \varnothing$.

(i) Given $h > 0$, the *h-tubular neighbourhood* of Ω is the open set

$$U_h(\Omega) = \{x \in D \,:\, d_\Omega(x) < h\}. \tag{3.11}$$

(ii) We say that the set Ω has *Bounded Exterior Local Curvature* if there exists $h = h(\Omega) > 0$ such that

$$\nabla d_\Omega \in BV_{loc}(U_h(\Omega))^N \quad \text{(resp. } BV(U_h(\Omega))^N). \tag{3.12}$$

All C^2 domains with a compact boundary belong to the three categories. The norms $\|D^2 b_\Omega\|_{M^1(U_h(\partial\Omega))}$, $\|D^2 d_{\complement\Omega}\|_{M^1(U_h(\complement\Omega))}$, and $\|D^2 d_\Omega\|_{M^1(U_h(\Omega))}$ are all decreasing as h goes to zero. The limit is particularly interesting since it singles out the behaviour of the gradient in a shrinking neighbourhood of the boundary $\partial\Omega$.

Example 3.1. If $\Omega \subset R^N$ has a compact boundary which is a C^2 $(N-1)$-submanifold of $\mathbb{R}^N$, then

$$\lim_{h \searrow 0} \|D^2 b_\Omega\|_{M^1(U_h(\partial\Omega))} = 0.$$

Example 3.2. Let $\Omega = \{x_i\}_{i=1}^I$ be I distinct points in R^N. Then $\partial\Omega = \Omega$ and

$$\lim_{h \searrow 0} \|D^2 b_\Omega\|_{M^1(U_h(\partial\Omega))} = \begin{cases} 2\,I - 1, & N = 1 \\ 0, & N \geq 2. \end{cases}$$

Example 3.3. Let Ω be a line in R^N of length $L > 0$, then $\partial\Omega = \Omega$ and

$$\lim_{h \searrow 0} \|D^2 b_\Omega\|_{M^1(U_h(\partial\Omega))} = \begin{cases} 2\,L, & N = 2 \\ 0, & N \geq 3. \end{cases}$$

Example 3.4. Let $N = 2$. For the finite square and the ball of finite radius

$$\lim_{h \searrow 0} \|\Delta d_\Omega\|_{M^1(U_h(\partial\Omega))} = \mathcal{H}^1(\partial\Omega),$$

where $\mathcal{H}^1$ is the one-dimensional Hausdorff measure (cf. [13]).

Also by using $U_h(\Omega)$ and Δb_Ω, we can extract information about the skeleton of Ω.

Example 3.5. Let Ω be the unit square in R^2, then

$$\lim_{h \searrow 0} \|\Delta b_\Omega\|_{M^1(U_h(\Omega))} = \frac{\sqrt{2}}{2}\,\mathcal{H}^1(\mathrm{Sk}_{\mathrm{int}}(\Omega)),$$

where $\mathcal{H}^1$ is the one-dimensional Hausdorff measure and $\mathrm{Sk}(\Omega) = \mathrm{Sk}_{\mathrm{int}}(\Omega)$ is the skeleton of Ω made up of the two interior diagonals (cf. [13]). It seems that in general

$$\lim_{h \searrow 0} \|\Delta b_\Omega\|_{M^1(U_h(\Omega))} = \int_{\mathrm{Sk}(\Omega)} |\,[\nabla b_\Omega] \cdot n|\, dH_1$$

where $[\nabla b_\Omega]$ is the jump in ∇b_Ω and n is the unit normal to $\mathrm{Sk}(\Omega)$ (if it exists!).

In general sets with a bounded local curvature do not have a boundary with zero Lebesgue measure as can be seen from the following example.

Example 3.6. Let B be the open unit ball centered in 0 of $\mathbb{R}^2$ and define

$$\Omega = \{x \in B \,:\, x \text{ with rational coordinates}\}.$$

Then $\partial\Omega = \overline{B}$, $b_\Omega = d_B$ and for all $h > 0$

$$\nabla b_\Omega \in BV(U_h(\partial\Omega))^2$$

and

$$< \Delta b_\Omega, \varphi > \int_{\partial B} \varphi\, dx + \int_{\complement B} \frac{1}{|x|} \varphi\, dx.$$

The Lebesgue measure of the boundary is zero for all Lipschitian subsets Ω of D. It is also true for the following family of subsets of D.

Theorem 3.1. *Let D be a bounded open subset of $\mathbb{R}^N$ and let $\Omega \subset D$ be such that $\partial\Omega \neq \varnothing$. Assume that there exist $h > 0$ and $p \geq 1$ such that*

$$\int_{U_h(\partial\Omega)} |D^2 b_\Omega|^p \, dx < \infty. \tag{3.13}$$

Then

$$|\nabla b_\Omega|^2 = 1 \quad \text{and} \quad \chi_{\partial\Omega} = 0 \quad \text{a.e. in } \mathbb{R}^N \quad \text{and} \quad \mathrm{m}(\partial\Omega) = 0. \tag{3.14}$$

Moreover when $p > N$, then $W^{2,p}(U_h(\partial\Omega)) \subset C^{1,\lambda}(U_h(\partial\Omega))$ for all λ, $0 < \lambda \leq 1 - N/p$ and the boundary of $\partial\Omega$ is Hölderian of class $C^{1,\lambda}$.

Proof. (i) Consider the function $|\nabla b_\Omega|^2$. Since $|\nabla b_\Omega| \leq 1$, then $\nabla b_\Omega \in W^{1,p}(U_h(\partial\Omega))^N \cap L^\infty(U_h(\partial\Omega))^N$ and $|\nabla b_\Omega|^2 \in W^{1,p}(U_h(\partial\Omega)) \cap L^\infty(U_h(\partial\Omega))$. But for almost all x, we know from [13] that $\nabla b_\Omega(x)$ is differentiable,

$$|\nabla b_\Omega(x)| = \begin{cases} 0, & x \in \partial\Omega \\ 1, & x \notin \partial\Omega \end{cases} \tag{3.15}$$

and

$$|\nabla b_\Omega(x)| = 1 - \chi_{\partial\Omega}(x).$$

Necessarily

$$\nabla(|\nabla b_\Omega|^2)(x) = 0 \quad \text{a.e. in } U_h(\partial\Omega)$$

Since $\partial\Omega$ is compact, there exists a finite sequence of distinct points $x_i \in \partial\Omega$, $i \in I$, such that

$$\partial\Omega \subset \bigcup_{i\in I} B(x_i, h).$$

Define the following partition of the set I of indices

$$I_0 = \{i \in \{1, \ldots, n\} \,:\, \exists x \in B(x_i, h),\, \nabla b_\Omega(x) = 0\}$$
$$I_1 = \{i \in \{1, \ldots, n\} \,:\, \exists x \in B(x_i, h),\, \nabla b_\Omega(x) \,\exists \text{ and } |\nabla b_\Omega(x)| = 1\}.$$

Therefore for all $i \in I_0$, $\nabla b_\Omega(x) = 0$ in $B(x_i, h)$ and since b_Ω is Lipschizian and $b_\Omega(x_i) = 0$, then $b_\Omega = 0$ in $B(x_i, h)$ and $B(x_i, h) \subset \partial\Omega$. Similarly for all $i \in I_1$, $|\nabla b_\Omega(x)| = 1$ in $B(x_i, h)$. As a result $\partial\Omega$ can be partionned into two compact parts

$$\partial\Omega = (\partial\Omega)^0 \cup (\partial\Omega)^1$$

since

$$(\partial\Omega)^0 \subset V^0 \overset{\text{def}}{=} \bigcup_{i\in I_0} B(x_i, h) \quad (\partial\Omega)^1 \subset V^1 \overset{\text{def}}{=} \bigcup_{i\in I_1} B(x_i, h)$$

and

$$V^0 \cap V^1 = \varnothing.$$

But for $i \in I_0$

$$B(x_i, h) \subset \partial\Omega \quad \Rightarrow \quad \bigcup_{i\in I_0} B(x_i, h) \subset \partial\Omega$$

and

$$V^0 \subset V^0 \cap \partial\Omega = (\partial\Omega)^0 \subset V^0 \quad \Rightarrow \quad (\partial\Omega)^0 = V^0.$$

Therefore V^0 is both open and closed. It cannot be $\mathbb{R}^N$ since Ω is bounded. So $V^0 = \varnothing$, $|\nabla b(x)| = 1$ in $U_h(\partial\Omega)$, and $\chi_{\partial\Omega} = 0$. This proves that $\mathrm{m}(\partial\Omega) = 0$.

(ii) In view of (3.13), $b_\Omega \in W^{2,p}(U_h(\partial\Omega))$ for $p > N$. Given ε, $0 < \varepsilon < h$, there exists $\rho_{\varepsilon,h} \in \mathcal{D}(\mathbb{R}^N)$ such that

$$\rho_{\varepsilon,h} = \begin{cases} 1, & x \in \overline{U_{h-\varepsilon}(\partial\Omega)} \\ 0, & x \in \overline{U_h(\partial\Omega)\backslash U_{h-\frac{\varepsilon}{2}}(\partial\Omega)}. \end{cases}$$

As a result

$$\rho_{\varepsilon,h}\, b_\Omega \in W_0^{2,p}(U_h(\partial\Omega))$$

and from [1] (cf. Thm 5.4, Part III, p. 98)

$$\begin{aligned} & \rho_{\varepsilon,h} b_\Omega \in C^{1,\lambda}(\,\overline{U_h(\partial\Omega)}\,), \quad 0 < \lambda \le 1 - N/p \\ \Rightarrow\quad & b_\Omega \in C^{1,\lambda}(\,\overline{U_{h-\varepsilon}(\partial\Omega)}\,), \quad 0 < \lambda \le 1 - N/p \\ \Rightarrow\quad & b_\Omega \in C^{1,\lambda}(U_h(\partial\Omega)), \quad 0 < \lambda \le 1 - N/p \end{aligned}$$

since ε can be made arbitrarily small. □

Remark 3.1. In dimension $N = 2$ the condition $b_\Omega \in W^{2,p}(N(\partial\Omega))$ for some bounded neighbourhood $N(\partial\Omega)$ of $\partial\Omega$ is equivalent to $\Delta b_\Omega \in L^p(N(\partial\Omega))$. Recall that $\mathrm{m}(\partial\Omega) = 0$, $|\nabla b_\Omega(x)|^2 = 1$, and $D^2 b_\Omega(x)\, \nabla b_\Omega(x) = 0$. Hence

$$\begin{aligned} \partial^2_{12} b_\Omega(x) = \partial^2_{21} b_\Omega(x) &= -\partial_1 b_\Omega(x)\, \partial_2 b_\Omega(x)\, \Delta b_\Omega(x) \\ \partial^2_{11} b_\Omega(x) &= (\partial_2 b_\Omega(x))^2\, \Delta b_\Omega(x) \\ \partial^2_{22} b_\Omega(x) &= (\partial_1 b_\Omega(x))^2\, \Delta b_\Omega(x). \end{aligned}$$

Now if $\Delta b_\Omega \in M^1(D)$ can we conclude that $\nabla b_\Omega \in BV(D)^2$? We know for instance that for compact convex sets, the Steiner-Minkowski formula still holds and that for C^2-domains

$$\mathrm{m}(B(\Omega, \lambda)) = \mathrm{m}(\Omega) + \mathcal{H}^1(\partial\Omega)\, \lambda + \int_{\partial\Omega} \Delta b_\Omega(x)\, dx\, \frac{\lambda^2}{2}.$$

Somehow the integral in front λ^2 must still make sense for general convex domains with corners. □

Another important property which is characterized by the function b_Ω is the convexity.

Theorem 3.2. *Let Ω be a subset of $\mathbb{R}^N$ such that $\partial\Omega \neq \varnothing$.*

(i) $\overline{\Omega}$ is convex if and only if b_Ω is convex.

(ii) If $\overline{\Omega}$ is convex, then ∇b_Ω and ∇d_Ω belong to $BV_{loc}(\mathbb{R}^N)^N$. $(D^2 b_\Omega)_{ii}$ and $(D^2 d_\Omega)_{ii}$ are non-negative Radon measures. $D^2 b_\Omega(x)$ and $D^2 d_\Omega(x)$ exist almost everywhere and when they exist

$$\begin{aligned} |b_\Omega(y) - b_\Omega(x) - \nabla b_\Omega(x) \cdot (y - x) - \frac{1}{2}(y - x) \cdot D^2 b_\Omega(x)(y - x)| \\ = o(|y - x|^2) \\ |d_\Omega(y) - d_\Omega(x) - \nabla d_\Omega(x) \cdot (y - x) - \frac{1}{2}(y - x) \cdot D^2 d_\Omega(x)(y - x)| \\ = o(|y - x|^2). \end{aligned}$$

Proof. (i) From [13]. (ii) From Aleksandrov's theorem (cf. for instance [17]). □

In particular if Ω is convex and $\partial\Omega$ is of class C^2, then for any $X \in \partial\Omega$, there exists a strictly convex neighbourhood $N(X)$ of X such that

$$b_\Omega \in C^2(N(X)) \tag{3.16}$$

$$\forall x \in N(X), \forall \xi \in \mathbb{R}^N, \quad D^2 b(x)\xi \cdot \xi \geq 0. \tag{3.17}$$

This is related to the notion of *strong elliptic boundary* in shell theory:

$$\exists c > 0, \forall x \in \partial\Omega, \forall \xi \in \mathbb{R}^N, \xi \cdot \nabla b(x) = 0 \quad D^2 b(x)\,\xi \cdot \xi \geq c|\xi|^2$$

(cf for instance [11]). Note that there is no information in vectors ξ's which are normal to $\partial\Omega$. All this motivates the introduction of the following natural notions.

Definition 3.5. Let Ω be a subset of $\mathbb{R}^N$ such that $\partial\Omega \neq \varnothing$.

(i) The set Ω is *locally convex (resp. locally strictly convex)* if for each $X \in \partial\Omega$ there exists a strictly convex neighbourhood $N(X)$ of X such that

$$b_\Omega \text{ is convex (resp. strictly convex) in } N(X).$$

(ii) The set Ω is *semiconvex* if

$$\exists \alpha \geq 0, \quad b_\Omega(x) + \alpha|x|^2 \text{ is convex in } \mathbb{R}^N.$$

(iii) The set Ω is *locally semiconvex* if for each $X \in \partial\Omega$ there exists a strictly convex neighbourhood $N(X)$ of X such that

$$\exists \alpha \geq 0, \quad b_\Omega(x) + \alpha|x|^2 \text{ is convex in } N(X).$$

Remark 3.2. When Ω has a compact C^2 boundary, $D^2 b_\Omega$ is bounded in a bounded neighbourhood of $\partial\Omega$ and Ω is necessarily locally semiconvex. When Ω is a locally semiconvex set, for each $X \in \partial\Omega$ there exists a strictly convex neighbourhood $N(X)$ of X such that $\nabla b_\Omega \in BV(N(X))^N$. If in addition $\partial\Omega$ is compact, then there exists $h > 0$ such that $\nabla b_\Omega \in BV(U_h(\partial\Omega))^N$. □

Remark 3.3. If we fix a constant $\beta > 0$ and consider all the subsets of D which are semiconvex with constant $0 \leq \alpha \leq \beta$, then this set is closed for the uniform and the $W^{1,p}$, $1 \leq p < \infty$, topologies. □

4 Compactness results

In Shape Optimization, compactness theorems are used to establish the existence of optimal domains. In this section we present sharper versions of the compactness theorems from [13] where the global boundedness condition on the generalized curvatures of the level sets of Ω is relaxed to a local condition in a neighbourhood of the boundary of the set Ω.

4.1 $W^{1,p}$-complementary topology

When $\overline{D}$ is compact, $C^c_d(D)$ is compact for the uniform topology and closed in the $W^{1,p}(D)$-topology $(1 \le p < \infty)$. In particular

$$d_{\complement\Omega_n} \to d_{\complement\Omega} \quad \text{in } W^{1,p}(D)\text{-strong}$$

implies that

$$d_{\complement\Omega_n} \to d_{\complement\Omega} \quad \text{in } C^0(\overline{D})\text{-strong} \tag{4.1}$$

$$\chi_{\text{int }\Omega_n} = |\nabla d_{\complement\Omega_n}| \to |\nabla d_{\complement\Omega}| = \chi_{\text{int }\Omega} \quad \text{in } L^p(D)\text{-strong}. \tag{4.2}$$

For the $W^{1,p}(D)$-topology we recall the following compactness theorem from [13].

Theorem 4.1. *Let $\{\Omega_n\}$ be a sequence of open subsets of the fixed open hold-all D. Assume that there exists $c > 0$ such that*

$$\forall n, \quad \|D^2 d_{\complement\Omega_n}\|_{M^1(D)} \;\le\; c. \tag{4.3}$$

Then there exists a subsequence $\{\Omega_{n_k}\}$ and an open subset Ω of D such that for all p, $1 \le p < \infty$,

$$d_{\complement\Omega_{n_k}} \to d_{\complement\Omega} \text{ in } W^{1,p}(D). \tag{4.4}$$

Moreover

$$\|D^2 d_{\complement\Omega}\|_{M^1(D)} \;\le\; \liminf_{n\to\infty} \;\|D^2 d_{\complement\Omega_n}\|_{M^1(D)} \;\le\; c, \tag{4.5}$$

$\chi_\Omega \in BV(D)$, and for all $\varphi \in \mathcal{D}^0(D)^{N\times N}$

$$< D^2 d_{\complement\Omega_n}, \varphi > \to < D^2 d_{\complement\Omega}, \varphi > . \tag{4.6}$$

The global condition (4.3) can be weakened by relaxing it to a local one in a neighbourhood of the boundary of each set of the sequence.

Theorem 4.2. *Let $\{\Omega_n\}$ be a sequence of non-empty open subsets of the fixed hold-all D. Assume that there exist $h > 0$ and $c > 0$ such that*

$$\forall n, \quad \|D^2 d_{\complement\Omega_n}\|_{M^1(U_h(\partial\Omega_n))} \;\le\; c. \tag{4.7}$$

Then there exists a subsequence $\{\Omega_{n_k}\}$ and a subset Ω^* of D, $\partial\Omega^* \neq \varnothing$, such that for all p, $1 \le p < \infty$,

$$b_{\Omega_{n_k}} \to b_{\Omega^*} \text{ in } W^{1,p}(D), \tag{4.8}$$

$$d_{\complement\Omega_{n_k}} \to d_{\complement\Omega} \text{ in } W^{1,p}(D), \quad \Omega \overset{def}{=} \text{int } \Omega^* \tag{4.9}$$

$$\|D^2 d_{\complement\Omega}\|_{M^1(U_h(\partial\Omega^*))} \;\le\; \liminf_{k\to\infty} \;\|D^2 d_{\complement\Omega_{n_k}}\|_{M^1(U_h(\partial\Omega_{n_k}))} \;\le\; c \tag{4.10}$$

and χ_Ω and $\chi_{\complement\Omega} \in BV(U_h(\complement\Omega))$.

Remark 4.1. In the above theorem we could have used the neighbourhood $U_h(\complement\Omega)$ instead of $U_h(\partial\Omega^*)$. Note that $\partial\Omega \subset \partial\Omega^* \subset \complement\Omega$. It must be emphasized that even if the limit set $\Omega^* \subset D$ is not empty since $\partial\Omega^* \neq \varnothing$, the set $\Omega \subset D$ could be empty since $\Omega = \text{int } \Omega^*$. This results from the convergence of the sequence $\{d_{\complement\Omega_n}\}$, $\complement\Omega_n \neq \varnothing$, to $d_{\complement\Omega}$, $\complement\Omega \neq \varnothing$, rather than $\{b_{\Omega_n}\}$, $\partial\Omega_n \neq \varnothing$, to b_{Ω^*}, $\partial\Omega^* \neq \varnothing$. All what is known is that $\Omega \neq \mathbb{R}^N$, which was already true since $\Omega \subset D \neq \mathbb{R}^N$. □

Proof. (i) By assumption $\Omega_n \neq \varnothing$, $\complement\Omega_n \supset \complement D \neq \varnothing$. So the sequences $\{b_{\Omega_n}\}$, $\{d_{\Omega_n}\}$ and $\{\nabla d_{\complement\Omega_n}\}$ are all pointwise uniformly bounded in the bounded hold-all D. There exists a subsequence, still indexed by n, and a set Ω^* in D such that

$$b_{\Omega_n} \to b_{\Omega^*} \text{ and } d_{\complement\Omega_n} \to d_{\complement\Omega^*} \text{ in } C^0(\overline{D})\text{-strong and } H^1(D)\text{-weak.}$$

For all $\epsilon > 0$, $0 < 2\varepsilon < h$, there exists $N > 0$ such that for all $n \geq N$

$$b_{\Omega^*}(x) \leq b_{\Omega_n}(x) + \varepsilon \tag{4.11}$$
$$b_{\Omega_n}(x) \leq b_{\Omega^*}(x) + \varepsilon \tag{4.12}$$
$$d_{\partial\Omega_n}(x) = |b_{\Omega_n}(x)| \leq |b_{\Omega^*}(x)| = d_{\partial\Omega^*}(x) + \varepsilon. \tag{4.13}$$

Therefore

$$U_{h-\varepsilon}(\partial\Omega^*) \subset U_h(\partial\Omega_n) \tag{4.14}$$
$$\Omega^*\backslash U_{h-\varepsilon}(\partial\Omega^*) \subset \Omega_n\backslash U_{h-2\varepsilon}(\partial\Omega_n) \tag{4.15}$$
$$\complement\Omega^*\backslash U_{h-\varepsilon}(\partial\Omega^*) \subset \complement\Omega_n\backslash U_h(\partial\Omega_n). \tag{4.16}$$

The first property is obvious. For the second note that

$$\Omega^*\backslash U_{h-\varepsilon}(\partial\Omega^*) = \{x : b_{\Omega^*}(x) \leq -(h-\varepsilon)\}$$
$$\Omega_n\backslash U_{h-2\varepsilon}(\partial\Omega_n) = \{x : b_{\Omega_n}(x) \leq -(h-2\varepsilon)\}$$

and use (4.12) to obtain (4.15). As for the third one

$$\complement\Omega^*\backslash U_{h-\varepsilon}(\partial\Omega^*) = \{x : b_{\Omega^*}(x) \geq h-\varepsilon\}$$
$$\complement\Omega_n\backslash U_h(\partial\Omega_n) = \{x : b_{\Omega_n}(x) \geq h\}$$

and use (4.11) to obtain (4.16).

From (4.9) and (4.14) for all $n \geq N$

$$\|D^2 d_{\complement\Omega_n}\|_{M^1(U_{h-\varepsilon}(\partial\Omega^*))} \leq c$$

and from Theorem 4.1 there exists a subsequence, still indexed by n, such that

$$d_{\complement\Omega_n} \to d_{\complement\Omega^*} \text{ in } W^{1,1}(U_{h-\varepsilon}(\partial\Omega^*))\text{-strong}$$

and hence in $H^1(U_{h-\varepsilon}(\partial\Omega^*))$-strong. Now for $n \geq N$

$$|\nabla d_{\complement\Omega_n}(x)| = 1 \text{ a.e. in } \Omega_n \supset \Omega_n\backslash U_{h-2\varepsilon}(\partial\Omega_n) \supset \Omega^*\backslash U_{h-\varepsilon}(\partial\Omega^*)$$
$$|\nabla d_{\complement\Omega^*}(x)| = 1 \text{ a.e. in } \operatorname{int}\Omega^* \supset \Omega^*\backslash U_{h-\varepsilon}(\partial\Omega^*).$$

Therefore since $\nabla d_{\complement\Omega_n}$ weakly converges to $\nabla d_{\complement\Omega^*}$

$$\int_{\Omega^*\backslash U_{h-\varepsilon}(\partial\Omega^*)} |\nabla d_{\complement\Omega_n} - \nabla d_{\complement\Omega^*}|^2\, dx \to 0 \text{ as } n \to \infty.$$

Similarily for $n \geq N$

$$|\nabla d_{\complement\Omega_n}(x)| = 0 \text{ a.e. in } \complement\Omega_n \supset \complement\Omega_n\backslash U_h(\partial\Omega_n) \supset \complement\Omega^*\backslash U_{h-\varepsilon}(\partial\Omega^*)$$
$$|\nabla d_{\complement\Omega^*}(x)| = 0 \text{ a.e. in } \complement\Omega^* \supset \complement\Omega^*\backslash U_{h-\varepsilon}(\partial\Omega^*)$$

and

$$\int_{\complement\Omega^*\setminus U_{h-\varepsilon}(\partial\Omega^*)} |\nabla d_{\complement\Omega_n} - \nabla d_{\complement\Omega^*}|^2\, dx = 0.$$

As a result putting the three parts together

$$d_{\complement\Omega_n} \to d_{\complement\Omega^*} \text{ in } H^1(D)\text{-strong}$$

and hence for all $p \geq 1$ in $W^{1,p}(D)$-strong (cf. [13] proof of Theorem 3.6 (iii)). Finally define the open set Ω as int $\Omega^* = \complement\overline{\complement\Omega^*}$ since

$$d_{\complement\Omega^*} = d_{\overline{\complement\Omega^*}} = d_{\complement\Omega}.$$

This completes the first part of the proof.

(ii) (Inequality (4.10)) From (4.14), for each $\varepsilon > 0$, $0 < 2\,\varepsilon < h$, there exists $N(\varepsilon) > 0$,

$$\forall n \geq N(\varepsilon), \quad U_{h-\varepsilon}(\partial\Omega^*) \subset U_h(\partial\Omega_n).$$

For all $\Phi \in \mathcal{D}(U_{h-\varepsilon}(\partial\Omega^*))^{N\times N}$

$$\int_{U_{h-\varepsilon}(\partial\Omega^*)} \nabla d_{\complement\Omega_n} \cdot \overrightarrow{\operatorname{div}}\, \Phi\, dx = \int_{U_h(\partial\Omega_n)} \nabla d_{\complement\Omega_n} \cdot \overrightarrow{\operatorname{div}}\, \Phi\, dx$$

and

$$\left| \int_{U_{h-\varepsilon}(\partial\Omega^*)} \nabla d_{\partial\Omega_n} \cdot \overrightarrow{\operatorname{div}}\, \Phi\, dx \right| \leq \|D^2 d_{\complement\Omega_n}\|_{M^1(U_h(\partial\Omega_n))}\, \|\Phi\|_{C^0(U_h(\partial\Omega_n))}$$

$$\leq\ c\, \|\Phi\|_{C^0(U_{h-\varepsilon}(\partial\Omega^*))}$$

$$\Rightarrow \quad \|D^2 d_{\complement\Omega_n}\|_{M^1(U_{h-\varepsilon}(\partial\Omega^*))}\ \leq\ c.$$

Therefore for all $\varepsilon > 0$, $0 < 2\,\varepsilon < h$,

$$\|D^2 d_{\complement\Omega}\|_{M^1(U_{h-\varepsilon}(\partial\Omega^*))}\ \leq\ \liminf_{n\to\infty} \|D^2 d_{\complement\Omega_n}\|_{M^1(U_{h-\varepsilon}(\partial\Omega^*))} \leq c$$

and

$$\forall \Phi \in \mathcal{D}(U_{h-\varepsilon}(\partial\Omega^*))^{N\times N}, \quad \left| \int_{U_h(\partial\Omega^*)} \nabla d_{\complement\Omega} \cdot \overrightarrow{\operatorname{div}}\, \Phi\, dx \right|\ \leq\ c\|\Phi\|_{C^0(U_h(\partial\Omega^*))}.$$

For each $\Phi \in \mathcal{D}(U_h(\partial\Omega^*))^{N\times N}$

$$\operatorname{supp}\, \Phi \subset U_h(\partial\Omega^*)$$

and there exists $\varepsilon = \varepsilon(\Phi) > 0$ such that

$$\operatorname{supp}\, \Phi \subset U_{h-\varepsilon}(\partial\Omega^*)$$

From the previous inequality

$$\forall \Phi \in \mathcal{D}(U_h(\partial\Omega^*))^{N\times N}, \quad \left| \int_{U_h(\partial\Omega^*)} \nabla d_{\complement\Omega} \cdot \overrightarrow{\operatorname{div}}\, \Phi\, dx \right|\ \leq\ c\|\Phi\|_{C^0(U_h(\partial\Omega^*))}$$

and this proves (4.10). Moreover there exists $N = N(\varepsilon(\Phi)) \geq 1$ such that

$$\lim_{N\leq n\to\infty} < D^2 d_{\complement\Omega_n}, \Phi > = < D^2 d_{\complement\Omega}, \Phi > .$$

Finally

$$\nabla_{\complement d_\Omega} \in BV(U_h(\partial\Omega^*))^N$$

and since

$$d_{\complement\Omega}(x) = 0 \quad \text{in } \complement\Omega \quad \text{and} \quad \nabla d_{\complement\Omega} \in BV(U_h(\complement\Omega))^N.$$

From Theorem 3.7 in [13] $\chi_{\complement\Omega}$ and $\chi_\Omega = 1 - \chi_{\complement\Omega}$ belong to $BV(U_h(\complement\Omega))$. □

4.2 $W^{1,p}$-oriented distance topology

When $\overline{D}$ is compact, $C_b(D)$ is compact for the uniform topology and closed in the $W^{1,p}(D)$-topology $(1 \le p < \infty)$. In particular

$$b_{\Omega_n} \to b_\Omega \quad \text{in } W^{1,p}(D)\text{-strong}$$

implies that

$$d_{\complement\Omega_n} = b^-_{\Omega_n} \to b^-_\Omega = d_{\complement\Omega} \quad \text{in } C^0(\overline{D})\text{-strong} \tag{4.17}$$

$$d_{\Omega_n} = b^+_{\Omega_n} \to b^+_\Omega = d_\Omega \quad \text{in } C^0(\overline{D})\text{-strong}$$

$$d_{\partial\Omega_n} = |b_{\Omega_n}| \to |b_\Omega| = d_{\partial\Omega} \quad \text{in } C^0(\overline{D})\text{-strong}$$

$$\chi_{\text{int }\Omega_n} = |\nabla b^-_{\Omega_n}| \to |\nabla b^-_\Omega| = \chi_{\text{int }\Omega} \quad \text{in } L^p(D)\text{-strong}$$

$$\chi_{\partial\Omega_n} = |\nabla b_{\Omega_n}| \to |\nabla b_\Omega| = \chi_{\partial\Omega} \quad \text{in } L^p(D)\text{-strong.} \tag{4.18}$$

However even if the sets Ω_n's are open, the limit set Ω need not be open.

We recall the following compactness theorem from [13].

Theorem 4.3. *Let $\{\Omega_n\}$ be a sequence of subsets of the fixed hold-all D such that $\partial\Omega_n \neq \varnothing$. Assume that there exists a constant $c > 0$ such that*

$$\forall n, \quad \|D^2 b_{\Omega_n}\|_{M^1(D)} \;\le\; c. \tag{4.19}$$

There exists a subsequence $\{\Omega_{n_k}\}$ and a subset Ω of D, $\partial\Omega \neq \varnothing$, such that for all p, $1 \le p < \infty$,

$$b_{\Omega_{n_k}} \to b_\Omega \text{ in } W^{1,p}(D). \tag{4.20}$$

Moreover for all $\varphi \in \mathcal{D}^0(D)^{N\times N}$

$$< D^2 b_{\Omega_{n_k}}, \varphi > \to < D^2 b_\Omega, \varphi >, \tag{4.21}$$

and

$$\|D^2 b_\Omega\|_{M^1(D)} \;\le\; \liminf_{n\to\infty} \;\|D^2 b_{\Omega_n}\|_{M^1(D)} \;\le\; c. \tag{4.22}$$

As in the previous section we now relax condition (4.19) to a local condition in a *h-tubular* neighbourhood of the boundaries.

Theorem 4.4. *Let $\{\Omega_n\}$ be a sequence of subsets of D such that $\partial\Omega_n \neq \varnothing$. Assume that there exist $h > 0$ and $c > 0$ such that*

$$\forall n, \quad \|D^2 b_{\Omega_n}\|_{M^1(U_h(\partial\Omega_n))} \le c. \tag{4.23}$$

Then there exists a subsequence $\{b_{\Omega_{n_k}}\}$ and a subset Ω of D with $\partial\Omega \neq \varnothing$ such that for all p, $1 \le p < \infty$,

$$b_{\Omega_{n_k}} \to b_\Omega \quad \text{in } W^{1,p}(D)\text{-strong.} \tag{4.24}$$

Moreover

$$\|D^2 b_\Omega\|_{M^1(U_h(\partial\Omega))} \;\le\; \liminf_{n\to\infty} \;\|D^2 b_{\Omega_n}\|_{M^1(U_h(\partial\Omega_n))} \;\le\; c \tag{4.25}$$

and for all $\varphi \in \mathcal{D}^0(U_h(\partial\Omega))^{N\times N}$

$$< D^2 b_{\Omega_{n_k}}, \varphi > \to < D^2 b_\Omega, \varphi > . \tag{4.26}$$

Furthermore

$$\chi_{\partial\Omega} \in BV(U_h(\partial\Omega)).$$

Proof. (i) Since the sequences $\{b_{\Omega_n}\}$ and $\{\nabla b_{\Omega_n}\}$ are both pointwise uniformly bounded, there exists a subsequence, still indexed by n, and a subset Ω of D with $\partial\Omega \neq \varnothing$ such that

$$b_{\Omega_n} \to b_\Omega \quad \text{in } W^{1,2}(D)\text{-weak and in } C^0(D)\text{-strong.}$$

Clearly we can apply the compactness Theorem 4.3 in $U_h(\partial\Omega)$ and there exists a subsequence of the subsequence, still indexed by n, such that

$$b_{\Omega_n} \to b_\Omega \quad \text{in } W^{1,2}(U_h(\partial\Omega))\text{-strong.}$$

(ii) Start with the subsequence in (i). For any ε, $0 < \varepsilon < h$,

$$\exists N > 0,\ \forall n \geq N, \quad d_{\partial\Omega}(x) = |b_\Omega|(x) < |b_{\Omega_n}|(x) + \varepsilon = d_{\partial\Omega_n}(x) + \varepsilon.$$

So for all $x \in \mathbb{R}^N \backslash U_h(\partial\Omega)$ and $n \geq N$

$$d_{\partial\Omega_n}(x) \geq d_{\partial\Omega}(x) - \varepsilon \geq h - \varepsilon > 0$$

and

$$\forall n \geq N, \quad \mathbb{R}^N \backslash U_h(\partial\Omega) \subset \mathbb{R}^N \backslash \partial\Omega_n.$$

Notice that for all $n \geq N$

$$|\nabla b_{\Omega_n}(x)| = 1 \text{ a.e. in } \mathbb{R}^N \backslash \partial\Omega_n \supset R^N \backslash U_h(\partial\Omega)$$
$$|\nabla b_\Omega(x)| = 1 \text{ a.e. in } \mathbb{R}^N \backslash \partial\Omega \supset R^N \backslash U_h(\partial\Omega).$$

Therefore by weak convergence of ∇b_{Ω_n} to ∇b_Ω in $L^2(D\backslash U_h(\partial\Omega))$, for all $n \geq N$

$$\int_{D\backslash U_h(\partial\Omega)} |\nabla b_{\Omega_n} - \nabla b_\Omega|^2\, dx = \int_{D\backslash U_h(\partial\Omega)} |\nabla b_{\Omega_n}|^2 + |\nabla b_\Omega|^2 - 2\,\nabla b_{\Omega_n} \cdot \nabla b_\Omega\, dx$$

goes to zero as n goes to infinity. Hence

$$b_{\Omega_n} \to b_\Omega \quad \text{in } W^{1,2}(D\backslash U_h(\partial\Omega))\text{-strong.}$$

But since the b's and their gradients are uniformly pointwise bounded, this convergence is true for all p, $1 \leq p < \infty$, and combining this last result with the one of part (i), we obtain the strong convergence in $W^{1,p}(D)$.

To prove (4.25) and (4.26) we use the same techniques as in the proof of Theorem 4.2. □

Corollary 4.1. *Let $\{\Omega_n\}$ be a sequence of subsets of D for which there exist $h > 0$, $p \geq 1$ and $c > 0$ such that*

$$\forall n, \quad \int_{U_h(\partial\Omega_n)} |D^2 b_{\Omega_n}(x)|^p\, dx \leq c. \tag{4.27}$$

Then, in addition to the conclusions of Theorem 4.4, $\mathrm{m}(\partial\Omega) = \mathrm{m}(\partial\Omega_n) = 0$ and

$$\int_{U_h(\partial\Omega)} |D^2 b_\Omega(x)|^p\, dx \;\leq\; c. \tag{4.28}$$

Moreover for $p > N$ the sets Ω_n's and the limit set Ω have Hölderian boundaries of class $C^{1,\lambda}$ for all λ, $0 < \lambda \leq 1 - N/p$.

Proof. The first part of the corollary is a direct consequence of the Theorem 4.4. Since D is bounded, condition (4.27) for some $p \geq 1$ implies the same condition for $p = 1$. Hence condition (4.23) in Theorem 4.4 is verified and its conclusions follow. The properties of the boundaries follow from 3.1 □

5 A continuity of the solution of the Dirichlet boundary value problem with respect to smooth domains

Assume that

$$d_{\complement\Omega_n} \to d_{\complement\Omega} \text{ in } C^0(D) \tag{5.1}$$

for a sequence $\{\Omega_n\}$ of open subsets of D and an open subset Ω of D (here Ω can possibly be empty). Associate with each n the solution y_n of the homogeneous Dirichlet problem

$$\begin{aligned} &\exists y_n = y(\Omega_n) \in H_0^1(\Omega_n), \quad \forall \varphi \in H_0^1(\Omega_n) \\ &\int_{\Omega_n} \nabla y_n \cdot \nabla \varphi - f\,\varphi\,dx = 0. \end{aligned} \tag{5.2}$$

Introduce for any open subset Ω of D the closed linear subspace

$$H_0^1(\Omega; D) = \overline{\mathcal{D}(\Omega; D)}^{H^1} \tag{5.3}$$

of $H_0^1(D)$ where

$$\mathcal{D}(\Omega; D) = \{\varphi \in \mathcal{D}(D) \,:\, \text{supp}\,\varphi \subset \Omega\}. \tag{5.4}$$

As a consequence

$$\{\varphi_{|\Omega} \,:\, \varphi \in H_0^1(\Omega; D)\} = H_0^1(\Omega). \tag{5.5}$$

This defines a unique extension by zero in $H_0^1(D)$ of each element y_n of $H_0^1(\Omega_n)$. For simplicity this extension will also be denoted y_n.

The sequence of extensions by zero of the solutions y_n's to problem (5.2) is uniformly bounded in $H_0^1(D)$. Hence there exists $y^* \in H_0^1(D)$ and a bounded subsequence, still indexed by n, such that

$$y_n \rightharpoonup y^* \text{ in } H_0^1(D)\text{-weak}.$$

If $y(\Omega)$ is the solution of the homogeneous Dirichlet problem on Ω

$$\begin{aligned} &\exists y = y(\Omega) \in H_0^1(\Omega), \quad \forall \varphi \in H_0^1(\Omega) \\ &\int_{\Omega} \nabla y \cdot \nabla \varphi\,dx - f\,\varphi\,dx = 0, \end{aligned} \tag{5.6}$$

can we conclude that $y^*_{|\Omega} = y(\Omega)$?

In view of (5.1) if the open domain Ω is non-empty it has the *compactivorous property*:

$$\forall K \text{ compact } \subset \Omega, \exists N > 0, \forall n > N, \quad K \subset \Omega_n.$$

Therefore for each $\varphi \in \mathcal{D}(\Omega)$, there exists $N > 0$ such that

$$\forall n > N, \quad \varphi \in \mathcal{D}(\Omega_n).$$

This property is sufficient to show that y^* verifies the variational equations (5.6). For each $\varphi \in \mathcal{D}(\Omega)$, its support $K \stackrel{\text{def}}{=} \text{supp}\,\varphi$ is compact in Ω. As a result of the uniform complementarity convergence of $d_{\complement\Omega_n}$

$$\exists N > 0, \forall n \geq N, \quad K \subset \Omega_n \quad \Rightarrow \varphi \in \mathcal{D}(\Omega_n).$$

Then for $n \geq N$

$$0 = \int_{\Omega_n} \nabla y_n \cdot \nabla \varphi - f\, \varphi\, dx = \int_D \nabla y_n \cdot \nabla \varphi - f\, \varphi\, dx$$

converges to

$$0 = \int_D \nabla y^* \cdot \nabla \varphi - f\, \varphi\, dx = \int_\Omega \nabla y^* \cdot \nabla \varphi - f\, \varphi\, dx$$

and by density of $\mathcal{D}(\Omega)$ in $H_0^1(\Omega)$

$$\begin{gathered} \exists y^* \in H_0^1(D), \quad \forall \varphi \in H_0^1(\Omega) \\ \int_\Omega \nabla y^* \cdot \nabla \varphi - f\, \varphi\, dx = 0. \end{gathered}$$

It remains to find under what conditions $y^* \in H_0^1(\Omega)$. The result is true in dimension $N = 1$. It follows from the fact that $H_0^1(\Omega_n) \subset C^0(\overline{\Omega_n})$ for all $n \geq 1$. In dimensions N higher than 1, the above result is no longer true and some additional assumptions are required on the sets Ω_n. In [24] for the dimension $N = 2$, this continuity was recovered by considering the smaller family

$$C_{d,\#}^c(D) = \{d_{\complement\Omega} : \Omega \text{ open in } D \text{ and } \#(\complement\Omega) \leq \ell\}$$

for some positive number $\ell \geq 0$ and

$$\#(\complement\Omega) = \text{ number of connected components of } \complement\Omega.$$

By lower semicontinuity of this function, $C_{d,\#}^c(D)$ is closed in the uniform complementary topology. As a result the limit set enjoys the same property. This result is specific of the dimension $N = 2$ and does not extend to higher dimensions.

For all $n \geq 1$, assume that $\partial\Omega_n \neq \varnothing$. Now since $y_n \in H_0^1(D)$ and $d_{\complement\Omega_n} \in W^{1,\infty}(D)$, then

$$y_n\, d_{\Omega_n} = 0 \quad \text{in } H_0^1(D).$$

The sequence $\{b_{\Omega_n}\}$ has a convergent subsequence and there exists Ω^*, $\partial\Omega^* \neq \varnothing$, such that

$$b_{\Omega_n} \rightharpoonup b_{\Omega^*} \quad \text{in } H^1(D)\text{-weak}.$$

As a result

$$\begin{aligned} d_{\Omega_n} &\to d_{\Omega^*} \quad \text{in } H^1(D)\text{-weak and } C^0(\overline{D})\text{-strong} \\ d_{\complement\Omega_n} &\to d_{\complement\Omega^*} \quad \text{in } H^1(D)\text{-weak and } C^0(\overline{D})\text{-strong} \\ d_{\partial\Omega_n} &\to d_{\partial\Omega^*} \quad \text{in } H^1(D)\text{-weak and } C^0(\overline{D})\text{-strong}. \end{aligned}$$

In particular $\overline{\complement\Omega^*} = \overline{\complement\Omega}$ which means that

$$\Omega = \text{int } \Omega^* \quad \text{and} \quad \partial\Omega \subset \partial\Omega^*.$$

Moreover

$$\begin{aligned} &0 = y_n\, d_{\Omega_n} \to y^*\, d_{\Omega^*} \quad \text{in } H_0^1(D)\text{-weak} \\ &y^*\, d_{\Omega^*} = 0 \quad \text{q.e. in } D \\ &y^* = 0 \quad \text{q.e. in } D \backslash \overline{\Omega^*}. \end{aligned}$$

For the family of sets characterized by Theorem 3.1 we have the desired continuity since the sequence of sets Ω_n and its limit Ω^* all have $C^{1,\lambda}$, $0 < \lambda \le 1 - N/p$ boundaries and necessarily

$$y^*_{|\partial\Omega^*} = 0.$$

Theorem 5.1. *Let $p > N$ be given. Let $\{\Omega_n\}$, $\partial\Omega_n \neq \varnothing$, be a sequence of open subsets of D such that*

$$d_{\complement\Omega_n} \to d_{\complement\Omega} \quad \text{in } C^0(D)\text{-strong} \tag{5.7}$$

for some open subset Ω (possibly empty) of D. Assume that there exist $c > 0$ and $h > 0$ such that

$$\forall n, \quad \int_{U_h(\partial\Omega_n)} |D^2 b_{\Omega_n}(x)|^p \, dx < c. \tag{5.8}$$

Then the domains $y(\Omega_n)$ and $\Omega = \text{int}\,\Omega^*$ are all of class $C^{1,\lambda}$, $0 < \lambda \le N/p$, and for the solutions $y(\Omega_n)$ of the Dirichlet problem (5.2)

$$y(\Omega_n) \to y(\Omega) \qquad \text{in } H^1_0(D) \tag{5.9}$$

where $y(\Omega)$ is the solution of problem (5.6) in $H^1_0(\Omega; D)$ or

$$y(\Omega)_{|\Omega} \in H^1_0(\Omega) \tag{5.10}$$

is the solution of the homogeneous Dirichlet problem (5.6) in the domain Ω.

Proof. In view of (5.8) from Theorem 4.4 and Corollary 4.1, there exists a subsequence of $\{b_{\Omega_n}\}$, still indexed by n, and a set $\Omega^* \subset D$, $\partial\Omega^* \neq \varnothing$, such that

$$b_{\Omega_n} \to b_{\Omega^*} \quad \text{in } W^{1,p}(D) - \text{strong}$$

and the sets Ω^* and $\{\Omega_n\}$'s all have Hölderian boundaries of class $C^{1,\lambda}$, $0 < \lambda \le N/p$. in particular

$$d_{\complement\Omega_n} = b^-_{\Omega_n} \to b^-_{\Omega^*} = d_{\complement\Omega^*} \quad \text{in } C^0(D)$$

and

$$d_{\complement\Omega^*} = d_{\complement\Omega} \quad \Longrightarrow \quad \overline{\complement\Omega^*} = \overline{\complement\Omega} \quad \Longrightarrow \quad \Omega = \text{int}\,\Omega^*.$$

Therefore Ω^* is of class $C^{1,\lambda}$. We conclude that

$$y^* = 0 \quad \text{on } \partial\Omega^*$$

and $y^* \in H^1_0(\Omega)$. Hence y^* is the solution of (5.6). To complete the proof we show that the whole sequence converges to $y(\Omega) \in H^1_0(\Omega)$ in the $H^1_0(D)$-strong topology. This is readily seen by noting that

$$\int_D |\nabla y_n|^2 \, dx = \int_D \chi_{\Omega_n} \, f \, y_n \, dx \to \int_D \chi_\Omega \, f \, y^* \, dx = \int_D |\nabla y^*|^2 \, dx.$$

Hence

$$\int_D |\nabla y_n - \nabla y^*|^2 \, dx = \int_D |\nabla y_n|^2 \, dx + \int_D |\nabla y^*|^2 \, dx - 2 \int_D \nabla y_n \cdot \nabla y^* \, dx$$

and the right-hand side converges to zero as n goes to infinity. $\square$

The classical *Uniform Cone Condition* in [10] and more recently *Flat Cone Conditions* in [5] have been sucessfully used to obtain both continuity and existence results. The flat cone conditions are capacity conditions (see also [6] for the use of the uniform Wiener condition). These constraints are such that the linear tangent space associated with the admissible family of domains at a given element Ω do not readily lend themselves to conditions of the variational type or provide directions in descent algorithms.

The conditions given in the above theorem are of a different type. They do not involve pointwise constraints on the boundaries of the domains. They make use of the compactness property associated with the boundedness of the total variation of the Hessian matrix of the oriented distance function of subsets Ω of D (cf. [13]) in a neighbourhood of its boundary.

References

[1] MR.A. Adams, *Sobolev spaces*, Academic Press, New York, London, 1975

[2] M. Bernadou, *Méthodes d'éléments finis pour les problèmes de coques minces*, Masson, Paris, Milan, Barcelone, 1994.

[3] H. Blum, *A transformation for extracting view description of shapes* in "Models for perception of speech and visual form", W. Wathen-Dunn, ed., pp. 362–380 , MIT Press, Cambridge Mass. 1967.

[4] D. Bucur, *Capacitary extension and two dimensional shape optimization*, Preprint INLN #93.24, Institut Non Linéaire de Nice, Sophia-Antipolis, France, 1993.

[5] D. Bucur and J.-P. Zolésio, *Flat cone condition and shape analysis*, in "Control of partial differential equations", G. Da Prato and L. Tubaro, eds, pp. 37-48, Marcel Dekker, New York 1994.

[6] D. Bucur and J.-P. Zolésio, *Wiener's criterion and shape continuity for the Dirichlet problem*, Preprint INLN #94.32, Institut Non Linéaire de Nice, Sophia-Antipolis, France, May 1994.

[7] D. Bucur and J.-P. Zolésio, *Optimisation de forme sous contrainte capacitaire*, C.R. Acad. Sc. Paris Sér. I, **318** (1994), 795-800.

[8] D. Bucur and J.-P. Zolésio, *N-dimensional shape optimization under capacitary constraint*, J. Differential Equations. accepted (Preprint INLN #93.22, Institut Non Linéaire de Nice, Sophia-Antipolis, France, 1993).

[9] D. Bucur and J.-P. Zolésio, *Continuité par rapport au domaine dans le problème de Neumann*, C.R. Acad. Sc. Paris Sér. I, **319** (1994), 57-60.

[10] D. Chenais, *On the existence of a solution in a domain identification problem*, J. Math. Anal. Appl. **52** (1975), 189–289

[11] Ph. G. Ciarlet et V. Lods, *Ellipticité des équations membranaires d'une coque uniformément elliptique*, C.R. Acad. Sc. Paris Sér. I, **318** (1994), 195–200.

[12] P.-E. Danielsson, *Euclidean distance mapping*, Computer Graphics and Image Processing **14** (1980), 227-248.

[13] M. C. Delfour and J.-P. Zolésio, *Shape analysis via distance functions*, J. Functional Analysis **123** (1994), 129-201.

[14] M. C. Delfour and J.-P. Zolésio, *On a variational equation for thin shells*, in "Control and Optimal Design of Distributed Parameter Systems, J. Lagnese, D.L. Russell, L. White, eds, pp. 25–37, Springer-Verlag, Berlin, Heidelberg, New York, Tokyo 1994.

[15] M. C. Delfour and J.-P. Zolésio, *A boundary differential equation for thin shells*, J. Differential Equations to appear (Jan. 1994).

[16] M. C. Delfour and J.-P. Zolésio, *Tangential differential equations for dynamical thin/shallow shells*, CRM Report 2177, Dec. 1994, to appear in J. Differential Equations.

[17] L.C. Evans and R.F. Gariepy, *Measure theory and fine properties of functions*, CRC Press, Boca Raton 1992.

[18] D. Gilbarg and N.S. Trudinger, *Elliptic Partial Differential Equations of Second Order*, Springer-Verlag, Berlin, Heidelberg, New York, Tokyo 1983.

[19] C.M. Hoffmann *How to construct the skeleton of CGS objects*, in "Proc. 4th IMA Conference, The Mathematics of Surfaces", Oxford University Press, Bath, England." To appear

[20] C.M. Hoffmann *Algebraic and numerical techniques for offsets and blends*, in "Computation of Curves and Surfaces", W. Dahmen et als., eds, pp. 499-528, Kluwer 1990.

[21] G. Matheron,: *Examples of Topological Properties of Skeletons*, in "Image Analysis and Mathematical Morphology" J. Serra, ed, pp 217–238, Academic Press, London 1988.

[22] A. Rivière, *Classification de points d'un ouvert d'un espace euclidien relativement à la distance au bord, étude topologique et quantitative des classes obtenues*, thèse de doctorat, 18 novembre 1987, U. de Paris-Sud, Centre d'Orsay.

[23] S. Stifter, *The Roider method: a method for static and dynamic collision detection*, in "Issues in Robotics and Nonlinear Geometry", C. Hoffmann, ed., JAI Press, to appear.

[24] V. S̃verák, *On optimal shape design*, J. Math. Pur. et Appl., to appear.

[25] S. Yokoi, J.-I. Toriwaki, and T. Fukumura, *On Generalized Distance Transformation of Discretized Pictures*, Pattern Anal. and Machine Intel. PAMI-3 (1981), 424–443.