Differential equations for linear shells : comparison between intrinsic and classical models

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Abstract

The oriented boundary (signed, algebraic) distance function of a set can remarkably describe the geometric properties of its boundary. It also serves as a basis to elaborate an intrinsic tangential differential calculus on a C^2 submanifold of $\mathbb{R}^N$ of codimension one. This has made possible the development of new intrinsic models for thin or shallow shells. In this paper we give the correspondence between intrinsic tangential derivatives and classical covariant derivatives using Christoffel symbols. Then we show how the classical models of Naghdi, Koiter and the asymptotic membrane model can be expressed in terms of intrinsic operators, opening the way to comparisons of models and a more tractable approach to optimization, control and sensitivity analysis of shells.

KEYWORKS: shell, intrinsic differential calculus, distance function.

Résumé

La fonction distance orientée à la frontière d'un ensemble peut décrire de façon remarquable les proprétés géométriques de sa frontière. Elle sert aussi de base à l'élaboration d'un calcul différentiel tangentiel sur une sous-variété différentiable de codimension un dans $\mathbb{R}^N$ de classe C^2. Ceci a permis de développer de nouveaux modèles intrinsèques des coques minces ou peu profondes. Dans cet article on donne la correspondance entre les dérivées tangentielles intrinsèques et les dérivées covariantes utilisant les symboles de Christoffel. On montre ensuite comment les modèles classiques de Naghdi, Koiter, et le modèle asymptotique membranaire peuvent s'exprimer en fonction d'opérateurs intrinsèques, ouvrant la voie à la comparaison de modèles et à une approche plus souple à l'optimisation, au contrôle et à la sensitivité de forme des coques.

MOTS CLÉS :coque, calcul différentiel tangentiel, fonction distance.

1 Introduction

In a previous paper (cf. Delfour and Zolésio [5]), we have presented a new intrinsic model for linear $(N-1)$-dimensional thin/shallow shells in $\mathbb{R}^N$ using the tangential differential calculus, the oriented boundary (resp. algebraic or signed) distance function, and linear elasticity. This model is an extension to thin/shallow shells of the "natural theory" and the Love-Kirchhoff theory of plates (cf. for instance P. Germain [1] or R. Valid [1]). The constructions are quite general and more complex rheologies can be considered.

Its main advantage is that it is mathematically more tractable than currently available models which use local coordinates systems and Christoffel symbols. It uses an approximation of the strain tensor by a quadratic expression in the thickness variable. This slight increase in complexity is motivated by the fact that in so doing we first obtain an intrinsic *Korn's inequality* for shells. Secondly the kernel of the "approximate" intrinsic strain tensor coincides with the set of "rigid displacements". Finally the Love-Kirchhoff theory of shells comes out of the analysis as a special case of the natural theory by looking at the same variational equation over some closed linear subspace of the Hilbert space V associated with the natural theory.

In this paper we emphasize the intrinsic character of the techniques developped in previous papers and throw a bridge with classical approaches using covariant derivatives and Christoffel symbols. This link was so far missing making comparisons difficult between our intrinsic model and the more classical models of Naghdi, Koiter, and the asymptotic equations.

In § 2 we recall definitions and notation. In § 3 we recall earlier results on intrinsic tangential differential operators and introduce a new tangential operator which naturally occurs in asymptotic membrane equations. In § 4 we illustrate the general intrinsic techniques for the Laplace-Beltrami operator which is generic of the scalar case. In § 5 we recall and sharpen the main results for static linear intrinsic shells. In § 6 we provide an extensive "dictionary" between the intrinsic tangential differential operators and the covariant differential ones. We discuss the open question of the a priori difference between the intrinsic Korn's inequality and the one obtained when the shell is completely described by a single map. The intrinsic version gives an additional term, but no example has so far been constructed to justify the presence of that term. The same issue also arises for Poincaré's inequality. Finally in § 7 we show how to obtain the asymptotic membrane equations from the intrinsic model. It coincides with the existing model that can be found in Ciarlet and Sanchez-Palencia [1]. For convenience the tedious computations to express the classical models in terms of intrinsic tangential operators have been put in Appendix A.

2 Definitions and notation

2.1 Oriented boundary distance function

Let $\mathbb{R}^N$ be the N-dimensional Euclidean space for some integer $N > 1$ (in practice $N = 3$). Let Ω be a subset of $\mathbb{R}^N$ with a boundary $\partial\Omega$ which is a C^2 $(N-1)$-dimensional submanifold of $\mathbb{R}^N$. Associate with Ω the *oriented boundary distance function*

$$b_\Omega(x) \stackrel{\text{def}}{=} d_\Omega(x) - d_{\complement\Omega}(x) \tag{2.1}$$

where $\complement\Omega = \{x \in \mathbb{R}^N : x \notin \Omega\}$ and d_A is the usual distance function to a subset A of $\mathbb{R}^N$. This function captures all the geometrical properties of the boundary $\partial\Omega$. For $k \geq 2$ a domain Ω has a C^k boundary $\partial\Omega$ if and only if in each point $X \in \partial\Omega$ there exists a bounded open neighbourhood $N(X)$ of X such that $b_\Omega \in C^k(N(X))$.

At each point X of $\partial\Omega$, its gradient $\nabla b_\Omega(X)$ coincides with the unitary exterior normal field n to $\partial\Omega$ and the eigenvalues of the symmetrical matrix of second order partial derivatives $D^2 b_\Omega$ are 0 and the *principal curvatures*, $\kappa_i, 1 \le i \le N-1$, of the surface $\partial\Omega$. The trace of $D^2 b_\Omega(X)$ is the *mean curvature*

$$H(X) \stackrel{\text{def}}{=} \text{tr}(D^2 b_\Omega(X)) = \Delta b_\Omega(X), \tag{2.2}$$

up to a multiplying factor which is used as a normalization factor to make the mean curvature of the unit sphere equal to one in all dimensions. The trace of the matrix of cofactors $M(D^2 b_\Omega)$ is the *total* or *Gaussian curvature*

$$K(X) \stackrel{\text{def}}{=} \text{tr}\, M(D^2 b_\Omega(X)). \tag{2.3}$$

The reader is referred to Delfour and Zolésio [1] for more details on the properties of the function b_Ω and to Gilbarg and Trudinger [1] for the study of curvature via distance functions.

Since the domain Ω is fixed throughout this paper, from now on the function b_Ω will be denoted by b. For each $X \in \partial\Omega$, the *projection mapping* $p\colon N(X) \to \partial\Omega$ is obtained directly from the oriented distance function b as

$$p(x) = x - b(x)\nabla b(x). \tag{2.4}$$

This definition is independent of the choice of $N(X)$ and X. It only uses the fact that $\nabla b(x)$ exists. Its Jacobian matrix is given by

$$Dp(x) = I - b(x)D^2 b(x) - \nabla b(x)\, {}^*\nabla b(x), \tag{2.5}$$

where ${}^*\nabla b(x)$ is the transposed of the vector $\nabla b(x)$ and I is the identity matrix. For $x \in N(X)$, the *linear projector* onto the *tangent plane* $T_{p(x)}\partial\Omega$ at the point $p(x)$ of $\partial\Omega$ is given by

$$P(x) = I - \nabla b(x)\, {}^*\nabla b(x). \tag{2.6}$$

2.2 Definition of the "shell"

The motivation for this definition arises from the theory of shells. Yet it is a general construction for a domain of thickness $2h$ around an $(N-1)$-submanifold of $\mathbb{R}^n$ which is usually refered to as the "mean surface" in the theory of shells. For convenience we use the terminology "shell" as a generic term, but we shall consider differential equations defined on the underlying submanifold which are not necessarily the shell equations.

A shell is characterized by its mean surface Γ and its thickness (function) $\tilde{h}$. The *mean surface* Γ of the shell is a bounded open domain in the $(N-1)$-submanifold $\partial\Omega$ of $\mathbb{R}^N$. When $\Gamma = \partial\Omega$ (hence $\partial\Omega$ is compact), the shell has no boundary. When $\Gamma \subsetneqq \partial\Omega$, the (relative) boundary $\partial_{\partial\Omega}\Gamma$ is assumed to be uniformly Lipschitzian in $\partial\Omega$.

Since Γ is bounded and $\partial\Omega$ is C^2, there exist $h > 0$ and a bounded neighbourhood

$$S_h \stackrel{\text{def}}{=} \{x \in \mathbb{R}^N \colon p(x) \in \Gamma, |b(x)| < h\}, \tag{2.7}$$

where b is C^2. In view of the assumptions on Γ and h, the set S_h is a bounded open domain in $\mathbb{R}^N$ with a Lipschitzian boundary.

When $\Gamma \subsetneqq \partial\Omega$, S_h has a *lateral boundary*

$$\Sigma_h = \{x \in \mathbb{R}^N \colon p(x) \in \partial_{\partial\Omega}\Gamma, |b(x)| < h\} \tag{2.8}$$

which is an $(N-1)$-dimensional surface normal to the mean surface Γ.

In practice the mean surface Γ will be given first and the underlying assumption will be the existence of an appropriate domain Ω with the above properties. It is important to keep in mind that we use the distance function $b = b_\Omega$ and not the distance function to Γ.

2.3 Flow of the gradient of b and local coordinates

Since $\nabla b \in C^1(S_h)$, consider the *flow mapping* $T_z = T_z(\nabla b)$, defined by

$$T_z(X) = x(z), \qquad \begin{cases} \frac{dx}{dz}(z) = \nabla b(x(z)), & |z| < h, \\ x(0) = X. \end{cases} \tag{2.9}$$

It is a homeomorphism from Γ onto $\Gamma_z = \{x \in \mathbb{R}^N : b(x) = z, p(x) \in \Gamma\}$. In particular

$$T_z(X) = X + z\nabla b(X) \tag{2.10}$$

for $|z| < h$. This induces a "curvilinear coordinate system" $(X, z) \in \Gamma \times]-h, h[$ in S_h. The points on the level set Γ_z are given by $\{X + z\nabla b(X) : X \in \Gamma\}$ and for each $(X, z) \in \Gamma \times]-h, h[$

$$\nabla b(T_z(X)) = \nabla b(X + z\nabla b(X)) = \nabla b(X). \tag{2.11}$$

We have the following identities and properties on Γ:

$$p \circ T_z = p, \quad b \circ T_z = z, \qquad DT_z = I + zD^2b. \tag{2.12}$$

In particular $\det DT_z(X)$ is a polynomial of degree at most $N-1$ and

$$\det DT_z(X) = \begin{cases} 1 + z\Delta b(X), & \text{for } N = 2 \\ 1 + z\Delta b(X) + z^2 \ \operatorname{tr} M(D^2b(X)), & \text{for } N = 3 \end{cases} \tag{2.13}$$

where $M(D^2b(X))$ is the cofactor matrix of the matrix $D^2b(X)$. It will be useful to introduce the notation

$$j(z) \stackrel{\text{def}}{=} \det DT_z(X) = \sum_{i=0}^{N-1} K_i z^i \tag{2.14}$$

where the K_i's are functions of X on Γ, $K_0 = 1$, $K_1 = H$ for $N \geq 2$, and $K_{N-1} = K$ for $N \geq 3$.

3 Intrinsic tangential calculus

For any scalar function $w\colon \Gamma \to \mathbb{R}$, denote by $\nabla_\Gamma w$ the *tangential gradient*

$$\nabla_\Gamma w = \nabla W\big|_\Gamma - \frac{\partial W}{\partial n} n \tag{3.1}$$

defined in terms of an extension W of w to S_h. It can be shown that this definition is independent of the choice of the extension W and that $\nabla_\Gamma w(X)$ is the projection of ∇W onto the tangent plane $T_X\Gamma$ to Γ in X. It is easy to check that

$$\nabla(w \circ p) = Dp(\nabla_\Gamma w) \circ p = [I - bD^2b]\nabla_\Gamma w \circ p \tag{3.2}$$

and that $\nabla(w \circ p) = \nabla_\Gamma w$ on Γ. The *tangential Jacobian matrix* of a vector $v\colon \Gamma \to \mathbb{R}^N$ is defined through its transposed

$$^*D_\Gamma v = (\nabla_\Gamma v_1, \ldots, \nabla_\Gamma v_N) \tag{3.3}$$

in terms of the column tangential gradients. In particular

$$\begin{aligned} ^*D(v \circ p) &= (\nabla_\Gamma v_1, \ldots, \nabla_\Gamma v_N) \circ p \\ &= Dp(^*D_\Gamma v) \circ p = [I - bD^2b](^*D_\Gamma v) \circ p \end{aligned} \tag{3.4}$$

$$D(v \circ p) = (D_\Gamma v) \circ p Dp = (D_\Gamma v) \circ p[I - bD^2 b] \tag{3.5}$$

and on Γ, $D(v \circ p) = D_\Gamma v$. In general we have the following identity

$$^*D_\Gamma v \nabla b = \nabla_\Gamma (v \cdot \nabla b) - D^2 b\, v \quad \text{on } \Gamma. \tag{3.6}$$

In the same way define the *tangential divergence* as

$$\operatorname{div}_\Gamma v \stackrel{\text{def}}{=} \operatorname{tr} D_\Gamma(v) \tag{3.7}$$

or equivalently in term of an extension V of v to a neighbourhood of Γ

$$\operatorname{div}_\Gamma v \stackrel{\text{def}}{=} \operatorname{div} V|_\Gamma - DVn \cdot n. \tag{3.8}$$

It is easy to verify that

$$\operatorname{div}(v \circ p) = \operatorname{div}_\Gamma v \circ p - b \operatorname{tr}[D_\Gamma v) \circ p D^2 b] \tag{3.9}$$

and $\operatorname{div}(v \circ p)_{|\Gamma} = \operatorname{div}_\Gamma v$. Similarily the *tangential strain tensor* is defined as

$$\varepsilon_\Gamma(v) \stackrel{\text{def}}{=} \frac{1}{2}(D_\Gamma v + {}^*D_\Gamma v) \tag{3.10}$$

$$\begin{aligned} \varepsilon(v \circ p) &\stackrel{\text{def}}{=} \frac{1}{2}(D(v \circ p) + {}^*D(v \circ p)) \\ &= \varepsilon_\Gamma(v) \circ p - \frac{b}{2}[D_\Gamma(v) \circ p D^2 b + D^2 b^* D_\Gamma(v) \circ p] \end{aligned} \tag{3.11}$$

$$\varepsilon_\Gamma(v) = \varepsilon(v \circ p)_{|\Gamma}. \tag{3.12}$$

In view of identities (3.2), (3.5) and (3.9) the composition of $\operatorname{div}_\Gamma$ and ∇_Γ yields the *Laplace-Beltrami* operator

$$\Delta_\Gamma w \stackrel{\text{def}}{=} \operatorname{div}_\Gamma(\nabla_\Gamma w) \quad (= \Delta(w \circ p)_{|\Gamma}). \tag{3.13}$$

Similarily the *matrix of tangential second order derivatives* is defined as

$$D^2_\Gamma w \stackrel{\text{def}}{=} D_\Gamma(\nabla_\Gamma w) \quad (\Rightarrow D^2(w \circ p)_{|\Gamma} = D^2_\Gamma w - D^2 b\, \nabla_\Gamma w\, {}^*\nabla b) \tag{3.14}$$

and it is readily seen that

$$\varepsilon(\nabla(w \circ p))_{|\Gamma} = \varepsilon_\Gamma(\nabla_\Gamma(w)) - \frac{1}{2}[D^2 b\, \nabla_\Gamma w\, {}^*\nabla b + \nabla b\, {}^*(D^2 b\, \nabla_\Gamma w)]. \tag{3.15}$$

Another important tangential operator is

$$\varepsilon^P_\Gamma(v) \stackrel{\text{def}}{=} P \varepsilon_\Gamma(v) P. \tag{3.16}$$

By definition

$$\varepsilon^P_\Gamma(v) = \varepsilon_\Gamma(v) - [\varepsilon_\Gamma(v) \nabla b\, {}^*\nabla b + \nabla b^* (\varepsilon_\Gamma(v) \nabla b)] \tag{3.17}$$

and if we denote by w and u the normal and tangential omponents of v

$$w \stackrel{\text{def}}{=} v \cdot \nabla b \quad \text{and} \quad u \stackrel{\text{def}}{=} v - w \nabla b \tag{3.18}$$

then

$$\varepsilon^P_\Gamma(v) = \varepsilon^P_\Gamma(u) + w D^2 b \tag{3.19}$$

and in view of (2.6)

$$\varepsilon^P_\Gamma(u) = \varepsilon_\Gamma(u) + \frac{1}{2}[D^2 b\, u\, {}^*\nabla b + \nabla b^* (D^2 b\, u)]. \tag{3.20}$$

4 An example of elliptic problem on a submanifold

With the tangential calculus, elliptic problems on a C^2 $(N-1)$-dimensional submanifold $\partial\Omega$ of $\mathbb{R}^N$ can be handled in much the same way as their counterparts in $\mathbb{R}^N$. This will now be illustrated on a Laplace-Beltrami operator with variable coefficients.

4.1 The underlying function spaces

In § 3 we have shown how tangential differential operators can be defined for scalar functions $w\colon \Gamma \to \mathbb{R}$ with $\Gamma = \partial\Omega$,

$$\nabla_\Gamma w = \nabla(w \circ p)|_\Gamma, \tag{4.1}$$

and for vector functions $v\colon \Gamma \to \mathbb{R}^N$

$$D_\Gamma v = D(v \circ p)|_\Gamma, \quad \operatorname{div}_\Gamma v = \operatorname{div}(v \circ p)|_\Gamma. \tag{4.2}$$

Denote by $C(\Gamma)$ the space of continuous functions on Γ and define

$$C^1(\Gamma) \overset{\text{def}}{=} \{w \in C(\Gamma)\colon \nabla_\Gamma w \in C(\Gamma)^N\}, \tag{4.3}$$

$$C^2(\Gamma) \overset{\text{def}}{=} \{w \in C^1(\Gamma)\colon \nabla_\Gamma w \in C^1(\Gamma)^N\}, \tag{4.4}$$

and so on by induction for all n, $1 \le n < \infty$,

$$C^n(\Gamma) \overset{\text{def}}{=} \{w \in C^{n-1}(\Gamma)\colon \nabla_\Gamma w \in C^{n-1}(\Gamma)^N\}, \tag{4.5}$$

and

$$C^\infty(\Gamma) \overset{\text{def}}{=} \bigcap_{1 \le n} C^n(\Gamma). \tag{4.6}$$

When Γ is a connected domain, then for $w \in C^1(\Gamma)$

$$\nabla_\Gamma w = 0 \Rightarrow \nabla(w \circ p) = [I - bD^2 b]\nabla_\Gamma w \circ p = 0$$

in a neighboorhood $N(X)$ of each point $X \in \Gamma$. Hence there exists a constant such that

$$w \circ p = c \text{ in } U(X) \Rightarrow w(X) = c \text{ in } U(X) \cap \Gamma.$$

Thus $\nabla_\Gamma w$ really plays the same role as the gradient in $\mathbb{R}^N$.

Assume now that Γ is a bounded open subset of $\partial\Omega$ with relative boundary $\partial\Gamma$ in the $(N-1)$-dimensional C^2 submanifold $\partial\Omega$ of $\mathbb{R}^N$ with Lipschitzian boundary $\partial\Gamma$ in $\partial\Omega$. Define the first Sobolev space by density

$$H^1(\Gamma) \overset{\text{def}}{=} \overline{C^\infty(\bar{\Gamma})}^{H^1}, \tag{4.7}$$

with the usual H^1-norm

$$\|w\|^2_{H^1(\Gamma)} \overset{\text{def}}{=} \|w\|^2_{L^2(\Gamma)} + \|\nabla_\Gamma w\|^2_{L^2(\Gamma)^N}. \tag{4.8}$$

By induction for $1 \le n < \infty$

$$H^n(\Gamma) \overset{\text{def}}{=} \{w \in H^{n-1}(\Gamma)\colon \nabla_\Gamma w \in H^{n-1}(\Gamma)^N\}. \tag{4.9}$$

With a similar definition, the Sobolev spaces $W^{n,p}(\Gamma)$, $1 \leq n < \infty$, $1 \leq p < \infty$, can be defined in the same way. In the same spirit the subspaces

$$\mathcal{D}^0(\Gamma) \stackrel{\text{def}}{=} \left\{w \in C(\partial\Omega) \colon \exists K \text{ (relatively) compact in } \partial\Omega, \operatorname{supp} w \subset K\right\} \tag{4.10}$$

and by induction for all $n \geq 1$

$$\mathcal{D}^n(\Gamma) \stackrel{\text{def}}{=} \{w \in \mathcal{D}^{n-1}(\Gamma) \colon \nabla_\Gamma w \in \mathcal{D}^{n-1}(\Gamma)^{N-1}\} \tag{4.11}$$

and

$$\mathcal{D}(\Gamma) \stackrel{\text{def}}{=} \bigcap_{n \geq 1} \mathcal{D}^n(\Gamma) \tag{4.12}$$

The associated Sobolev spaces are

$$H_0^n(\Gamma) = \overline{\mathcal{D}(\Gamma)}^{H^n}, \qquad n \geq 1. \tag{4.13}$$

The distributional derivatives are defined as

$$\langle \nabla_\Gamma w, \Phi \rangle = -\int_\Gamma w \operatorname{div}_\Gamma \Phi \, d\Gamma + \int_\Gamma H w \Phi \cdot \nabla b \, d\Gamma, \tag{4.14}$$

for all $\Phi \in \mathcal{D}(\Gamma)^N$. It is readily seen that if

$$\Phi_t = \Phi - \Phi \cdot \nabla b \nabla b,$$

is the tangential component of Φ, then

$$\langle \nabla_\Gamma w, \Phi \rangle = -\int_\Gamma f \operatorname{div}_\Gamma \Phi_t \, d\Gamma, \tag{4.15}$$

and it is sufficient to use vectors $\Phi \in \mathcal{D}(\Gamma)^N$ such that $\Phi \cdot \nabla b$ in the definition of $\nabla_\Gamma w$.

4.2 The Laplace-Beltrami operator

Let the functions

$$a_{ij} \in L^\infty(\Gamma), \qquad 1 \leq i, j \leq N, \tag{4.16}$$

and $f \in L^2(\Gamma)$ be given. Assume that

$$\exists \alpha > 0, \quad \forall \xi \in \mathbb{R}^n, \sum_{i,j=1}^N a_{ij}(x) \xi^j \xi^i \geq \alpha |\xi|^2, \tag{4.17}$$

and that

$$\int_\Gamma f \, d\Gamma = 0. \tag{4.18}$$

Assume that the domain Γ is connected. Consider the variational problem: to find $u \in H^1(\Gamma)$ such that for all $v \in H^1(\Gamma)$

$$\langle Au, v \rangle = \int_\Gamma f v \, d\Gamma, \tag{4.19}$$

where

$$\langle Au, v\rangle = \sum_{i,j=1}^{N} \int_{\Gamma} a_{ij}(x)(\nabla_\Gamma u)_j(\nabla_\Gamma v)_i \, d\Gamma, \tag{4.20}$$

or if a is the matrix a_{ij}

$$\langle Au, v\rangle = \int_{\Gamma} a\nabla_\Gamma u \cdot \nabla_\Gamma v \, d\Gamma. \tag{4.21}$$

In view of (4.17)

$$\forall v \in H^1(\Gamma), \quad \langle Av, v\rangle \geq \alpha \|\nabla_\Gamma v\|^2_{L^2},$$

and the kernel of the operator A is given by

$$N = \{w \in H^1(\Gamma) \colon \nabla_\Gamma w = 0\}. \tag{4.22}$$

This space coincides with the constants since

$$\nabla(w \circ p) = [I - bD^2 b]\nabla_\Gamma w \circ p = 0 \Rightarrow \exists c, \quad w \circ p = c \text{ in } N_h(\Gamma).$$

As a result we obtain the "standard" result under condition (4.18)

$$\begin{cases} \exists! u \in H^1(\Gamma)/\mathbb{R}, & \forall v \in H^1(\Gamma) \\ \int_\Gamma a\nabla_\Gamma u \cdot \nabla_\Gamma v \, dx = \int_\Gamma f v \, d\Gamma \end{cases} \tag{4.23}$$

or equivalently (4.23) has a solution up to an additive constant.

If the coefficients $a_{ij} \in W^{1,\infty}(\Gamma)$, then

$$\begin{cases} -\operatorname{div}_\Gamma(a(x)\nabla_\Gamma u) + H(x)\, a(x)\nabla_\Gamma u \cdot \nabla b = f \text{ in } \Gamma \\ a(x)\nabla_\Gamma u \cdot \nu = 0 \text{ on } \partial\Gamma, \end{cases} \tag{4.24}$$

when ν is the unit exterior normal to the boundary $\partial\Gamma$ of Γ in $\partial\Omega$ (that is $\nu \cdot \nabla b = 0$). When $a = I$ then we recover the Laplace-Beltrani operator

$$\begin{cases} -\Delta_\Gamma u = f \text{ in } \Gamma \\ \nabla_\Gamma u \cdot \nu = 0 \text{ on } \partial\Gamma. \end{cases}$$

Therefore we have the same picture as in $\mathbb{R}^N$ up to the fact that the familiar differential operators are now tangential.

5 Intrinsic linear model for shells

For simplicity we shall work with a shell S_h of constant thickness. As in our previous paper we first make the following assumption on the displacement vector as in the "natural theory" of plates (cf. for instance P. Germain [1]).

5.1 Prelimimary results

Assumption 1. At each point x of the shell the *displacement vector* $V(x)$ is of the form

$$V(x) = e \circ p(x) + b(x)\ell \circ p(x), \quad x \in S_h, \tag{5.1}$$

for vector-valued mappings e and ℓ from Γ to $\mathbb{R}^N$) such that $\ell(X)$ is a tangential vector, that is $\ell(X)$ belongs to the tangent space $T_X\Gamma$ at X for each X on Γ or equivalently

$$\ell(X) \cdot \nabla b(X) = 0, \quad \forall X \in \Gamma, \tag{5.2}$$

where $\cdot$ denotes the inner product in $\mathbb{R}^N$. $\square$

With the help of the tangential calculus the Jacobian matrix DV in S_h is given by

$$DV = [D_\Gamma e \circ p + bD_\Gamma \ell \circ p + \ell \circ p \,{}^*\nabla b][I - bD^2 b] \tag{5.3}$$

$$DV \nabla b = \ell \circ p \tag{5.4}$$

and the *strain tensor* $\varepsilon(V)$ by

$$\begin{aligned} 2\varepsilon(V) &= D(V) + {}^*D(V) \\ &= [D_\Gamma(e) \circ p + bD_\Gamma(\ell) \circ p + \ell \circ p \,{}^*\nabla b][I - bD^2 b] \\ &\quad + [I - bD^2 b]^*[D_\Gamma(e) \circ p + bD_\Gamma(\ell) \circ p + \ell \circ p \,{}^*\nabla b] \end{aligned} \tag{5.5}$$

$$2\varepsilon(V)\nabla b = [I - bD^2 b][2\varepsilon_\Gamma(e) \circ p \nabla b + \ell \circ p] \tag{5.6}$$

where $\varepsilon_\Gamma(e)$ is the *tangential strain tensor* defined in (3.10). In the (X, z) coordinate system the above expression becomes

$$DV \circ T_z = [D_\Gamma e + zD_\Gamma \ell + \ell \,{}^*\nabla b][I + zD^2 b]^{-1} \tag{5.7}$$

$$DV \circ T_z \nabla b = \ell$$

$$\begin{aligned} 2\varepsilon(V) \circ T_z &= [D_\Gamma(e) + zD_\Gamma(\ell) + \ell \,{}^*\nabla b][I + zD^2 b]^{-1} \\ &\quad + [I + zD^2 b]^{-1*}[D_\Gamma(e) + zD_\Gamma(\ell) + \ell \,{}^*\nabla b] \end{aligned} \tag{5.8}$$

$$2\varepsilon(V) \circ T_z \nabla b = [I + zD^2 b]^{-1}[2\varepsilon_\Gamma(e)\nabla b + \ell]. \tag{5.9}$$

The identity

$$2\varepsilon_\Gamma(e)\nabla b + \ell = 0 \text{ on } \Gamma$$

characterizes the *Love-Kirchhoff* models. When Γ is C^3, this identity can be written as

$$\nabla_\Gamma(e \cdot \nabla b) - D^2 b \, e + \ell = 0 \quad \text{on } \Gamma$$

extending to shells the identity (with $D^2 b = 0$) which can be found in P. Germain [1] for plates.

The nonlinear part of $\varepsilon(V) \circ T_z$ with respect to the variable z is contained in the matrix $[I + zD^2 b]^{-1}$. So for $h\|D^2 b\| < 1$, that is

$$h \max_{1 \le i \le N-1} |\kappa_i(X)| < 1, \quad \forall X \in \Gamma,$$

the inverse is given by

$$[I + zD^2 b]^{-1} = \sum_{i=0}^{\infty} (-D^2 b)^i z^i, \tag{5.10}$$

and we get

$$\varepsilon(V) \circ T_z = \sum_{i=0}^{\infty} \varepsilon^i z^i, \tag{5.11}$$

where

$$\begin{aligned}
2\varepsilon^0 &= 2\varepsilon_\Gamma(e) + \ell^* \nabla b + \nabla b^* \ell & (5.12)\\
2\varepsilon^1 &= 2\varepsilon_\Gamma(\ell) - D_\Gamma(e) D^2 b - D^2 b^* D_\Gamma(e) & (5.13)\\
2\varepsilon^2 &= [D_\Gamma(\ell) - D_\Gamma(e) D^2 b](-D^2 b) & (5.14)\\
&\quad + (-D^2 b)[{}^* D_\Gamma(\ell) - D^2 b^* D_\Gamma(e)] &\\
2\varepsilon^n &= [D_\Gamma(\ell) - D_\Gamma(e) D^2 b](-D^2 b)^{n-1} & (5.15)\\
&\quad + (-D^2 b)^{n-1}[{}^* D_\Gamma(\ell) - D^2 b^* D_\Gamma(e)] &
\end{aligned}$$

for $n \geq 3$. We quote the following central result which provides a critical information on where to truncate the infinite sum.

Theorem 5.1. *Assume that e and ℓ belong to $H^1(\Gamma)^N$.*

(i) We have the following equivalence:

$$\varepsilon(V) = 0 \quad \text{in } S_h \tag{5.16}$$

if and only if

$$\varepsilon^0(e, \ell) = \varepsilon^1(e, \ell) = \frac{1}{2}[{}^* D_\Gamma \ell D^2 b + D^2 b \, D_\Gamma \ell] = 0 \quad \text{on } \Gamma, \tag{5.17}$$

if and only if

$$\varepsilon^0(e, \ell) = \varepsilon^1(e, \ell) = \varepsilon^2(e, \ell) = 0 \quad \text{on } \Gamma, \tag{5.18}$$

if and only if there exists a vector $a \in \mathbb{R}^N$ and an $N \times N$ matrix B such that

$$\ell(X) = B \nabla b, \quad e(X) = a + BX, \quad \forall X \in \Gamma, \tag{5.19}$$

where B is such that

$$B + {}^* B = 0. \tag{5.20}$$

In addition (5.19) and (5.20) imply that ℓ is tangential and $\nabla b \cdot B \nabla b = 0$ on Γ.

(ii) For all z, $|z| < h$, $X \in \Gamma$,

$$\varepsilon(V) \circ T_z = \varepsilon^0 + [I + zD^2 b]^{-1} \{ z[\varepsilon^1 + \varepsilon^1 z D^2 b + z D^2 b \, \varepsilon^1] + z^2 \varepsilon^2 \} [I + zD^2 b]^{-1}$$

and

$$\varepsilon^2 = -\varepsilon^1 D^2 b - D^2 b \, \varepsilon^1 - D^2 b \, \varepsilon^0 D^2 b + \frac{1}{2}[{}^* D_\Gamma(\ell) D^2 b + D^2 b \, D_\Gamma(\ell)] \tag{5.21}$$

and for $n \geq 2$

$$\varepsilon^{n+1} = -\varepsilon^n D^2 b - D^2 b \, \varepsilon^n - D^2 b \, \varepsilon^{n-1} D^2 b. \tag{5.22}$$

5.2 The second order model in the thickness variable

In order to preserve the so-called *rigid displacements* the series can be truncated as

$$\tilde{\varepsilon}_b(V) \circ T_z \stackrel{\text{def}}{=} \varepsilon^0(e,\ell) + \varepsilon^1(e,\ell)z + \varepsilon^2(e,\ell)z^2 \tag{5.23}$$

or

$$\begin{aligned}&\tilde{\varepsilon}_c(V) \circ T_z \\ &\stackrel{\text{def}}{=} \varepsilon^0(e,\ell) + \varepsilon^1(e,\ell)z + \frac{1}{2}[{}^*D_\Gamma(\ell)D^2b + D^2b\, D_\Gamma(\ell)]z^2.\end{aligned} \tag{5.24}$$

Both models have been investigated in Delfour and Zolésio [6] [7] and a complete theory is available. It is natural to associate with $\tilde{\varepsilon}_b$ and $\tilde{\varepsilon}_c$ the following Hilbert spaces

$$\mathcal{H} = \{(e,\ell) \in L^2(\Gamma)^N \times L^2(\Gamma)^N : \ell \cdot n = 0 \text{ on } \Gamma\} \tag{5.25}$$

$$\mathcal{V} = \left\{(e,\ell) \in \mathcal{H} \colon \varepsilon^i \in L^2(\Gamma)^{N\times N}, 0 \le i \le 2\right\} \tag{5.26}$$

$$\mathcal{N} = \left\{(e,\ell) \in \mathcal{V} \colon \varepsilon^i = 0 \text{ on } \Gamma, 0 \le i \le 2\right\} \tag{5.27}$$

and

$$\mathcal{N} = \{(e,\ell) \in \mathcal{V} \colon e(X) = a + BX,\ \ell = B\nabla b, \quad \forall a \in \mathbb{R}^N, \tag{5.28}$$

$$\forall B \text{ an } N \times N \text{ matrix such that } B + {}^*B = 0\} \tag{5.29}$$

with norms

$$|(e,\ell)|^2_{\mathcal{H}} = |e|^2_{L^2(\Gamma)} + |\ell|^2_{L^2(\Gamma)} \tag{5.30}$$

$$\|(e,\ell)\|^2_{\mathcal{V}} = |(e,\ell)|^2_{\mathcal{H}} + \sum_{i=0}^{2} \|\varepsilon^i(e,\ell)\|^2_{L^2(\Gamma)}. \tag{5.31}$$

In order to fully specify the above spaces we make use of the following *Korn's inequality*.

Theorem 5.2. *Assume that Γ is a bounded open domain in the C^2 $(N-1)$-dimensional submanifold $\partial\Omega$ of $\mathbb{R}^N$ with a Lipschitzian boundary $\partial_{\partial\Omega}\Gamma$ in $\partial\Omega$. As h goes to zero*, there exists a constant $c(h) > 0$ such that for all $(e,\ell) \in \mathcal{V}$

$$\begin{aligned}&\int_\Gamma 2h[|\ell|^2 + \|D_\Gamma(e)\|^2] + 2\frac{h^3}{3}\|D_\Gamma(\ell)\|^2\, d\Gamma \\ &\le c(h)^2 \int_\Gamma 2h|e|^2 + 2\frac{h^3}{3}|\ell|^2 + 2h\|\varepsilon^0(e,\ell)\|^2 \\ &\qquad + 2\frac{h^3}{3}\|\varepsilon^1(e,\ell)\|^2 + 2\frac{h^5}{5}\|\varepsilon^2(e,\ell)\|^2\, d\Gamma,\end{aligned} \tag{5.32}$$

where

$$\|A\|^2 = \sum_{i,j=1}^{N} A_{ij}A_{ji}, \quad |a|^2 = \sum_{i=1}^{N} a_i^2.$$

In particular

$$\mathcal{V} = \left\{(e,\ell) \in H^1(\Gamma)^N \times H^1(\Gamma)^N : \ell \cdot\ n = 0\right\}. \tag{5.33}$$

Moreover there exists $c > 0$ such that

$$\begin{aligned}\int_\Gamma \|D_\Gamma(e)\|^2 \le c^2 \int_\Gamma |e|^2 + \|\varepsilon_\Gamma(e)\|^2 \\ + \|D_\Gamma(e)D^2b + D^2b^* D_\Gamma(e)\|^2 \, d\Gamma\end{aligned} \tag{5.34}$$

$$H^1(\Gamma)^N = \left\{ e \in L^2(\Gamma)^N : \begin{array}{l} \varepsilon_\Gamma(e) \in L^2(\Gamma)^{N\times N} \\ D_\Gamma(e)D^2b + D^2b^* D_\Gamma(e) \in L^2(\Gamma)^{N\times N} \end{array} \right\}.$$

Remark . When Γ is defined through a single map Φ, we shall see in § 6.6 that a tangential Korn's inequality (6.59) holds with only ε_Γ. Does this result depend on the parametrization by a single map Φ or is it true in general? So far no example has been constructed to show that the mixed term is necessary. □

Remark . (Spherical shells) For a spherical shell $\{x \in \mathbb{R}^N : |x| = R\}$ of radius R in $\mathbb{R}^N$

$$b(x) = |x| - R, \quad \nabla b(x) = \frac{x}{|x|}, \quad p(x) = R\,\nabla b(x).$$

As a result

$$Dp(x) = R\,D^2b(x), \quad P(x) = |x|\,D^2b(x)$$

and since $P^2(x) = P(x)$

$$D^2b(x) = |x|\,(D^2b(x))^2.$$

If V is an extension of the vector $v : \Gamma \to \mathbb{R}^N$, then

$$\begin{aligned} D_\Gamma v &= DV_{|\Gamma}\,P = R\,DV_{|\Gamma}\,D^2b \\ D_\Gamma v\,D^2b &= DV_{|\Gamma}\,P\,D^2b = DV_{|\Gamma}\,R\,(D^2b)^2 = \frac{1}{R}\,D_\Gamma v \\ D_\Gamma v\,(D^2b)^2 &= DV_{|\Gamma}\,P\,(D^2b)^2 = \frac{1}{R^2}\,D_\Gamma v \end{aligned}$$

and necessarily

$$\begin{aligned} \frac{1}{2}[D_\Gamma v\,D^2b + D^2b^*D_\Gamma v] &= \frac{1}{R}\varepsilon_\Gamma(v) \\ \frac{1}{2}[D_\Gamma v\,(D^2b)^2 + (D^2b)^{2\,*}D_\Gamma v] &= \frac{1}{R^2}\,\varepsilon_\Gamma(v) \\ \varepsilon^0(e,\ell) &= \varepsilon_\Gamma(e) + \frac{1}{2}[\ell^*\nabla b + \nabla b^*\ell] \\ \varepsilon^1(e,\ell) &= \varepsilon_\Gamma(\ell) - \frac{1}{R}\,\varepsilon_\Gamma(e) \\ \varepsilon^2(e,\ell) &= -\frac{1}{R}\,\varepsilon^1(e,\ell) \end{aligned}$$

$$\tilde{\varepsilon}(V)\circ T_z = \left[1 - \frac{z}{R} + \left(\frac{z}{R}\right)^2\right] \varepsilon_\Gamma(e) + \frac{1}{2}[\ell^*\nabla b + \nabla b^*\ell] + \left[1 - \frac{z}{R}\right] z\,\varepsilon_\Gamma(\ell).$$

Setting $\ell = 0$ in (5.32) and dividing both sides by $2h$ we get

$$\int_\Gamma \|D_\Gamma(e)\|^2\,d\Gamma \le c(h)^2 \int_\Gamma |e|^2 + \left(1 + \frac{1}{3}\left(\frac{h}{R}\right)^2 + \frac{1}{5}\left(\frac{h}{R}\right)^4\right) \|\varepsilon_\Gamma(e)\|^2\,d\Gamma.$$

Therefore the mixed term can also disappear when Γ is not defined through a single map. □

Corollary . *Assume that e belongs to $H^1(\Gamma)^N$.*

(i) There exist $a \in \mathbb{R}^N$ and an $N \times N$ matrix B such that

$$e(X) = a + BX$$

where

$$^*B + B = 0 \quad \textit{and} \quad B\nabla b(X) = 0 \textit{ on } \Gamma, \tag{5.35}$$

if and only if

$$\varepsilon_\Gamma(e) = D_\Gamma e\, D^2 b + D^2 b^*\, D_\Gamma e = 0. \tag{5.36}$$

(ii) Assume that Γ verifies the following condition

$$\forall X \in \Gamma, \quad c \cdot \nabla b(X) = 0 \;\Rightarrow\; c = 0. \tag{5.37}$$

Then there exists $a \in \mathbb{R}^N$ such that

$$e(X) = a \tag{5.38}$$

if and only if

$$\varepsilon_\Gamma(e) = D_\Gamma e\, D^2 b + D^2 b^* D_\Gamma e = 0. \tag{5.39}$$

If e is zero on a piece Γ_0 of Γ with non-zero capacity, then $e(X) = 0$ in (7.39).

Remark . The reader is referred to the discussion at the end of Section 6.6 where it is pointed out that the term ε_Γ is not sufficient to obtain a tangential Poincaré's inequality. The mixed term $D_\Gamma e\, D^2 b + D^2 b^* D_\Gamma e$ seems to be essential here. □

Assume now the simplest rheological law

$$\sigma = 2\,\mu\,\tilde{\varepsilon} + \lambda\, \mathrm{tr}\,\tilde{\varepsilon}\, I, \quad \mu > 0,\ \lambda \geq 0. \tag{5.40}$$

Remark . It is interesting to note that if an hypothesis of the Naghdi type is used then since $\tilde{\varepsilon}\, n \cdot n = 0$

$$\sigma\, n \cdot n = 0 \quad \Rightarrow \quad \begin{array}{l} div_\Gamma\, e = 0,\ div_\Gamma\, \ell - \mathrm{tr}(D_\Gamma(e)\, D^2 b) = 0 \\ \text{and } \mathrm{tr}([D_\Gamma(\ell) - D_\Gamma(e)\, D^2 b]\, D^2 b) = 0. \end{array} \tag{5.41}$$

The first condition on e is some kind of *inextensibilty property* of the mean surface.

We now write the strain energy and obtain the associated bilinear operator A which is defined below in terms of five polynomials $\alpha(h)$ in h which are function on Γ and bilinear forms a_n. The polynomials $\alpha_n(h)$ of odd powers of h are defined as function of X on Γ as

$$\alpha_n(h) \stackrel{\text{def}}{=} h^{n+1} \sum_{i=0}^{N-1} [1 - (-1)^{n+i+1}] \frac{h^i}{n+i+1} K_i, \quad 0 \leq n \leq 4. \tag{5.42}$$

For $N = 3$

$$\begin{array}{l} \alpha_0 = 2\,h + 2\,\dfrac{h^3}{3}\,K \quad \alpha_1 = 2\,\dfrac{h^3}{3}\,H \quad \alpha_2 = 2\,\dfrac{h^3}{3} + 2\,\dfrac{h^5}{5}\,K \quad \alpha_3 = 2\,\dfrac{h^5}{5}\,H \\ \alpha_4 = 2\,\dfrac{h^5}{5} + 2\,\dfrac{h^7}{7}\,K. \end{array} \tag{5.43}$$

The spaces $\mathcal{H}$, $\mathcal{V}$, and $\mathcal{N}$, and their associated norms and seminorm have been defined in (7.25) to (7.31). Now introduce the bilinear operator $A\colon \mathcal{V} \to \mathcal{V}'$ and the linear operator $\mathcal{B}\colon L^2(\Gamma)^N \times L^2(\Gamma)^N \to \mathcal{H}'$: for all (e,ℓ) and $(\overline{e},\overline{\ell})$ in $\mathcal{V}$ and $\varepsilon^i = \varepsilon^i(e,\ell)$, $0 \le i \le 4$,

$$\langle A(e,\ell),(\overline{e},\overline{\ell})\rangle_{\mathcal{V}} = \sum_{n=0}^{4} \int_\Gamma \alpha_n(h) a_n((e,\ell),(\overline{e},\overline{\ell}))\, d\Gamma, \tag{5.44}$$

where

$$\begin{aligned}
a_0((e,\ell),(\overline{e},\overline{\ell})) &= 2\mu\varepsilon^0 \cdot\cdot\, \overline{\varepsilon}^0 + \lambda \operatorname{tr}\varepsilon^0 \operatorname{tr}\overline{\varepsilon}^0 \\
a_1((e,\ell),(\overline{e},\overline{\ell})) &= 2\mu[\varepsilon^0 \cdot\cdot\, \overline{\varepsilon}^1 + \overline{\varepsilon}^0 \cdot\cdot\, \varepsilon^1] + \lambda[\operatorname{tr}\varepsilon^0 \operatorname{tr}\overline{\varepsilon}^1 + \operatorname{tr}\overline{\varepsilon}^0 \operatorname{tr}\varepsilon^1] \\
a_2((e,\ell),(\overline{e},\overline{\ell})) &= 2\mu[\varepsilon^1 \cdot\cdot\, \overline{\varepsilon}^1 + \varepsilon^0 \cdot\cdot\, \overline{\varepsilon}^2 + \overline{\varepsilon}^0 \cdot\cdot\, \varepsilon^2] \\
&\quad + \lambda[\operatorname{tr}\varepsilon^1 \operatorname{tr}\overline{\varepsilon}^1 + \operatorname{tr}\varepsilon^0 \operatorname{tr}\overline{\varepsilon}^2 + \operatorname{tr}\overline{\varepsilon}^0 \operatorname{tr}\varepsilon^2] \\
a_3((e,\ell),(\overline{e},\overline{\ell})) &= 2\mu[\varepsilon^1 \cdot\cdot\, \overline{\varepsilon}^2 + \overline{\varepsilon}^1 \cdot\cdot\, \varepsilon^2] + \lambda[\operatorname{tr}\varepsilon^1 \operatorname{tr}\overline{\varepsilon}^2 + \operatorname{tr}\overline{\varepsilon}^1 \operatorname{tr}\varepsilon^2] \\
a_4((e,\ell),(\overline{e},\overline{\ell})) &= 2\mu\varepsilon^2 \cdot\cdot\, \overline{\varepsilon}^2 + \lambda \operatorname{tr}\varepsilon^2 \operatorname{tr}\overline{\varepsilon}^2.
\end{aligned} \tag{5.45}$$

Similarly introduce the continuous linear operator

$$\mathcal{B}\colon U \stackrel{\text{def}}{=} L^2(\Gamma)^N \times L^2(\Gamma)^N \to \mathcal{H}'$$

$$\langle \mathcal{B}(f,m),(e,\ell)\rangle_{\mathcal{H}} = \int_\Gamma \alpha_0(h)[f\cdot e + m\cdot \ell] + \alpha_1(h) f\cdot \ell\, d\Gamma. \tag{5.46}$$

By construction A is symmetrical and positive

$$\langle A(e,\ell),(\overline{e},\overline{\ell})\rangle_{\mathcal{V}} = \langle A(\overline{e},\overline{\ell}),(e,\ell)\rangle_{\mathcal{V}} \quad \langle A(e,\ell),(e,\ell)\rangle_{\mathcal{V}} \ge 0.$$

Lemma 5.1. *There exists $\overline{h} > 0$ and $\alpha > 0$ such that for all $0 < h < \overline{h}$*

$$\forall\,(e,\ell)\in\mathcal{V},\quad \langle A(e,\ell),(e,\ell)\rangle_{\mathcal{V}} \ge 2\mu h\alpha \sum_{n=0}^{2} h^{2n}\|\varepsilon^n(e,\ell)\|^2. \tag{5.47}$$

If the elements of the dual $\mathcal{H}'$ of $\mathcal{H}$ are identified with those of $\mathcal{H}$, then from the above Lemma A is a $\mathcal{V}$-$\mathcal{H}$ coercive operator.

Theorem 5.3. *Given $\overline{h} > 0$ as specified in Lemma 5.1 and assuming that the following condition is verified*

$$\forall\,(e,\ell)\in\mathcal{N},\quad \int_\Gamma \alpha_0(h)[f\cdot e + m\cdot\ell] + \alpha_1(h) f\cdot\ell\, d\Gamma, \tag{5.48}$$

then for all h, $0 < h \le \overline{h}$, there exists a unique solution $(\hat{e},\hat{\ell}) \in \mathcal{V}/\mathcal{N}$ to the variational equation:

$$\forall (e,\ell)\in\mathcal{V},\quad \langle A(\hat{e},\hat{\ell}),(e,\ell)\rangle_{\mathcal{V}} + \langle \mathcal{B}(f,m),(e,\ell)\rangle_{\mathcal{H}} = 0. \tag{5.49}$$

For a shell with boundary and homogeneous Dirichlet boundary conditions the results are analoguous to the ones of Theorem 5.3 without condition (5.48).

Theorem 5.4. *Given $\overline{h} > 0$ as specified in Lemma 5.1 and h, $0 < h \le \overline{h}$,* there exists a unique solution $(\hat{e},\hat{\ell}) \in \mathcal{V}_0$ to the variational equation: for all $(e,\ell)\in\mathcal{V}_0$

$$\langle A(\hat{e},\hat{\ell}),(e,\ell)\rangle_{\mathcal{V}} + \langle \mathcal{B}(f,m),(e,\ell)\rangle_{\mathcal{H}} = 0, \tag{5.50}$$

where

$$\mathcal{V}_0 = \left\{(e,\ell)\in H^1_0(\Gamma)^N \times H^1_0(\Gamma)^N : \ell\cdot n = 0\right\}.$$

Lemma 5.2. *Assume that Γ is a bounded open domain in the C^2 $(N-1)$-dimensional submanifold $\partial\Omega$ of $\mathbb{R}^N$ with a Lipschitzian boundary $\partial\Gamma = \partial_{\partial\Omega}\Gamma$ in $\partial\Omega$. For h sufficiently small*, there exists a constant $c = c(h) > 0$ such that for all $(e, \ell) \in \mathcal{V}_0$

$$\begin{aligned}&\int_\Gamma 2h|e|^2 + 2\frac{h^3}{3}|\ell|^2 + 2h[|\ell|^2 + \|D_\Gamma(e)\|^2] + 2\frac{h^3}{3}\|D_\Gamma(\ell)\|^2\, d\Gamma \\ &\leq c^2 \int_\Gamma 2h\|\varepsilon^0(e,\ell)\|^2 + 2\frac{h^3}{3}\|\varepsilon^1(e,\ell)\|^2 + 2\frac{h^5}{5}\|\varepsilon^2(e,\ell)\|^2\, d\Gamma\end{aligned} \tag{5.51}$$

and $\|\varepsilon_\Gamma(e)\| + \|\varepsilon^1(e,0)\|$ is an equivalent norm on $H_0^1(\Gamma)^N$.

6 Relationship between tangential and covariant derivatives

Associate with the space $\mathbb{R}^N$ an orthonormal basis $\{e_1, \ldots, e_N\}$ at the origin. Let Ω be a subset of $\mathbb{R}^N$ with a boundary $\partial\Omega$ of class C^2. Let Γ be a bounded subset of $\partial\Omega$ with relative boundary

$$\partial\Gamma \stackrel{\text{def}}{=} \partial_{\partial\Omega}\Gamma, \tag{6.1}$$

in the $(N-1)$-submanifold $\partial\Omega$ of $\mathbb{R}^n$.

6.1 Local coordinates

Assume the existence of a C^2-map

$$\xi' \stackrel{\text{def}}{=} (\xi^1, \ldots, \xi^{N-1}) \mapsto \Phi(\xi') \colon \bar{A} \subset \mathbb{R}^{N-1} \to \overline{\Gamma} \subset \mathbb{R}^N, \tag{6.2}$$

where A is a bounded open connected domain in $\mathbb{R}^{N-1}$ with Lipschitzian boundary ∂A (located on the same side of ∂A). Further assume that in each point the vectors

$$a_\alpha = \frac{\partial\Phi}{\partial\xi^\alpha}, \qquad 1 \leq \alpha \leq N-1, \tag{6.3}$$

are linearly independent in each point in $\bar{A}$. We shall follow the usual convention that a greek index ranges from 1 to $N-1$ and that a roman index ranges from 1 to N. The contravariant basis is defined as

$$a^i \cdot a_j = \delta_{ij}, \tag{6.4}$$

where " $\cdot$ " denotes the inner product in $\mathbb{R}^n$ and δ_{ij} the Kronecker index function. In addition we assume that Γ is oriented. For $N = 3$

$$a_3 = \frac{a_1 \times a_2}{|a_1 \times a_2|}. \tag{6.5}$$

Here for arbitrary $N's$ choose the domain Ω such that

$$a_N = a^N = -\nabla b_\Omega \circ \Phi. \tag{6.6}$$

6.2 Partial derivative of a scalar function and fundamental forms

Consider a C^1-function $w\colon \Gamma \to \mathbb{R}$ and its extension $W = w \circ p$ in a neighbourhood of Γ. By definition the *partial derivative* of w is given by

$$w_{,\alpha} \stackrel{\text{def}}{=} \frac{\partial}{\partial \xi^\alpha}(w \circ \Phi), \tag{6.7}$$

and

$$w_{,N} \stackrel{\text{def}}{=} 0, \tag{6.8}$$

since $w \circ \Phi$ is independent of the normal displacement to the submanifold $\Gamma \subset \partial\Omega$. Note that

$$w \circ \Phi = (w \circ p) \circ \Phi = W \circ \Phi$$

and consequently

$$\frac{\partial}{\partial \xi^\alpha}(w \circ \Phi) = (\nabla W \circ \Phi) \cdot \frac{\partial \Phi}{\partial \xi^\alpha}[(\nabla W \circ p) \circ \Phi] \cdot a_\alpha.$$

But

$$\nabla W \circ p = \nabla(w \circ p) \circ p = \nabla_\Gamma w \circ p,$$

and

$$\frac{\partial}{\partial \xi^\alpha}(w \circ \Phi) = [\nabla_\Gamma w \circ p \circ \Phi] \cdot a_\alpha = [\nabla_\Gamma w \cdot a_\alpha \circ \Phi^{-1}] \circ \Phi.$$

In the sequel it will be convenient to use the notation a_i for both a_i and $a_i \circ \Phi^{-1}$ whenever no confusion arises. Hence

$$w_{,\alpha} = [\nabla_\Gamma w \cdot a_\alpha] \circ \Phi. \tag{6.9}$$

Moreover

$$w_{,N} = 0 = [\nabla_\Gamma w \cdot a_N] \circ \Phi, \tag{6.10}$$

since

$$\nabla_\Gamma w \cdot \nabla b = 0.$$

This gives the first direct connection between the partial derivatives of $w \circ \Phi$ and the *directional tangential derivative* of w in the direction a_i.

This readily extends to C^1 vector functions $v\colon \Gamma \to \mathbb{R}^N$ by using the extension $V = v \circ p$ to a neighbourhood of Γ

$$v_{,\alpha} \stackrel{\text{def}}{=} \frac{\partial}{\partial \xi^\alpha}(v \circ \Phi), \quad v_{,N} \stackrel{\text{def}}{=} 0. \tag{6.11}$$

By the same technique

$$v_{,i} = [D_\Gamma v \circ \Phi] a_i. \tag{6.12}$$

As an illustration of the above identity, consider the *second fundamental form*

$$b_{\alpha\beta} \stackrel{\text{def}}{=} -a_\beta \cdot a_{N,\alpha}. \tag{6.13}$$

By definition

$$a_{N,\alpha} = D_\Gamma a_N \, a_\alpha = -D_\Gamma(\nabla b) \, a_\alpha.$$

But $\nabla b = \nabla b \circ p$

$$D_\Gamma(\nabla b) = D(\nabla b \circ p)|_\Gamma = D(\nabla b)|_\Gamma = D^2 b|_\Gamma,$$

and

$$a_{N,\alpha} = -D^2 b\, a_\alpha.$$

Finally

$$b_{\alpha\beta} = a_\beta \cdot D^2 b\, a_\alpha. \tag{6.14}$$

Note that the definition of $b_{\alpha\beta}$ readily extends to the whole space $\mathbb{R}^N$ as follows

$$b_{ij} \stackrel{\text{def}}{=} D^2 b\, a_i \cdot a_j, \tag{6.15}$$

and since $D^2 b \nabla b = 0$

$$b_{Ni} = b_{iN} = 0. \tag{6.16}$$

Therefore the second fundamental form coincides with the bilinear form associated with the matrix $D^2 b$. In a similar way consider the *third fundamental form*

$$c_{\alpha\beta} \stackrel{\text{def}}{=} b_\alpha^\lambda\, b_{\lambda\beta} \tag{6.17}$$

where

$$b_i^j \stackrel{\text{def}}{=} a^j \cdot D^2 b\, a_i. \tag{6.18}$$

From (6.14)

$$\begin{aligned} a^\lambda \cdot a^\beta b_{\beta\alpha} &= a^\lambda \cdot a^\beta\, a_\beta \cdot D^2 b\, a_\alpha \\ &= a^\lambda \cdot [D^2 b\, a_\alpha - D^2 b\, a_\alpha \cdot a_N\, a^N]. \end{aligned}$$

But $D^2 b\, a_N = -D^2 b \nabla b = 0$ and necessarily

$$b_\alpha^\lambda = a^\lambda \cdot D^2 b\, a_\alpha, \tag{6.19}$$

which can also be extended to indices equal to N. Therefore

$$c_{\alpha\beta} = a^\lambda \cdot D^2 b\, a_\alpha\, a_\lambda \cdot D^2 b\, a_\beta = D^2 b\, a_\alpha \cdot D^2 b\, a_\beta,$$

and

$$c_{\alpha\beta} = a_\beta \cdot (D^2 b)^2 a_\alpha. \tag{6.20}$$

By extending $c_{\alpha\beta}$ to $\mathbb{R}^N$

$$c_{ij} = a_i \cdot D^2 b\, a_j \quad \text{and} \quad c_{Ni} = c_{iN} = 0.$$

6.3 Christoffel symbols

By definition

$$\Gamma^\alpha_{\beta\gamma} \stackrel{\text{def}}{=} a^\alpha \cdot a_{\beta,\gamma}, \tag{6.21}$$

and if we recall that for any vector function v on Γ, $v_{,N} = 0$, this definition extends to $\mathbb{R}^N$

$$\Gamma^i_{jk} \stackrel{\text{def}}{=} a^i \cdot a_{j,k}. \tag{6.22}$$

Hence from (6.12)

$$\Gamma^i_{jk} = a^i \cdot D_\Gamma(a_j \circ \Phi^{-1}) \circ \Phi\, a_k \quad \text{and} \quad \Gamma^i_{jN} = 0. \tag{6.23}$$

For simplicity we shall denote $D_\Gamma(a_j \circ \Phi^{-1}) \circ \Phi$ by $D_\Gamma a_j$. Moreover

$$a^i \cdot a_j = \delta_{ij} \Rightarrow \frac{\partial}{\partial \xi^\gamma}(a^i \cdot a_j) = 0,$$

and for all γ

$$D_\Gamma a^i \, a_\gamma \cdot a_j + D_\Gamma a_j \, a_\gamma \cdot a^i = 0,$$

which yields

$$\Gamma^i_{j\gamma} = a^i \cdot D_\Gamma a_j a_\gamma = -D_\Gamma a^i \, a_\gamma \cdot a_j = -{}^*D_\Gamma a^i \, a_j \cdot a_\gamma. \tag{6.24}$$

For a C^2-mapping

$$a_{\gamma,\beta} = \frac{\partial}{\partial \xi^\beta}\frac{\partial}{\partial \xi^\gamma}\Phi = \frac{\partial}{\partial \xi^\gamma}\frac{\partial}{\partial \xi^\beta}\Phi = a_{\beta,\gamma}, \tag{6.25}$$

and

$$\Gamma^i_{\beta\gamma} = \Gamma^i_{\gamma\beta}. \tag{6.26}$$

Finally from (6.22), (6.23) and (6.24)

$$\begin{aligned}\Gamma^i_{Nj} &= -D_\Gamma a^i \, a_j \cdot a_N = a_j \cdot {}^*D_\Gamma a^i \, \nabla b \\ &= a_j \cdot [-D^2 b \, a^i + \nabla_\Gamma (a^i \cdot \nabla b)].\end{aligned}$$

But $a^i \cdot \nabla b = \delta_{i,N}$ is constant and

$$\Gamma^i_{Nj} = -a^i \cdot D^2 b \, a_j = -b^i_j. \tag{6.27}$$

6.4 Covariant derivatives

We cover the vector case, but all this readily extends to tensors of higher order. So for $v \colon \Gamma \to \mathbb{R}^N$

$$v_\alpha|_\gamma \overset{\text{def}}{=} v_{\alpha,\gamma} - \Gamma^\lambda_{\alpha\gamma} v_\lambda \tag{6.28}$$

$$v^\alpha|_\gamma \overset{\text{def}}{=} v^\alpha_{,\gamma} + \Gamma^\alpha_{\lambda\gamma} v^\lambda. \tag{6.29}$$

Recalling that $\Gamma^i_{jN} = 0$ and that $v_{\alpha,N} = v^\alpha_{,N} = 0$, the above definitions are extendable and

$$v_\alpha|_N = 0 = v^\alpha|_N. \tag{6.30}$$

They also make sense for $\alpha = N$. For any two vector functions $u, v \colon \Gamma \to \mathbb{R}^N$

$$\begin{aligned}\frac{\partial}{\partial \xi^\gamma}(u \circ \Phi \cdot v \circ \Phi) &= (u \circ \Phi)_{,\gamma} \cdot v \circ \Phi + u \circ \Phi \cdot (v \circ \Phi)_{,\gamma} \\ \frac{\partial}{\partial \xi^\gamma}(u \circ \Phi \cdot v \circ \Phi) &= ({}^*D_\Gamma u \circ \Phi \, v + {}^*D_\Gamma v \circ \Phi \, u) \cdot a_\gamma.\end{aligned} \tag{6.31}$$

Again for simplicity we shall write $D_\Gamma u$ instead of $D_\Gamma u \circ \Phi$. Hence

$$v_{\alpha,\gamma} \overset{\text{def}}{=} \frac{\partial}{\partial \xi^\gamma}(v \circ \Phi \cdot a_\alpha) = D_\Gamma v \, a_\gamma \cdot a_\alpha + v \cdot D_\Gamma a_\alpha a_\gamma,$$

and from (6.27)

$$\begin{aligned}v_\alpha|_\gamma &= D_\Gamma v \, a_\gamma \cdot a_\alpha + v \cdot D_\Gamma a_\alpha a_\gamma - a^\lambda \cdot D_\Gamma a_\alpha a_\gamma v_\lambda \\ &= D_\Gamma v \, a_\gamma \cdot a_\alpha + (v - a^\lambda v_\lambda) \cdot D_\Gamma a_\alpha a_\gamma \\ &= D_\Gamma v \, a_\gamma \cdot a_\alpha + a^N \, v_N \cdot D_\Gamma a_\alpha a_\gamma \\ &= D_\Gamma v \, a_\gamma \cdot a_\alpha - {}^*D_\Gamma a_\alpha \nabla b \cdot a_\gamma v_N.\end{aligned}$$

Recall from (3.6) that

$$^*D_\Gamma a_\alpha \nabla b = \nabla_\Gamma(a_\alpha \cdot \nabla b) - D^2 b\, a_\alpha = -D^2 b\, a_\alpha.$$

Finally

$$\boxed{v_\alpha|_\gamma = D_\Gamma v\, a_\gamma \cdot a_\alpha + D^2 b\, a_\alpha \cdot a_\gamma v_N = [^*D_\Gamma v + D^2 b\, v_N] a_\alpha \cdot a_\gamma}. \tag{6.32}$$

Similarily we can show that

$$v^\alpha|_\gamma = D_\Gamma v\, a_\gamma \cdot a^\alpha + D^2 b\, a^\alpha \cdot a_\gamma v_N. \tag{6.33}$$

Equivalently

$$^*D_\Gamma v\, a_\alpha \cdot a_\gamma = v_\alpha|_\gamma - b_{\alpha\gamma} v_N \tag{6.34}$$

$$^*D_\Gamma v\, a^\alpha \cdot a_\gamma = v^\alpha|_\gamma - b^\alpha_\gamma v_N \tag{6.35}$$

For tangential vector fields ($v \cdot \nabla b = 0$), the covariant derivatives coincide with the bilinear form generated by $^*D_\Gamma v$. For non tangent field we have an additional term which arises from the fact that in the definition of $v_\alpha|_\gamma$ and $v^\alpha|_\gamma$ the summation over λ ranges from 1 to $N-1$ missing the normal component v_N.

6.5 A few useful formulas

In the theory of shells some identities will repeatedly occur. We summarize them below.

Theorem 6.1. *For all u and v in $H^1(\Gamma)^N$*

$$\varepsilon_\Gamma(v)\, a_\alpha \cdot a_\beta = \frac{1}{2}(v_\alpha|_\beta + v_\beta|_\alpha) - b_{\alpha\beta} v_N, \tag{6.36}$$

$$\varepsilon_\Gamma(v)\, a_\alpha \cdot a^\beta = \frac{1}{2}(v^\alpha|_\beta + v^\beta|_\alpha) - b^\beta_\alpha v_N, \tag{6.37}$$

$$(D^2 b\, D_\Gamma v)\, a_\alpha \cdot a_\beta = b^\gamma_\beta [v_\gamma|_\alpha - b_{\alpha\gamma} v_N] = b^\gamma_\beta v_\gamma|_\alpha - c_{\alpha\beta} v_N, \tag{6.38}$$

$$(D_\Gamma v D^2 b)\, a_\alpha \cdot a_\beta = b^\gamma_\alpha [v_\beta|_\gamma - b_{\beta\gamma} v_N] = b^\gamma_\alpha v_\beta|_\gamma - c_{\alpha\beta} v_N, \tag{6.39}$$

where

$$c^\beta_\alpha = c_{\alpha\gamma} a^\gamma_\beta, \quad c^{\alpha\beta} = c_{\alpha\gamma} a^{\gamma\beta}. \tag{6.40}$$

Moreover

$$a_\beta \cdot \varepsilon_\Gamma(u)\, a^\beta = \operatorname{tr} \varepsilon_\Gamma(u) = \operatorname{div}_\Gamma u, \tag{6.41}$$

$$\begin{aligned} & a^\alpha \cdot \varepsilon_\Gamma(u) a_\beta\, \varepsilon_\Gamma(v)\, a^\beta \cdot a_\alpha \\ &= \varepsilon^0(u, -2\varepsilon_\Gamma(u)\nabla b) \cdot\cdot\, \varepsilon^0(v, -2\varepsilon_\Gamma(v)\nabla b) \\ &= \varepsilon_\Gamma(u) \cdot\cdot\, \varepsilon_\Gamma(v) - 2\varepsilon_\Gamma(u)\nabla b \cdot \varepsilon_\Gamma(v)\nabla b \end{aligned} \tag{6.42}$$

where

$$\varepsilon^0(u, \ell) \stackrel{def}{=} \varepsilon_\Gamma(u) + \frac{1}{2}(\ell^* \nabla b + \nabla b^* \ell). \tag{6.43}$$

Proof. (i) Identities (6.36) and (6.37) directly follow from (6.34)–(6.35). For (6.38)

$$
\begin{aligned}
(D^2b\, D_\Gamma v)\, a_\alpha \cdot a_\beta &= D_\Gamma v\, a_\alpha \cdot D^2 b\, a_\beta = D_\Gamma v\, a_\alpha \cdot a_\gamma\, a^\gamma \cdot D^2 b\, a_\beta \\
&= b^\gamma_\beta [v_\gamma|_\alpha - b_{\alpha\gamma} v_N] = b^\gamma_\beta v_\gamma|_\alpha - c_{\alpha\beta} v_N, \\
(D_\Gamma v D^2 b)\, a_\alpha \cdot a_\beta &= D^2 b\, a_\alpha \cdot {}^*D_\Gamma v\, a_\beta = D^2 b\, a_\alpha \cdot a^\gamma a_\gamma \cdot {}^*D_\Gamma v\, a_\beta \\
&= b^\gamma_\alpha [v_\beta|_\gamma - b_{\beta\gamma} v_N] = b^\gamma_\alpha v_\beta|_\gamma - c_{\alpha\beta} v_N.
\end{aligned}
$$

(ii) For the other three formulas recall that the greek indices are summed from 1 to $N-1$. By definition

$$
\begin{aligned}
\operatorname{tr} \varepsilon_\Gamma(u) &= e_i \cdot \varepsilon_\Gamma(u) e_i = e_i \cdot a^j\, a_j \cdot \varepsilon_\Gamma(u)\, a^\ell\, a_\ell \cdot e_i \\
&= a_j \cdot \varepsilon_\Gamma(u)\, a^\ell\, a^j \cdot e_i\, e_i \cdot a_\ell \\
&= a_j \cdot \varepsilon_\Gamma(u)\, a^\ell\, a^j \cdot a_\ell = a_j \cdot \varepsilon_\Gamma(u)\, a^\ell\, \delta_{j\ell} \\
&= a_j \cdot \varepsilon_\Gamma(u)\, a^j = a_\beta \cdot \varepsilon_\Gamma(u)\, a^\beta,
\end{aligned}
$$

since

$$
a_N \cdot \varepsilon_\Gamma(u)\, a^N = \nabla b \cdot \varepsilon_\Gamma(u) \nabla b = 0.
$$

For (6.42)

$$
\begin{aligned}
\varepsilon_\Gamma(u) \cdot\cdot\, \varepsilon_\Gamma(v) &= \varepsilon_\Gamma(u) e_i \cdot e_j\, \varepsilon_\Gamma(v) e_i \cdot e_j \\
&= [\varepsilon_\Gamma(u)\, a_k\, a^k \cdot e_i] \cdot (a^\ell\, a_\ell \cdot e_j)\, [\varepsilon_\Gamma(v)\, a^m\, a_m \cdot e_i] \cdot (a_n\, a^n \cdot e_j) \\
&= \varepsilon_\Gamma(a_k) \cdot a^\ell\, \varepsilon_\Gamma(v)\, a^m \cdot a_n\, a^k \cdot e_i\, a_\ell \cdot e_j\, a_m \cdot e_i\, a^n \cdot e_j \\
&= \varepsilon_\Gamma(a_k) \cdot a^\ell\, \varepsilon_\Gamma(v)\, a^m \cdot a_n\, a^k \cdot a_m\, a_\ell \cdot a^n \\
&= \varepsilon_\Gamma(a_k) \cdot a^\ell\, \varepsilon_\Gamma(v)\, a^m \cdot a_n\, \delta_{km} \delta_{\ell n} \\
&= \varepsilon_\Gamma(u)\, a_m \cdot a^\ell\, \varepsilon_\Gamma(v)\, a^m \cdot a_\ell.
\end{aligned}
$$

But

$$
\begin{aligned}
&\varepsilon_\Gamma(u)\, a_N \cdot a^\ell\, \varepsilon_\Gamma(v)\, a^N \cdot a_\ell + \varepsilon_\Gamma(u)\, a_m \cdot a^N\, \varepsilon_\Gamma(v)\, a^m \cdot a_N \\
&\quad = \varepsilon_\Gamma(u) \nabla b \cdot a^\ell\, a_\ell \cdot\cdot\, \varepsilon_\Gamma(v) \nabla b + \varepsilon_\Gamma(u) \nabla b \cdot a_m\, a^m \cdot \varepsilon_\Gamma(v) \nabla b \\
&\quad = 2 \varepsilon_\Gamma(u) \nabla b \cdot \varepsilon_\Gamma(v) \nabla b,
\end{aligned}
$$

and

$$
\varepsilon_\Gamma(u) \cdot\cdot\, \varepsilon_\Gamma(v) = \varepsilon_\Gamma(u)\, a_\beta \cdot a^\alpha\, \varepsilon_\Gamma(v)\, a^\beta \cdot a_\alpha + 2 \varepsilon_\Gamma(u) \nabla b \cdot \varepsilon_\Gamma(v) \nabla b.
$$

Now by definition (6.43) of ε^0

$$
\begin{aligned}
&\varepsilon^0(u, -2\varepsilon_\Gamma(u)\nabla b) \cdot\cdot\, \varepsilon^0(v, -2\varepsilon_\Gamma(v)\nabla b) \\
&= \big[\varepsilon_\Gamma(u) - (\varepsilon_\Gamma(u) \nabla b\, {}^*\nabla b + \nabla b^* (2 \varepsilon_\Gamma(u) \nabla b))\big] \\
&\quad \cdot\cdot\, \big[\varepsilon_\Gamma(v) - (\varepsilon_\Gamma(v) \nabla b^* \nabla b + \nabla b^* (\varepsilon_\Gamma(v) \nabla b))\big] \\
&= \varepsilon_\Gamma(u) \cdot\cdot\, \varepsilon_\Gamma(v) - 2 \varepsilon_\Gamma(u) \nabla b \cdot \varepsilon_\Gamma(v) \nabla b \\
&\quad - 2 \varepsilon_\Gamma(v) \nabla b \cdot \varepsilon_\Gamma(u) \nabla b + 2 \varepsilon_\Gamma(u) \nabla b \cdot \varepsilon_\Gamma(v) \nabla b \\
&= \varepsilon_\Gamma(u) \cdot\cdot\, \varepsilon_\Gamma(v) - 2 \varepsilon_\Gamma(u) \nabla b \cdot \varepsilon_\Gamma(v) \nabla b = \varepsilon_\Gamma(u) a_\beta \cdot a^\alpha\, \varepsilon_\Gamma(v)\, a^\beta \cdot a_\alpha. \quad \square
\end{aligned}
$$

6.6 More connections

To each $w\colon \Gamma \to \mathbb{R}$, associate $w \circ \Phi\colon \bar{A} \to \mathbb{R}$ and from (4.9)

$$\nabla_\Gamma w \cdot e_i = \nabla_\Gamma w \cdot a_\alpha\, a^\alpha \cdot e_i = w_{,\alpha}\, a^\alpha \cdot e_i \tag{6.44}$$

$$w_{,\alpha} = \nabla_\Gamma w \cdot a_\alpha = \nabla_\Gamma w \cdot e_i\, e_i \cdot a_\alpha \tag{6.45}$$

where $\{e_i\}$ is the orthonormal basis in $\mathbb{R}^N$. From this $\nabla_\Gamma w$ exists if and only if $\{w_{,\alpha}\}$ exists. In addition

$$|\nabla_\Gamma w| \le \left\{\sum_\alpha |w_{,\alpha}|^2\right\}^{1/2} \left\{\sum_\alpha |a^\alpha|^2\right\}^{1/2} \tag{6.46}$$

$$\left\{\sum_\alpha |w_{,\alpha}|^2\right\}^{1/2} \le |\nabla_\Gamma w| \left\{\sum_\alpha |a_\alpha|^2\right\}^{1/2} \tag{6.47}$$

and if

$$|a^\bullet| \stackrel{\text{def}}{=} \left\{\sum_\alpha |a^\alpha|^2\right\}^{1/2} \quad \text{and} \quad |a_\bullet| \stackrel{\text{def}}{=} \left\{\sum_\alpha |a_\alpha|^2\right\}^{1/2}$$

are bounded then

$$\nabla_\Gamma w \in L^2(\Gamma) \Leftrightarrow \nabla(w \circ \Phi) \in L^2(\bar{A}). \tag{6.48}$$

To each $v\colon \Gamma \to \mathbb{R}^N$, associate $v \circ \Phi\colon \bar{A} \to \mathbb{R}^N$. If v is tangential, $v(X) \cdot \nabla b(X) = 0$ on Γ, associate with v the vector function $\vec{v}\colon \bar{A} \to \mathbb{R}^{N-1}$

$$\vec{v}_\alpha \stackrel{\text{def}}{=} v_\alpha, \quad v_\alpha \stackrel{\text{def}}{=} (v \circ \Phi) \cdot a_\alpha. \tag{6.49}$$

From (6.28), (6.34) and (6.24)

$$v_{\alpha,\beta} = [{}^*D_\Gamma v - {}^*D_\Gamma a^\lambda v_\lambda]\, a_\alpha \cdot \alpha_\beta = \nabla_\Gamma v_\alpha \cdot a_\beta \tag{6.50}$$

with the notation

$$(D\vec{v})_{\alpha\beta} = v_{\alpha,\beta} \tag{6.51}$$

there exists a constant $c > 0$ such that

$$\|D\vec{v}\|^2 \le c^2[\|D_\Gamma v\|^2 + |v|^2]. \tag{6.52}$$

Conversely

$$\begin{aligned} D_\Gamma v\, e_i \cdot e_j &= D_\Gamma v\, a_\ell \cdot a_m\, a^\ell \cdot e_i\, a^m \cdot e_j \\ &= D_\Gamma v\, a_\beta \cdot a_\alpha\, a^\beta \cdot e_i\, a^\alpha \cdot e_j + D_\Gamma v\, a_\beta \cdot a_N\, a^\beta \cdot e_i\, a^N \cdot e_j \\ &= [v_{\alpha,\beta} + D_\Gamma\, a^\lambda a_\beta \cdot a_\alpha v_\lambda]\ a^\beta \cdot e_i\ a^\alpha \cdot e_j - D^2 b\, v \cdot a_\beta\ a^\beta \cdot e_i\ \partial_j b \end{aligned}$$

$$D_\Gamma v\, e_i \cdot e_j = v_{\alpha,\beta}\ a^\beta \cdot e_i\ a^\alpha \cdot e_j + D_\Gamma\, a^\lambda e_i \cdot e_j\ v_\lambda \tag{6.53}$$

and there exists a constant $c > 0$ such that

$$\|D_\Gamma v\|^2 \le c^2[\|D\vec{v}\|^2 + |\vec{v}|^2]. \tag{6.54}$$

Similarly from (6.50)

$$\varepsilon(\vec{v})_{\alpha\beta} = [\varepsilon_\Gamma(v) - \varepsilon_\Gamma(a^\lambda) v_\lambda]\, a_\alpha \cdot a_\beta \tag{6.55}$$

where by symmetry $\varepsilon_\Gamma(a^\lambda)\, a_\alpha \cdot a_\beta = \Gamma^\lambda_{\alpha\beta} = \Gamma^\lambda_{\beta\alpha}$ and

$$\|\varepsilon(\vec{v})\|^2 \le c^2[\|\varepsilon_\Gamma(v)\|^2 + |v|^2] \tag{6.56}$$

$$\|\varepsilon_\Gamma(v)\|^2 \le c^2[\|\varepsilon(\vec{v})\|^2 + |\vec{v}|^2]. \tag{6.57}$$

In $\mathbb{R}^{N-1}$ we have a Korn's inequality: there exists $c > 0$ such that for all $\vec{v}$

$$\|D\vec{v}\| \le c[\|\varepsilon(\vec{v})\|^2 + |\vec{v}|^2]^{1/2}. \tag{6.58}$$

Therefore in view of (6.54), (6.58) and (6.56) for all v tangential

$$\|D_\Gamma v\| \le c[\|\varepsilon_\Gamma(v)\|^2 + |v|^2]^{1/2}. \tag{6.59}$$

However this does not extend to Poincaré's inequality. We have

$$|\vec{v}| \le c\|\varepsilon(\vec{v})\|$$

but from (6.56)

$$|v| \le c[\|\varepsilon_\Gamma(v)\|^2 + |v|^2]^{1/2} \tag{6.60}$$

since even for $\vec{v}$ which are zero on $\partial\Gamma$ the term in v_λ does not vanish in identity (6.55) unless $\Gamma^\lambda_{\alpha\beta} = 0$. This is true for the plate and the cylinder (with cylindrical coordinates) in $\mathbb{R}^3$. We have seen in Lemma 5.2 that there exists a constant $c > 0$ such that

$$\int_\Gamma |v|^2 + \|D_\Gamma v\|^2 \, d\Gamma \le c^2 \int_\Gamma \|\varepsilon_\Gamma(v)\|^2 + \|D_\Gamma\, e\, D^2 b + D^2 b\,{}^* D_\Gamma e\|^2 \, d\Gamma.$$

From this discussion it would seem that it takes more than $\varepsilon_\Gamma(v)$ to get a Poincaré's inequality in the tangential calculus. This is a fundamental issue which has not yet received a complete answer.

7 Some classical linear models

In this section we use the material from §6 to rewrite the linear models of Naghdi and Koiter and the asymptotic model in the tangential notation. For the classical models we use the notation and definitions given in M. Bernadou [1], and in Ciarlet and Sanchez-Palencia [1].

7.1 Naghdi's linear model

We use the variational forms and associated definitions from M. Bernadou [1, Chapter I, §3]. The details of the transcription are given in Appendix A.1. We have

$$\begin{aligned}\frac{hE}{1+\nu}\int_\Gamma & \varepsilon^0(e, \ell - \nabla_\Gamma(e\cdot\nabla b)) \cdot\cdot\, \varepsilon^0(\bar{e}, \bar{\ell} - \nabla_\Gamma(\bar{e}\cdot\nabla b)) \\ & + \frac{h^2}{12}\varepsilon^1_t(e,\ell) \cdot\cdot\, \varepsilon^1_t(\bar{e},\bar{\ell}) \\ & + \frac{1}{1-\nu}\left\{\mathrm{div}_\Gamma\, e\; \mathrm{div}_\Gamma\, \bar{e} + \frac{h^2}{12}\mathrm{div}_\Gamma\, \ell\; \mathrm{div}_\Gamma\, \bar{\ell}\right\} d\Gamma \\ = \int_\Gamma & p\cdot\bar{e}\, d\Gamma + \int_{\partial\Gamma} N\cdot\bar{e} - M\cdot\bar{\ell}\, d\gamma,\end{aligned} \tag{7.1}$$

where

$$\varepsilon^0(v,\zeta) \stackrel{\text{def}}{=} \varepsilon_\Gamma(v) + \frac{1}{2}[\zeta\, {}^*\nabla b + \nabla b\, {}^*\zeta], \tag{7.2}$$

$$\varepsilon_t^1(v,\zeta) \stackrel{\text{def}}{=} \varepsilon_\Gamma(\zeta) + \frac{1}{2}[D^2 b\, D_\Gamma v +^* D_\Gamma v\, D^2 b] - \frac{1}{2}[D^2 b\, \zeta\, {}^*\nabla b + \nabla b\, {}^*(D^2 b\, \zeta)]. \tag{7.3}$$

The detailed computations are given in Appendix A.1 with

$$h = e, \quad \ell = -\beta, \quad e = u, \quad \bar{\ell} = -\delta, \quad \bar{e} = v. \tag{7.4}$$

The difference with our intrinsic model is that Naghdi uses the additional assumption $\tilde{\sigma}\, n \cdot n = 0$ which yields some type of inextensibility condition on e and ℓ as was pointed out in section 5.2.

7.2 Koiter's linear model

It is the same model as Naghdi's model with

$$\ell + 2\varepsilon_\Gamma(e)\nabla b = 0. \tag{7.5}$$

In particular

$$0 = \ell + 2\varepsilon_\Gamma(e)\nabla b = \ell + \nabla_\Gamma(e \cdot \nabla b) - D^2 b\, e. \tag{7.6}$$

7.3 Asymptotic membrane equation model

It is specified in Appendix A.3 with the coefficients λ and μ such that $\lambda \geq 0$ and $\mu > 0$ (cf. Ciarlet and Sanchez-Palencia [1].

$$\begin{aligned}\int_\Gamma & 4\mu\, \varepsilon^0(e, -2\varepsilon_\Gamma(e)\nabla b) \cdot\cdot\, \varepsilon^0(\bar{e}, -2\varepsilon_\Gamma(\bar{e})\nabla b) \\ &+ \frac{4\mu\lambda}{\lambda + 2\mu} \operatorname{tr} \varepsilon^0(e, -2\varepsilon_\Gamma(e)\nabla b) \operatorname{tr} \varepsilon^0(\bar{e}, -2\varepsilon_\Gamma(\bar{e})\nabla b)\, d\Gamma = \int_\Gamma f \cdot \bar{e}\, d\Gamma.\end{aligned} \tag{7.7}$$

The identification with the variables of Appendix A.3 are

$$e = \eta, \quad \bar{e} = \zeta. \tag{7.8}$$

Recall from (3.16) and (6.42) that

$$\varepsilon^0(e, -2\varepsilon_\Gamma(e)\nabla b) = \varepsilon_\Gamma^P(e), \tag{7.9}$$

and hence

$$\int_\Gamma 4\mu\, \varepsilon_\Gamma^P(e) \cdot\cdot\, \varepsilon_\Gamma^P(\bar{e}) + \frac{4\mu\lambda}{\lambda + 2\mu} \operatorname{tr} \varepsilon_\Gamma^P(e)\, \operatorname{tr} \varepsilon_\Gamma^P(\bar{e})\, d\Gamma = \int_\Gamma f \cdot \bar{e}\, d\Gamma. \tag{7.10}$$

8 Asymptotic membrane equations from the intrinsic model

In this section we study the asymptotic membrane behaviour of the natural model of the shell as the thickness h goes to zero. We show that the equations are the same as those in Ciarlet and Sanchez-Palencia [1]). The equations for the normal component w of e and the tangential part u of e are coupled when $D^2 b \neq 0$. When $D^2 b = 0$ they are uncoupled and the normal component is arbitrary. In both cases the variable ℓ is given explicitly in terms of u, w and the function m.

The attentive reader will have noticed that the coefficient α_n which appears in expression (5.44) of the operator A is proportional to h^{n+1}, $0 \leq n \leq 4$. Dividing both sides of the variational equation (5.49) by $2h$ and going to the limit as h goes to zero yields a new variational equation whose solution or solutions seem to be the candidate or candidates for the asymptotic limit of the solution $(e(h), \ell(h))$ of (5.49) at $h = 0$.

Define

$$\underline{\alpha}_n = \alpha_n/(2\,h), \quad 0 \leq n \leq 4, \tag{8.1}$$

$$\begin{aligned} a(h;(e,\ell),(\overline{e},\overline{\ell})) &= \sum_{n=0}^{4} \int_\Gamma \underline{\alpha}_n a_n((e,\ell),(\overline{e},\overline{\ell}))\, d\Gamma \\ b(h;(f,m),(\overline{e},\overline{\ell})) &= \int_\Gamma \underline{\alpha}_0 (f\cdot\overline{e} + m\cdot\overline{\ell}) + \underline{\alpha}_1 f\cdot\overline{\ell}\, d\Gamma. \end{aligned} \tag{8.2}$$

Equation (5.49) can be rewritten with the above bilinear forms as follows

$$a(h;(e(h),\ell(h)),(e,\ell)) + b(h;(f,m),(e,\ell)) = 0. \tag{8.3}$$

For $h = 0$ we formally get the following variational equation for $(\hat{e}_0, \hat{\ell}_0) = (\hat{e}(0), \hat{\ell}(0))$

$$a(0;(\hat{e}_0,\hat{\ell}_0),(e,\ell)) + b(0;(f,m),(e,\ell)) = 0. \tag{8.4}$$

From (5.43) and (8.1), only $\underline{\alpha}(0)$ is different from zero and

$$a(0;(e,\ell),(\overline{e},\overline{\ell})) = \int_\Gamma 2\mu\varepsilon_0 \cdot\cdot\, \overline{\varepsilon}_0 + \lambda \operatorname{tr}\varepsilon_0 \operatorname{tr}\overline{\varepsilon}_0\, d\Gamma. \tag{8.5}$$

Recall that

$$\varepsilon^0(e,\ell) = \varepsilon_\Gamma(e) + \frac{1}{2}[\ell^*\nabla b + \nabla b^*\ell], \quad \operatorname{tr}\varepsilon^0(e,\ell) = \operatorname{tr}\varepsilon_\Gamma(e) = \operatorname{div}_\Gamma(e). \tag{8.6}$$

From Lemma 5.1 setting $h = 0$ in (5.47) after dividing both sides by $2h$ we also have

$$a(0;(e,\ell),(e,\ell)) \geq \mu\alpha \|\varepsilon^0(e,\ell)\|_{L^2}^2. \tag{8.7}$$

This suggests to work in the space

$$\mathcal{V}_0^\bullet = \left\{(e,\ell) \in L^2(\Gamma)^N \times L^2(\Gamma)^N : \ell\cdot\nabla b = 0 \quad \text{and} \quad \varepsilon^0(e,\ell) \in L^2(\Gamma)^{N\times N}\right\}, \tag{8.8}$$

and to introduce the operator

$$\langle A_0(e,\ell),(\overline{e},\overline{\ell})\rangle_{\mathcal{V}_0^\bullet} = \int_\Gamma 2\mu\varepsilon^0 \cdot\cdot\, \overline{\varepsilon}^0 + \lambda \operatorname{tr}\varepsilon^0 \operatorname{tr}\overline{\varepsilon}^0\, d\Gamma. \tag{8.9}$$

In general if we denote by u and w the tangential and normal components of e

$$\varepsilon^0(e,\ell) = \varepsilon_\Gamma(u) + w\,D^2 b + \frac{1}{2}[(\nabla_\Gamma w + \ell)^*\nabla b + \nabla b^*(\nabla_\Gamma w + \ell)]$$

and

$$\varepsilon^0(e,\ell)\,\nabla b = \frac{1}{2}\,{}^*D_\Gamma(u)\,\nabla b + \frac{1}{2}(\nabla_\Gamma w + \ell) = \frac{1}{2}(\nabla_\Gamma w + \ell - D^2 b\, u).$$

This implies that $w \in H_0^1(\Gamma)$ and hence that $\varepsilon_\Gamma(u) \in L^2(\Gamma)^{N\times N}$. So we are back to the issue of the intrinsic Korn's inequality. When it is true with only the term $\varepsilon_\Gamma(u)$, then $u \in H^1(\Gamma)^N$. In the general intrinsic case we only know that

$$u \in \{u \in L^2(\Gamma)^N \;:\; \varepsilon_\Gamma(u) \in L^2(\Gamma)^{N\times N}\}.$$

In order to focus on the asymptotic problem we assume that both Korn's and Poincaré's inequalities hold with only the ε_Γ term and we shall use the space

$$\mathcal{V}_0^\bullet = \left\{(e,\ell) \in H_0^1(\Gamma)^N \times L^2(\Gamma)^N : \ell \cdot \nabla b = 0 \quad \text{and} \quad \varepsilon^0(e,\ell) \in L^2(\Gamma)^{N\times N}\right\}, \tag{8.10}$$

From the coercivity condition (8.7) it is clear that the variational equation (8.4) will at best have a solution in the quotient space

$$\mathcal{V}_0^\bullet/\mathcal{N}_0^\bullet, \quad \mathcal{N}_0^\bullet = \text{Ker}\ \varepsilon^0. \tag{8.11}$$

This naturally induces the following condition on the linear form in (8.4): there exists a constant $c > 0$ such that for all $(e,\ell) \in \mathcal{V}_0^\bullet$

$$b(0;(f,m),(e,\ell)) \le c\|\varepsilon^0(e,\ell)\|. \tag{8.12}$$

It will be convenient to introduce the notation

$$L(e,\ell) = \ell + 2\varepsilon_\Gamma(e)\nabla b \tag{8.13}$$

for the Love-Kirchhoff term and consider the following decomposition of ε^0

$$\varepsilon^0(e,\ell) = \varepsilon^0(e,-2\varepsilon_\Gamma(e)\nabla b) + \varepsilon^0(0,L(e,\ell)),$$

where

$$\varepsilon^0(e,-2\varepsilon_\Gamma(e)\nabla b) = \varepsilon_\Gamma(e) - \varepsilon_\Gamma(e)\nabla b^*\nabla b - \nabla b^*\nabla b\varepsilon_\Gamma(e) = P\varepsilon(e)P. \tag{8.14}$$

Note that

$$\text{tr}\,\varepsilon^0(e,-2\varepsilon_\Gamma(e)\nabla b) = \text{tr}\,\varepsilon_\Gamma(e) = \text{div}_\Gamma\, e. \tag{8.15}$$

For $L = \ell + 2\varepsilon_\Gamma(e)\nabla b$ and $\overline{L} = \overline{\ell} + 2\varepsilon_\Gamma(\overline{e})\nabla b$, compute

$$\begin{aligned}
\varepsilon^0(e,\ell) \cdot\cdot\, \varepsilon^0(\overline{e},\overline{\ell}) &= \varepsilon^0(e,-2\varepsilon_\Gamma(e)\nabla b) \cdot\cdot\, \varepsilon^0(\overline{e},-2\varepsilon_\Gamma(\overline{e})\nabla b) \\
&\quad + \varepsilon^0(e,-2\varepsilon_\Gamma(e)\nabla b) \cdot\cdot\, \varepsilon^0(0,\overline{L}) \\
&\quad + \varepsilon^0(0,L) \cdot\cdot\, \varepsilon^0(\overline{e},-2\varepsilon_\Gamma(\overline{e})\nabla b) + \varepsilon^0(0,L) \cdot\cdot\, \varepsilon^0(0,\overline{L}).
\end{aligned}$$

Observing that

$$\begin{aligned}
&\varepsilon^0(0,L) \cdot\cdot\, \varepsilon^0(0,\overline{L}) = \frac{1}{2}[L^*\nabla b + \nabla b^* L] \cdot\cdot\, \frac{1}{2}[\overline{L}^*\nabla b + \nabla b^*\overline{L}] = \frac{1}{2} L \cdot \overline{L} \\
&\varepsilon^0(e,-2\varepsilon_\Gamma(e)\nabla b) \cdot\cdot\, \varepsilon^0(0,\overline{L}) \\
&\quad = [\varepsilon_\Gamma(e) - \varepsilon_\Gamma(e)\nabla b^*\nabla b - \nabla b^*\nabla b\varepsilon_\Gamma(e)] \cdot\cdot\, \frac{1}{2}[\overline{L}^*\nabla b + \nabla b^*\overline{L}] \\
&\quad = [\varepsilon_\Gamma(e)\nabla b - \varepsilon_\Gamma(e)\nabla b^*\nabla b\nabla b - \nabla b^*\nabla b\,\varepsilon_\Gamma(e)\nabla b] \cdot \overline{L} = 0
\end{aligned}$$

we finally obtain

$$\begin{aligned}\varepsilon^0(e,\ell)\cdot\cdot\,\varepsilon^0(\overline{e},\overline{\ell}) = \quad & \varepsilon^0(e,-2\varepsilon_\Gamma(e)\nabla b)\cdot\cdot\,\varepsilon^0(\overline{e},-2\varepsilon_\Gamma(\overline{e})\nabla b) \\ & + \frac{1}{2}(\ell+2\varepsilon_\Gamma(e)\nabla b)\cdot(\overline{\ell}+2\varepsilon_\Gamma(\overline{e})\nabla b)\end{aligned} \tag{8.16}$$

$$a(0;(e,\ell),(\overline{e},\overline{\ell})) \tag{8.17}$$

$$\begin{aligned}= \int_\Gamma \Bigg\{ 2\mu\Bigg[& \varepsilon^0(e,-2\varepsilon_\Gamma(e)\nabla b)\cdot\cdot\,\varepsilon^0(\overline{e},-2\varepsilon_\Gamma(\overline{e})\nabla b) \\ & + \frac{1}{2}(\ell+2\varepsilon_\Gamma(e)\nabla b)\cdot(\overline{\ell}+2\varepsilon_\Gamma(\overline{e})\nabla b)\Bigg] \\ & + \lambda\,\mathrm{div}_\Gamma(e)\,\mathrm{div}_\Gamma(\overline{e})\Bigg\}\,d\Gamma.\end{aligned} \tag{8.18}$$

With expression (8.18) for the binear form $a(0;(e,\ell),(\overline{e},\overline{\ell}))$ and inequality (8.7) we have an existence theorem.

Theorem 8.1. *(i) Given f and m in $L^2(\Gamma)^N$ verifying*

$$\forall (e,\ell)\in\mathcal{N}_0^\bullet, \quad \int_\Gamma f\cdot e + m\cdot\ell\,d\Gamma = 0, \quad m\cdot\nabla b = 0 \quad \text{on } \Gamma, \tag{8.19}$$

there exists a unique $(e_0,\ell_0)\in\mathcal{V}_0^\bullet/\mathcal{N}_0^\bullet$ such that for all $e\in H_0^1(\Gamma)^N$

$$\begin{aligned}\int_\Gamma \Bigg\{ 2\mu\Bigg[& \varepsilon^0(e_0,-2\varepsilon_\Gamma(e_0)\nabla b)\cdot\cdot\,\varepsilon^0(e,-2\varepsilon_\Gamma(e)\nabla b)\Bigg] \\ & + \lambda\,\mathrm{div}_\Gamma(e_0)\,\mathrm{div}_\Gamma(e) + f\cdot e - m\cdot 2\varepsilon_\Gamma(e)\nabla b\Bigg\}\,d\Gamma = 0\end{aligned} \tag{8.20}$$

and the associated tangential vector is given by

$$\ell_0 = -2\varepsilon_\Gamma(e_0)\nabla b - m/\mu \quad \text{on } \Gamma. \tag{8.21}$$

(ii) The normal component $w_0 = e_0\cdot\nabla b$ and the tangential part $u_0 = e_0 - w_0\nabla b$ of e_0 verify the following system of variational equations

$$\begin{aligned}\int_\Gamma \Bigg\{ 2\mu\Big[\varepsilon_\Gamma(u_0) + w_0 D^2 b\Big]\cdot\cdot\, D^2 b\, w + \lambda H(\mathrm{div}_\Gamma u_0 + H w_0) w \\ + f\cdot\nabla b\, w - m\cdot\nabla_\Gamma w\Bigg\}\,d\Gamma = 0, \quad \forall w\in H_0^1(\Gamma)\end{aligned} \tag{8.22}$$

$$\begin{aligned}\int_\Gamma \Bigg\{ 2\mu\Big[& \varepsilon^0(u_0,-D^2 b\, u_0)\cdot\cdot\,\varepsilon^0(u,-D^2 b\, u) \\ & + w_0\Big[2\mu D^2 b\cdot\cdot\,\varepsilon_\Gamma(u) + \lambda H\,\mathrm{div}_\Gamma u\Big] \\ & + \lambda\,\mathrm{div}_\Gamma u_0\,\mathrm{div}_\Gamma u + f\cdot u + D^2 b\, m\cdot u\Bigg\}\,d\Gamma = 0\end{aligned} \tag{8.23}$$

$$\forall u\in H_0^1(\Gamma)^N, \quad u\cdot\nabla b = 0 \quad \text{on } \Gamma$$

$$\ell_0 = -\nabla_\Gamma w_0 + D^2 b\, u_0 - m/\mu \quad on\ \Gamma. \tag{8.24}$$

Equation (8.22) is equivalent to

$$\begin{aligned} &\left[2\mu\, D^2 b \cdot\cdot\, D^2 b + \lambda H^2\right] w_0 \\ &= -\left[2\mu\, \varepsilon_\Gamma(u_0) \cdot\cdot\, D^2 b + \lambda H \operatorname{div}_\Gamma u_0 + f \cdot \nabla b + \operatorname{div}_\Gamma m\right]. \end{aligned} \tag{8.25}$$

For the plate ($D^2 b = 0$)*, condition* (8.19) *reduces to*

$$\begin{aligned} &\int_\Gamma f \cdot \nabla b\, w - m \cdot \nabla_\Gamma w\, d\Gamma = 0, \quad \forall w \in H_0^1(\Gamma), \\ &or\ f \cdot \nabla b + \operatorname{div}_\Gamma m = 0, \end{aligned} \tag{8.26}$$

the normal component w_0 *is an arbitrary element in* $H_0^1(\Gamma)$ *and the tangential component* $u_0 \in H_0^1(\Gamma)^N$ *is the unique solution of the variational equation*

$$\begin{cases} \displaystyle\int_\Gamma \left\{2\mu\, \varepsilon_\Gamma(u_0) \cdot\cdot\, \varepsilon_\Gamma(u) + \lambda \operatorname{div}_\Gamma u_0 \operatorname{div}_\Gamma u + f \cdot u\right\} d\Gamma = 0 \\ \forall u \in H^1(\Gamma)^N, \quad u \cdot \nabla b = 0 \quad on\ \Gamma, \quad \forall u \in H_0^1(\Gamma)^N. \end{cases} \tag{8.27}$$

(iii) Under the Love-Kirchhoff condition and assumption (8.19) *on* (f, m) *we get equations* (8.20), (8.22)–(8.23). For the plate we recover (8.27).

Remark . Note that the boundary condition on $\ell(h)$ has disappeared since the convergence takes place in $L^2(\Gamma)^N$. It is a typical case of singular perturbation on the term which contains the highest derivative of ℓ_0. Equations (8.22)–(8.23) are the asymptotic membrane equations (cf. Ciarlet and Sanchez-Palencia [1] and Ciarlet and Lods [1]). The case of the plate illustrates that in general this analysis does not give a full information on w_0 (cf. Delfour and Zolésio [7, eqs (6.8) to (6.14)]). In view of § 2.2 the notion of *uniformly elliptic mean surface* is equivalent to the uniform tangential positive definiteness of the matrix $D^2 b$

$$\exists \gamma > 0, \quad D^2 b(x)\xi \cdot \xi \geq \gamma |\xi|^2, \quad \forall \xi \in \mathbb{R}^N, \xi \cdot \nabla b = 0, \forall x \in \Gamma.$$

In loose terms this means that Γ is the subset of the boundary $\partial\Omega$ of a set Ω which is locally convex. It follows from Delfour and Zolésio [1, Theorems 5.4, 5.5 and 5.6] since $\partial\Omega$ is C^2 and a set Ω is convex if and only if b_Ω is convex. The condition that $D^2 b$ be non zero or equivalently that

$$D^2 b \cdot\cdot\, D^2 b = \sum_{i=1}^{N-1} \kappa_i^2 \neq 0$$

where the κ_i's are the principal curvatures of $\partial\Omega$ is weaker than the notion of uniform elliptic mean surface.

The uniqueness of solution to (8.20) in $\mathcal{V}_0^\bullet$ is related to the kernel of $\varepsilon^0(e, \ell)$ in $\mathcal{V}_0^\bullet$

$$\mathcal{N}_0^\bullet = \left\{(e, \ell) \in \mathcal{V}_0^\bullet \colon \varepsilon_\Gamma(e) + \frac{1}{2}(\ell^* \nabla b + \nabla b^* \ell) = 0\right\}$$

which can be different from $(0, 0)$. The elements of $\mathcal{N}_0^\bullet$ verify the Love-Kirchhoff condition

$$\forall (e, \ell) \in \mathcal{N}_0^\bullet, \quad 2\varepsilon_\Gamma(e)\nabla b + \ell = 0$$

It contains all rigid body motions of Theorem 5.1

$$e(X) = a + BX, \quad B + {}^*B = 0, \quad \ell(X) = B\nabla b$$

and possibly more elements. It can be characterized in terms of the tangential and normal components (u, w) of e.

Lemma 8.1. *The elements of $\mathcal{N}_0^\bullet$ elements of $\mathcal{V}_0^\bullet$ verify the following conditions*

$$\begin{aligned} &\ell = D^2b\, u - \nabla_\Gamma w, \quad u \cdot \nabla b = 0 \\ &\varepsilon_\Gamma(u) + wD^2b + \frac{1}{2}[\nabla b^*(D^2b\, u) + (D^2b\, u)^*\nabla b] = 0 \\ &\varepsilon_\Gamma(e) = \frac{1}{2}[\nabla b^*(\nabla_\Gamma w - D^2b\, u) + (\nabla_\Gamma w - D^2b\, u)^*\nabla b] = 0. \end{aligned} \tag{8.28}$$

In particular

$$wD^2b \cdot\cdot D^2b + \varepsilon_\Gamma(u) \cdot\cdot D^2b = 0 \qquad \textit{and } \operatorname{div}_\Gamma e = \operatorname{div}_\Gamma u + Hw = 0$$

and

$$\int_\Gamma Hw\, d\Gamma = 0.$$

If $D^2b \neq 0$ on Γ

$$\begin{aligned} u \in N_0^\bullet \overset{def}{=} \{u \in H_0^1(\Gamma)^N : u \cdot \nabla b = 0,& \\ &\varepsilon_\Gamma(u) - \frac{\varepsilon_\Gamma(u) \cdot\cdot D^2b}{D^2b \cdot\cdot D^2b} D^2b + \frac{1}{2}[\nabla b^*(D^2b\, u) + (D^2b\, u)^*\nabla b] = 0\}. \end{aligned}$$

For plates($D^2b = 0$)

$$\mathcal{N}_0^\bullet = \{((0, w), -\nabla_\Gamma w) : \forall w \in H_0^1(\Gamma)\} \tag{8.29}$$

since $\varepsilon_\Gamma(u) = 0$ and $u \in H_0^1(\Gamma)^N$ imply that $u = 0$.

Proof of Theorem 8.1. Part (i) of the proof is similar to the one of Theorem 5.3. From (8.4) and (8.18) setting $e = 0$ we get (8.21) which is used to eliminate ℓ_0. Hence we are left with the term

$$E^0 = \varepsilon_\Gamma(e_0) \cdot\cdot \varepsilon_\Gamma(e) - 2\varepsilon_\Gamma(e_0)\nabla b \cdot \varepsilon_\Gamma(e)\nabla b$$

in (8.18). Part (ii) requires the computation of the above term as a function of (u_0, w_0) and (u, w). First compute

$$\begin{aligned} \varepsilon_\Gamma(e) &= \varepsilon_\Gamma(u + w\nabla b) \\ &= \varepsilon_\Gamma(u) + wD^2b + \frac{1}{2}[\nabla b^*\nabla_\Gamma w + \nabla_\Gamma w^*\nabla b] \end{aligned}$$

and using (3.6)

$$\begin{aligned} &\varepsilon_\Gamma(e)\nabla b = \varepsilon_\Gamma(u)\nabla b + \frac{1}{2}\nabla_\Gamma w = \frac{1}{2}(\nabla_\Gamma w - D^2b\, u) \\ &\operatorname{div}_\Gamma(e) = \operatorname{div}_\Gamma(u) + Hw. \end{aligned}$$

Therefore

$$
\begin{aligned}
E^0 = &\left[\varepsilon_\Gamma(u_0) + w_0 D^2 b + \frac{1}{2}(\nabla_\Gamma w_0{}^* \nabla b + \nabla b^* \nabla_\Gamma w_0)\right] \\
&\cdot\cdot \left[\varepsilon_\Gamma(u) + w D^2 b + \frac{1}{2}(\nabla_\Gamma w^* \nabla b + \nabla b^* \nabla_\Gamma w)\right] \\
&- \frac{1}{2}(\nabla_\Gamma w_0 - D^2 b\, u_0) \cdot (\nabla_\Gamma w - D^2 b\, u) \\
= &\, \varepsilon_\Gamma(u_0) \cdot\cdot\, \varepsilon_\Gamma(u) + w_0 D^2 b \cdot\cdot\, \varepsilon_\Gamma(u) + w D^2 b \cdot\cdot\, \varepsilon_\Gamma(u_0) \\
&+ w_0 w D^2 b \cdot\cdot\, D^2 b + \nabla_\Gamma w_0 \cdot \nabla_\Gamma w - \frac{1}{2} D^2 b\, u_0 \cdot D^2 b\, u.
\end{aligned}
$$

Similarly

$$
\operatorname{div}_\Gamma e_0 \operatorname{div}_\Gamma e = (\operatorname{div}_\Gamma u_0 + H w_0)(\operatorname{div}_\Gamma u + H w).
$$

Going back to (8.20) with $u = 0$

$$
\int_\Gamma \Big\{ 2\mu \Big[(D^2 b \cdot\cdot\, \varepsilon_\Gamma(u_0) + w_0 D^2 b \cdot\cdot\, D^2 b)\, w \Big] + \lambda (\operatorname{div}_\Gamma u_0 + H w_0) H w + f \cdot \nabla b\, w - m \cdot \nabla_\Gamma w \Big\}\, d\Gamma.
$$

This is equivalent to

$$
2\mu[\varepsilon_\Gamma(u_0) + w_0 D^2 b] \cdot\cdot\, D^2 b + \lambda(\operatorname{div}_\Gamma u_0 + H w_0) H + f \cdot \nabla b + \operatorname{div}_\Gamma m
$$

which yields (8.25). Now set $w = 0$ in (8.20)

$$
\int_\Gamma \Big\{ 2\mu \Big[\varepsilon_\Gamma(u_0) \cdot\cdot\, \varepsilon_\Gamma(u) + w_0 D^2 b \cdot\cdot\, \varepsilon_\Gamma(u) - \frac{1}{2} D^2 b\, u_0 \cdot D^2 b\, u \Big] + \lambda (\operatorname{div}_\Gamma u_0 + H w_0) \operatorname{div}_\Gamma u + f \cdot u + m \cdot D^2 b\, u \Big\}\, d\Gamma
$$

and we get (8.23). □

References

M. Bernadou **[1]**, *Méthodes d'éléments finis pour les problèmes de coques minces*, Masson, Paris, Milan, Barcelone, 1994.

M. Bernadou, Ph.G. Ciarlet and B. Miara **[1]**, *Existence theorems for two-dimensional linear shell theories*, J. of Elasticity (1994), 111–138.
(Report R 92018, Laboratoire d'Analyse Numérique, Université Pierre et Marie Curie, Paris, France, October 1992) (to appear).

Ph. G. Ciarlet and V. Lods **[1]**, *Ellipticité des équations membranaires d'une coque uniformément elliptique*, C.R. Acad. Sci. Paris, Sér. I, **318** (1994), 195–200

Ph. G. Ciarlet and E. Sanchez-Palencia **[1]**, *Un théorème d'existence et d'unicité pour les équations des coques membranaires*, C.R. Acad. Sci. Paris, Sér. I, **317** (1993) 801–805

M.C. Delfour and J.P. Zolésio **[1],** *Shape analysis via oriented distance functions*, J. Functional Analysis **123** (1994), 129–201

[2], *Functional analytic methods in shape analysis*, Boundary Control and Variation (J.-P. Zolésio, ed.), Marcel Dekker, New-York, 1994, pp. 105–139.

[3], *Oriented distance function in shape analysis and optimization*, Control and Optimal Design of Distributed Parameter Systems (J. Lagnese, D.L. Russell, L. White, eds.), Springer-Verlag, Berlin, Heidelberg, New York, Tokyo, 1994, pp. 39–72.

[4], *On a variational equation for thin shells*, Control and Optimal Design of Distributed Parameter Systems (J. Lagnese, D.L. Russell, L. White, eds.), Springer-Verlag, Berlin, Heidelberg, New York, Tokyo, 1994, pp. 25–37.

[5], *A boundary differential equation for thin shells*, J. Differential Equations (1995), in press.

[6], *Modelling and control of thin/shallow shells*, Proc. IFIP Conference on Control of Partial Differential Equations (E. Casas and L.A. Fernández, eds.), Marcel Dekker, New York, 1995.

[7], *Tangential differential equations for dynamical thin/shallow shells*, J. Differential Equations, to appear.

Ph. Destuynder **[1],** *Modélisation des coques minces élastiques*, Masson, Paris, Milan, Barcelone, 1990.

[2], *Une théorie asymptotique des plaques minces en elasticité linéaire*, Masson, Paris, Milan, Barcelone, 1986

P. Germain **[1],** *Mécanique, tomes* I and II, Ellipses éditeur, Paris, 1986.

D. Gilbarg and N.S. Trudinger **[1],** *Elliptic partial differential equations of second Order*, Springer-Verlag, Berlin, Heidelberg, New York, Tokyo, 1983.

J.E. Lagnese and J.-L. Lions **[1],** *Modelling, analysis and control of thin plates*, RMA, Vol. 6, Masson, Paris, Milan, Barcelone, Mexico, 1988.

J.-L. Lions **[1],** *Contrôle optimal de systèmes gouvernés par des équations aux dérivées partielles*, Dunod, Gauthier-Villars, Paris, 1968.

J. Sokolowski and J.-P. Zolésio **[1],** *Introduction to shape optimization*: Shape sensitivity analysis, Springer-Verlag, New York, Berlin, Heidelberg, 1992.

R. Temam **[1],** *Monotone rearangement of a function and the Grad-Mercier equation of plasmas physics*, Recent methods in non linear analysis and application (E. De Giorgi, E. Magenes and U. Mosco, eds.), Pitagora Editrice, Bologna, 1979.

[2], *Problèmes mathématiques en plasticité*, Gauthier-Villars, Bordas, Paris, 1983.

R. Valid **[1],** *Mechanics of continuous media and analysis of structures*, North-Holland Series in Applied Mathematics and Mechanics, Amsterdam, New York, Oxford, 1981.

[2], *Le principe des travaux virtuels et les principes variationnels associés*, ONERA publication 1982-1, Chatillon, France, 1982.

[3], *Some aspects of the non linear shell theory*, Lecture Notes, Rio de Janeiro, August 1985

[4], *Le principe non linéaire d'énergie complémentaire en théorie des coques*, C.R. Acad. Sci. Paris, Sér. II **309** (1989), 1445–1449

J.P. Zolésio **[1],** *The Material Derivative (or Speed) Method for Shape Optimization*, Optimization of Distributed Parameter Structures, vol. II, (E.J. Haug and J. Céa, eds.), Sijthoff and Noordhoff, Alphen aan den Rijn, The Netherlands, 1981, pp. 1089–1151.

[2], *Domain variational formulation for free boundary problems*, Optimization of Distributed Parameter Structures, vol. II (E.J. Haug and J. Céa, eds.) Sijthoff and Noordhoff, Alphen aan den Rijn, The Netherlands, 1981, 1152–1194

A Naghdi, Koiter and asymptotic linear models

In this appendix we use the notation and definitions from the book of M. Bernadou [1]. Naghdi and Koiter's linear models can be found in Chapter I, Sections 3 and 4, respectively. The asymptotic model is the one given in Ciarlet and Sanchez-Palencia [1].

A.1 Naghdi's model

The displacement vector is

$$U = u + \xi^3 \beta_\alpha a^\alpha \tag{A.1}$$

where $-e < \xi^3 < e$, $2e$ is the thickness of the shell, u and β are maps from Γ to $\mathbb{R}^3$ and

$$\beta = \beta_\alpha a^\alpha, \quad \beta \cdot \nabla b = 0, \quad \beta_\alpha = \beta \cdot a_\alpha. \tag{A.2}$$

Throughout this appendix, we use the same notation for a vector $v \colon \Gamma \to \mathbb{R}^3$ and its 2-dimensional representation $v \circ \Phi$ in terms of the $1-2$ coordinates. We now go through several definitions and give their tangential equivalent

$$\varphi_\alpha(u) \stackrel{\text{def}}{=} u_{3,\alpha} + b^\lambda_\alpha u_\lambda, \tag{A.3}$$

$$\gamma_{\alpha\beta}(u) \stackrel{\text{def}}{=} \frac{1}{2}(u_\alpha|_\beta + u_\beta|_\alpha) - b_{\alpha\beta} u_3, \tag{A.4}$$

$$d_{\lambda\alpha}(u) \stackrel{\text{def}}{=} u_\lambda|_\alpha - b_{\lambda\alpha} u_3, \tag{A.5}$$

$$\chi_{\alpha\mu}(u,\beta) \stackrel{\text{def}}{=} \frac{1}{2}[\beta_\alpha|_\mu + \beta_\mu|_\alpha - b^\lambda_\alpha d_{\lambda\mu}(u) - b^\lambda_\mu d_{\lambda\alpha}(u)], \tag{A.6}$$

$$\varepsilon_{\alpha\beta}(u,\beta) \stackrel{\text{def}}{=} \gamma_{\alpha\beta}(u) + \xi^3 \chi_{\alpha\beta}(u,\beta). \tag{A.7}$$

Then

$$\begin{aligned}\varphi_\alpha(u) &= D_\Gamma u\, a_\alpha \cdot a_3 + u \cdot D_\Gamma\, a_3\, a_\alpha + a_\alpha \cdot D^2 b\, u \\ &= -a_\alpha \cdot {}^*D_\Gamma u \nabla b - u \cdot D^2 b\, a_\alpha + a_\alpha \cdot D^2 b\, u = a_\alpha \cdot D^2 b\, u\end{aligned}$$

and

$$\boxed{\varphi_\alpha(u) = a_\alpha \cdot D^2 b\, u = -D_\Gamma u\, a_\alpha \cdot \nabla b}. \tag{A.8}$$

Next

$$\boxed{\gamma_{\alpha\beta}(u) = \varepsilon_\Gamma(u)\, a_\alpha \cdot a_\beta}, \tag{A.9}$$

and

$$d_{\lambda\alpha}(u) = D_\Gamma u\, a_\alpha \cdot a_\lambda + D^2 b\, a_\lambda \cdot a_\alpha u_3 - D^2 b\, a_\alpha \cdot a_\lambda u_3,$$

$$\boxed{d_{\lambda\alpha}(u) = D_\Gamma u\, a_\alpha \cdot a_\lambda}. \tag{A.10}$$

Furthermore

$$\begin{aligned}\chi_{\alpha\mu}(u,\beta) &= \varepsilon_\Gamma(\beta)\, a_\alpha\cdot a_\mu - \frac{1}{2}[a_\alpha\cdot D^2b\, a^\lambda D_\Gamma u a_\mu\cdot a_\lambda + a_\mu\cdot D^2b\, a^\lambda D_\Gamma u\, a_\alpha\cdot a_\lambda]\\ &= \varepsilon_\Gamma(\beta)\, a_\alpha\cdot a_\mu - \frac{1}{2}[D^2b\, a_\alpha\cdot D_\Gamma u\, a_\mu + D^2b\, a_\mu\cdot D_\Gamma u\, a_\alpha],\end{aligned}$$

and

$$\boxed{\chi_{\alpha\mu}(u,\beta) = a_\alpha\cdot\left[\varepsilon_\Gamma(\beta) - \frac{1}{2}(D^2bD_\Gamma u + {}^*D_\Gamma uD^2b)\right] a_\mu}. \tag{A.11}$$

Finally

$$\varepsilon_{\alpha\beta}(u,\beta) = a_\alpha\cdot\varepsilon_\Gamma(u)\, a_\beta + \xi^3\left[a_\alpha\cdot\varepsilon_\Gamma(\beta)\, a_\beta - \frac{1}{2}a_\alpha\cdot(D^2bD_\Gamma u + {}^*D_\Gamma uD^2b)\, a_\beta\right],$$

and

$$\boxed{\varepsilon_{\alpha\beta}(u,\beta) = a_\alpha\cdot\left\{\varepsilon_\Gamma(u) + \xi^3\left[\varepsilon_\Gamma(\beta) - \frac{1}{2}(D^2b\, D_\Gamma u + {}^*D_\Gamma uD^2b)\right]\right\} a_\beta}. \tag{A.12}$$

The underlying variational equation for Naghdi's model consists in finding (u,β) such that for all (v,δ)

$$a^S((u,\beta),(v,\delta)) + b^S((u,\beta),(v,\delta)) = f^S(v,\delta), \tag{A.13}$$

where the bilinear forms a^S and b^S and the linear form f^S are given by

$$a^S((u,\beta),(v,\delta)) \tag{A.14}$$
$$\stackrel{\text{def}}{=} \int_\Gamma eE^{\alpha\beta\lambda\mu}\left\{\gamma_{\alpha\beta}(u)\gamma_{\lambda\mu}(v) + \frac{e^2}{12}\chi_{\alpha\beta}(u)\chi_{\lambda\mu}(v)\right\} d\Gamma$$
$$b^S((u,\beta),(v,\delta)) \tag{A.15}$$
$$\stackrel{\text{def}}{=} \int_\Gamma \frac{eEa^{\alpha\beta}}{2(1+\nu)}(\varphi_\alpha(u)+\beta_\alpha)(\varphi_\beta(v)+\delta_\beta)\, d\Gamma$$
$$f^S(v,\delta) \tag{A.16}$$
$$\stackrel{\text{def}}{=} \int_\Gamma p\cdot v\, d\Gamma + \int_{\partial\Gamma} N\cdot v - M\cdot\delta\, d\gamma$$

and

$$E^{\alpha\beta\lambda\mu} = \frac{E}{2(1+\nu)}\left[a^{\alpha\lambda}a^{\beta\mu} + a^{\alpha\mu}a^{\beta\lambda} + \frac{2\nu}{1-\nu}a^{\alpha\beta}a^{\lambda\mu}\right]. \tag{A.17}$$

First consider the term in γ of a^S

$$\begin{aligned}&E^{\alpha\beta\lambda\mu}\gamma_{\alpha\beta}(u)\gamma_{\lambda\mu}(v)\\ &= \frac{E}{2(1+\nu)}\left[a^{\alpha\lambda}a^{\beta\mu} + a^{\alpha\mu}a^{\beta\lambda} + \frac{2\nu}{1-\nu}a^{\alpha\beta}a^{\lambda\mu}\right]\gamma_{\alpha\beta}(u)\gamma_{\lambda\mu}(v).\end{aligned} \tag{A.18}$$

Observe that

$$A \stackrel{\text{def}}{=} a^{\alpha\lambda}a^{\beta\mu}\gamma_{\alpha\beta}(u)\gamma_{\lambda\mu}(v) = a^{\alpha\mu}a^{\beta\lambda}\gamma_{\alpha\beta}(u)\gamma_{\lambda\mu}(v).$$

So we only need to compute the term in the middle identity

$$
\begin{aligned}
A &\stackrel{\text{def}}{=} \varepsilon_\Gamma(u) a_\beta \cdot a_\alpha\, a^\alpha \cdot a^\lambda\, \varepsilon_\Gamma(v)\, a_\lambda \cdot a_\mu\, a^\mu \cdot a^\beta \\
&= \varepsilon_\Gamma(u)\, a_\beta \cdot [a^\lambda - a_3 a^3 \cdot a^\lambda]\, \varepsilon_\Gamma(v)\, a_\lambda \cdot [a^\beta - a_3 a^3 \cdot a^\beta] \\
&= \varepsilon_\Gamma(u)\, a_\beta \cdot a^\lambda\, a_\lambda \cdot \varepsilon_\Gamma(v)\, a^\beta \\
&= \varepsilon_\Gamma(u) \cdot\cdot\, \varepsilon_\Gamma(v) - 2\varepsilon_\Gamma(u)\nabla b \cdot \varepsilon_\Gamma(v)\nabla b = \varepsilon^0(u, -2\varepsilon_\Gamma(u)\nabla b) \cdot\cdot\, \varepsilon^0(v, -2\varepsilon_\Gamma(v)\nabla b).
\end{aligned}
$$

Now

$$
\begin{aligned}
B &\stackrel{\text{def}}{=} a^{\alpha\beta} a^{\lambda\mu} \gamma_{\alpha\beta}(u)\gamma_{\lambda\mu}(v) \\
&= \varepsilon_\Gamma(u)\, a_\alpha \cdot a_\beta\, a^\beta \cdot a^\alpha \varepsilon_\Gamma(v)\, a_\lambda \cdot a_\mu\, a^\mu \cdot a^\lambda \\
&= \varepsilon_\Gamma(u)\, a_\alpha \cdot a^\alpha\, \varepsilon_\Gamma(v)\, a_\lambda \cdot a^\lambda = \operatorname{tr} \varepsilon_\Gamma(u) \operatorname{tr} \varepsilon_\Gamma(v) \\
&\qquad\qquad\qquad\qquad\qquad\qquad\quad = \operatorname{tr} \varepsilon^0(u, -2\varepsilon_\Gamma(u)\nabla b) \operatorname{tr} \varepsilon^0(v, -2\varepsilon_\Gamma(v)\nabla b).
\end{aligned}
$$

Finally

$$
\boxed{
\begin{aligned}
\frac{2(1+\nu)}{E} E^{\alpha\beta\lambda\mu} &= 2\varepsilon^0(u, -2\varepsilon_\Gamma(u)\nabla b) \cdot\cdot\, \varepsilon^0(v, -2\varepsilon_\Gamma(v)\nabla b) \\
&\quad + \frac{2\nu}{1-\nu} \operatorname{tr} \varepsilon^0(u, -2\varepsilon_\Gamma(u)\nabla b) \operatorname{tr} \varepsilon^0(v, -2\varepsilon_\Gamma(v)\nabla b)
\end{aligned}
}.
$$

Next we turn to the term in χ. Again

$$
C \stackrel{\text{def}}{=} a^{\alpha\mu} a^{\beta\lambda} \chi_{\alpha\beta}(u,\beta)\chi_{\lambda\mu}(v,\delta) = a^{\alpha\lambda} a^{\beta\mu} \chi_{\alpha\beta}(u,\beta)\chi_{\lambda\mu}(v,\delta),
$$

and

$$
\begin{aligned}
C &= a_\alpha \cdot \left[\varepsilon_\Gamma(\beta) - \frac{1}{2}(D^2 b\, D_\Gamma u + {}^*D_\Gamma u D^2 b)\right] a_\beta\, a^\beta \cdot a^\lambda \\
&\quad \times a_\lambda \cdot \left[\varepsilon_\Gamma(\delta) - \frac{1}{2}(D^2 b\, D_\Gamma v + {}^*D_\Gamma v D^2 b)\right] a_\mu\, a^\mu \cdot a^\alpha \\
&= a_\alpha \cdot \left[\varepsilon_\Gamma(\delta) - \frac{1}{2}(D^2 b\, D_\Gamma v + {}^*D_\Gamma v D^2 b)\right]\, a^\lambda \\
&\quad \times a_\lambda \cdot \left[\varepsilon_\Gamma(\delta) - \frac{1}{2}(D^2 b\, D_\Gamma v + {}^*D_\Gamma v D^2 b)\right]\, a^\alpha \\
C &= \left[\varepsilon_\Gamma(\beta) - \frac{1}{2}(D^2 b D_\Gamma u + {}^*D_\Gamma u D^2 b)\right] \cdot\cdot \left[\varepsilon_\Gamma(\delta) - \frac{1}{2}(D^2 b D_\Gamma v + {}^*D_\Gamma v D^2 b)\right] \\
&\quad - 2\varepsilon_\Gamma(\beta)\nabla b \cdot \varepsilon_\Gamma(\delta)\nabla b.
\end{aligned}
$$

Define

$$
\begin{aligned}
\varepsilon_t^1(u, -\beta) \stackrel{\text{def}}{=} \varepsilon_\Gamma(-\beta) &+ \frac{1}{2}[D^2 b\, \beta^* \nabla b + \nabla b^* (D^2 b\, \beta)] \\
&+ \frac{1}{2}[D^2 b D_\Gamma u + {}^*D_\Gamma u D^2 b].
\end{aligned}
\tag{A.19}
$$

Then

$$
\begin{aligned}
&\varepsilon_t^1(u,-\beta)\cdot\cdot\,\varepsilon_t^1(v,-\delta)\\
&=\left[\varepsilon_\Gamma(-\beta)+\frac{1}{2}(D^2b\,D_\Gamma u+{}^*D_\Gamma uD^2b)\right]\cdot\cdot\left[\varepsilon_\Gamma(-\bar\beta)+\frac{1}{2}(D^2b\,D_\Gamma\bar u+{}^*D_\Gamma\bar uD^2b)\right]\\
&\quad+\left[\Gamma(-\beta)+\frac{1}{2}(D^2b\,D_\Gamma u+{}^*D_\Gamma uD^2b)\right]\cdot\cdot\,\frac{1}{2}[D^2b\,\bar\beta^*\nabla b+\nabla b^*(D^2b\,\bar\beta)]\\
&\quad+\frac{1}{2}[D^2b\,\beta^*\nabla b+\nabla b^*(D^2b\,\beta)]\cdot\cdot\left[\varepsilon_\Gamma(-\bar\beta)+\frac{1}{2}(D^2b\,D_\Gamma\bar u+{}^*D_\Gamma\bar uD^2b)\right]\\
&\quad+\frac{1}{4}[D^2b\,\beta^*\nabla b+\nabla b^*(D^2b\,\beta)]\cdot\cdot\,[D^2b\bar\beta^*\nabla b+\nabla b^*(D^2b\,\bar\beta)]\\
&=\left[\varepsilon_\Gamma(-\beta)+\frac{1}{2}(D^2b\,D_\Gamma u+{}^*D_\Gamma uD^2b)\right]\cdot\cdot\left[\varepsilon_\Gamma(-\bar\beta)+\frac{1}{2}(D^2b\,D_\Gamma\bar u+{}^*D_\Gamma\bar uD^2b)\right]\\
&\quad-\varepsilon_\Gamma(\beta)\nabla b\cdot D^2b\bar\beta-D^2b\,\beta\cdot\varepsilon_\Gamma(\bar\beta)\nabla b+\frac{1}{2}D^2b\,\beta\cdot D^2b\,\bar\beta\\
&=\left[\varepsilon_\Gamma(-\beta)+\frac{1}{2}(D^2b\,D_\Gamma u+{}^*D_\Gamma uD^2b)\right]\cdot\cdot\left[\varepsilon_\Gamma(-\bar\beta)+\frac{1}{2}(D^2b\,D_\Gamma\bar u+{}^*D_\Gamma\bar uD^2b)\right]\\
&\quad-\frac{1}{2}D^2b\,\beta\cdot D^2b\bar\beta.
\end{aligned}
$$

Therefore

$$C=\varepsilon_t^1(u,-\beta)\cdot\cdot\,\varepsilon_t^1(v,-\delta).$$

Next

$$
\begin{aligned}
D&\stackrel{\text{def}}{=}a^{\alpha\beta}\,a^{\lambda\mu}\chi_{\alpha\beta}(u,\beta)\chi_{\lambda\mu}(v,\delta)\\
&=a_\alpha\cdot[\chi_{\alpha\beta}(u,\beta)]a_\beta\,a^\beta\cdot a^\alpha a_\lambda\cdot[\chi_{\lambda\mu}(v,\delta)]a_\mu\,a^\mu\cdot a^\lambda\\
&=a_\alpha\cdot[\chi_{\alpha\beta}(u,\beta)]\,a^\alpha\,a_\lambda\cdot[\chi_{\lambda\mu}(v,\delta)]\,a^\lambda\\
&=\operatorname{tr}\left\{\varepsilon_\Gamma(-\beta)+\frac{1}{2}(D^2bD_\Gamma u+{}^*D_\Gamma uD^2b)\right\}\operatorname{tr}\left\{\varepsilon_\Gamma(-\delta)+\frac{1}{2}(D^2b\,D_\Gamma v+{}^*D_\Gamma vD^2b)\right\}\\
&=\operatorname{tr}\varepsilon_t^1(u,-\beta)\operatorname{tr}\varepsilon_t^1(v,-\delta).
\end{aligned}
$$

Finally

$$\frac{2(1+\nu)}{E}E^{\alpha\beta\lambda\mu}=2\varepsilon_t^1(u,-\beta)\cdot\cdot\,\varepsilon_t^1(v,-\delta)+\frac{2\nu}{1-\nu}\operatorname{tr}\varepsilon_t^1(u,-\beta)\operatorname{tr}\varepsilon_t^1(u,-\delta).$$

As for the bilinear form b^S the situation is simpler

$$
\begin{aligned}
F&\stackrel{\text{def}}{=}a^{\alpha\beta}(\varphi_\alpha(u)+\beta_\alpha)(\varphi_\beta(v)+\delta_\beta)\\
&=a^\beta\cdot\alpha^\alpha\,a_\alpha\cdot[D^2b\,u+\beta][D^2b\,v+\delta]\cdot a_\beta\\
&=(D^2b\,u+\beta)\cdot a^\beta\,a_\beta\cdot(D^2b\,v+\delta)=(D^2b\,u+\beta)\cdot(D^2b\,v+\delta).
\end{aligned}
$$

The term $2A+F$ can be slightly rearranged. To see this consider the following expression

$$G\stackrel{\text{def}}{=}\varepsilon^0(u,-\nabla_\Gamma(u\cdot\nabla b)-\beta)\cdot\cdot\,\varepsilon^0(v,-\nabla_\Gamma(v\cdot\nabla b)-\delta).$$

Substitute for $\nabla_\Gamma(u\cdot\nabla b)$ and $\nabla_\Gamma(v\cdot\nabla b)$ the expression

$$\nabla_\Gamma(u\cdot\nabla b)=2\varepsilon_\Gamma(u)\nabla b-D^2b\,u,\quad \nabla_\Gamma(v\cdot\nabla b)=2\varepsilon_\Gamma(v)\nabla b-D^2b\,v.$$

Then

$$\begin{aligned}
G &= \varepsilon^0(u, -2\varepsilon_\Gamma(u)\nabla b - D^2b\,u - \beta) \cdot\cdot\, \varepsilon^0(v, -2\varepsilon_\Gamma(v)\nabla b - D^2b\,v - \delta) \\
&= \varepsilon^0(u, -2\varepsilon_\Gamma(u)\nabla b) \cdot\cdot\, \varepsilon^0(v, -2\varepsilon_\Gamma(v)\nabla b) \\
&\quad - \varepsilon^0(0, D^2b\,u + \beta) \cdot\cdot\, \varepsilon^0(v, -2\varepsilon_\Gamma(v)\nabla b) \\
&\quad - \varepsilon^0(u, -2\varepsilon_\Gamma(u)\nabla b) \cdot\cdot\, \varepsilon^0(0, D^2b\,v + \delta) \\
&\quad + \varepsilon^0(0, D^2b\,u + \beta) \cdot\cdot\, \varepsilon^0(0, D^2b\,v + \delta).
\end{aligned}$$

But

$$\begin{aligned}
&\varepsilon^0(0, D^2b\,u + \beta) \cdot\cdot\, \varepsilon^0(v, -2\varepsilon_\Gamma(v)\nabla b) \\
&= \frac{1}{2}[(D^2b\,u + \beta)^*\nabla b + \nabla b^*(D^2b\,u + \beta)] \cdot\cdot\, \varepsilon^0(v, -2\varepsilon_\Gamma(v)\nabla b) \\
&= (D^2b\,u + \beta) \cdot \varepsilon^0(v, -2\varepsilon_\Gamma(v)\nabla b)\nabla b,
\end{aligned}$$

and

$$\begin{aligned}
\varepsilon^0(v, -2\varepsilon_\Gamma(v)\nabla b)\nabla b &= \big\{\varepsilon_\Gamma(v) - [\varepsilon_\Gamma(v)\nabla b^*\nabla b + \nabla b^*(\varepsilon_\Gamma(v)\nabla b)]\big\}\nabla b \\
&= \varepsilon_\Gamma(v)\nabla b - \varepsilon_\Gamma(v)\nabla b = 0.
\end{aligned}$$

Also

$$\begin{aligned}
&\varepsilon^0(0, D^2b\,u + \beta) \cdot\cdot\, \varepsilon^0(0, D^2b\,v + \delta) \\
&= \frac{1}{4}[(D^2b\,u + \beta)^*\nabla b + \nabla b^*(D^2b\,u + \beta)] \cdot\cdot\, [(D^2b\,v + \delta)^*\nabla b + \nabla b^*(D^2b\,v + \delta)] \\
&= \frac{1}{2}(D^2b\,u + \beta) \cdot (D^2b\,v + \delta).
\end{aligned}$$

Finally

$$2G = 2A + F.$$

Moreover

$$\operatorname{tr} \varepsilon^0(u, -\nabla_\Gamma(u \cdot \nabla b) - \beta) = \operatorname{tr} \varepsilon_\Gamma(u) = \operatorname{div}_\Gamma u.$$

Putting the two bilinear forms together

$$\begin{aligned}
&a^S((u,\beta),(v,\delta)) + b^S((u,\beta),(v,\delta)) \\
&= \frac{eE}{1+\nu}\int_\Gamma \varepsilon^0(u, -\nabla_\Gamma(u\cdot\nabla b) - \beta) \cdot\cdot\, \varepsilon^0(v, -\nabla_\Gamma(v\cdot\nabla b) - \delta) + \frac{e^2}{12}\varepsilon^1_t(u,-\beta) \cdot\cdot\, \varepsilon^1_t(v,-\delta) \\
&\quad + \frac{1}{1-\nu}\Bigg\{\operatorname{tr}\varepsilon^0(u, -\nabla_\Gamma(u\cdot\nabla b) - \beta)\operatorname{tr}\varepsilon^0(v, -\nabla_\Gamma(v\cdot\nabla b) - \delta) \\
&\qquad\qquad + \frac{e^2}{12}\operatorname{tr}\varepsilon^1_t(u,-\beta)\operatorname{tr}\varepsilon^1_t(v,-\delta)\Bigg\}\,d\Gamma \\
&= \frac{eE}{1+\nu}\int_\Gamma \varepsilon^0(u, -\nabla_\Gamma(u\cdot\nabla b) - \beta) \cdot\cdot\, \varepsilon^0(v, -\nabla_\Gamma(v\cdot\nabla b) - \delta) + \frac{e^2}{12}\varepsilon^1_t(u,-\beta) \cdot\cdot\, \varepsilon^1_t(v,-\delta) \\
&\quad + \frac{1}{1-\nu}\Bigg\{\operatorname{div}_\Gamma u\operatorname{div}_\Gamma v + \frac{e^2}{12}\operatorname{div}_\Gamma\beta\operatorname{div}_\Gamma\delta\Bigg\}\,d\Gamma.
\end{aligned}$$

Also

$$f^S(v,\delta) = \int_\Gamma p\cdot v\,d\Gamma + \int_{\partial\Gamma} N\cdot v - M\cdot\delta\,d\gamma.$$

A.2 Koiter's model

It is the same model as Naghdi's one with the following identity between the variables u and β

$$\beta_\alpha + \varphi_\alpha(u) = 0, \tag{A.20}$$

which can be rewritten as

$$a_\alpha \cdot [\beta - {}^*D_\Gamma u \nabla b] = 0 \quad \text{or} \quad \boxed{\beta - 2\varepsilon_\Gamma(u)\nabla b = 0} \tag{A.21}$$

or

$$\boxed{\beta = \nabla_\Gamma(u \cdot \nabla b) - D^2 b u}. \tag{A.22}$$

A.3 Asymptotic membrane equation model

It is characterized by the bilinear form

$$B(\zeta, \eta) \stackrel{\text{def}}{=} \int_\Gamma A^{\alpha\beta\rho\delta} \gamma_{\rho\sigma}(\zeta)\gamma_{\alpha\beta}(\eta)\, d\Gamma \tag{A.23}$$

$$A^{\alpha\beta\rho\delta} \stackrel{\text{def}}{=} \frac{4\lambda\mu}{\lambda + 2\mu} a^{\alpha\beta}\, a^{\rho\delta} + 2\mu[a^{\alpha\rho}\, a^{\beta\sigma} + a^{\alpha\sigma}\, a^{\beta\rho}] \tag{A.24}$$

$$\gamma_{\alpha\beta}(\eta) \stackrel{\text{def}}{=} \frac{1}{2}(\eta_{\alpha,\beta} + \eta_{\beta,\alpha}) - \Gamma^\rho_{\alpha\beta}\eta_\beta - b_{\alpha\beta}\eta_3. \tag{A.25}$$

Clearly

$$\gamma_{\alpha\beta}(\eta) = \frac{1}{2}(\eta_\alpha|_\beta + \eta_\beta|_\alpha) - b_{\alpha\beta}\eta_3 = a_\alpha \cdot \varepsilon_\Gamma(\eta)\, a_\beta.$$

But this is the same γ as in Naghdi's model. Therefore

$$\begin{aligned} B(\zeta, \eta) &= \int_\Gamma 4\mu\, \varepsilon^0(\zeta, -2\varepsilon_\Gamma(\zeta)\nabla b) \cdot\cdot\, \varepsilon^0(\eta, -2\varepsilon_\Gamma(\eta)\nabla b) \\ &\quad + \frac{4\lambda\mu}{\lambda + 2\mu} \operatorname{tr} \varepsilon^0(\zeta, -2\varepsilon_\Gamma(\zeta)\nabla b) + \operatorname{tr} \varepsilon^0(\eta, -2\varepsilon_\Gamma(\eta)\nabla b)\, d\Gamma \\ B(\zeta, \eta) &= \int_\Gamma 4\mu\, \varepsilon^0(\zeta, -2\varepsilon_\Gamma(\zeta)\nabla b) \cdot\cdot\, \varepsilon^0(\eta, -2\varepsilon_\Gamma(\eta)\nabla b) \\ &\quad + \frac{4\lambda\mu}{\lambda + 2\mu} \operatorname{div}_\Gamma \zeta \operatorname{div}_\Gamma \eta\, d\Gamma. \end{aligned} \tag{A.26}$$