Symmetry preserving quantization and self-similar potentials

Vyacheslav Spiridonov[*] [†]   Alexei Zhedanov[‡]

CRM-2297

sJuly 1995

---

[*]On leave of absence from the Institute for Nuclear Research, Russian Academy of Sciences, Moscow, Russia

[†]Centre de recherches mathématiques, Université de Montréal, C. P.6128, succ. Centre-ville, Montréal QC H3C 3J7.

[‡]Physics Department, Donetsk University, Donetsk, 340055, Ukraine.

Abstract

A particular one-dimensional model of classical mechanics obeying quadratic Poisson symmetry algebra is considered. After a special quantization the latter algebra is converted into the q-Weyl algebra describing symmetries of a specific Schrödinger operator. In this way one reconstructs the simplest self-similar potential having infinitely many bound states with eigenvalues forming one geometric series.

KEYWORKS: Poisson algebra, quantum algebra, self-similarity.

Résumé

Un modèle de mécanique classique à une dimension qui possède l'algèbre quadratique de Poisson est considéré. Après une quantification spéciale cette algèbre a la forme de l'algèbre de q-Weyl qui décrit les symétries d'opérateur spécifique de Schrödinger. Par cette procédure on a reconstruit le potentiel auto-similaire le plus simple qui possède les états liés de nombre infini avec des valeurs propres qui forment une série géométrique.

MOTS CLÉS : algèbre de Poisson, algèbre quadratique, auto-similarité.

Recently the specific (self-similar) one-dimensional reflectionless potentials with infinite number of bound states have been considered in [1-4]. Within the inverse scattering method these potentials correspond to some infinite soliton solutions of the Korteweg-de Vries equation. They obey non-trivial dynamical symmetries which are described by the q-Weyl (or q-oscillator), $su_q(1,1)$, or higher order polynomial quantum algebras [2]. Although the full structure of special functions associated with these potentials is not known, some of their properties have been derived rigorously. The aim of the present paper is to use the simplest self-similar potential for the description of a specific quantization scheme of some mechanical systems which we call the *symmetry preserving quantization.*

In short, the idea of such quantization consists in the following. Suppose that the classical dynamics of a mechanical system is described by the Hamiltonian obeying some non-trivial Poisson algebra of symmetries which is not necessarily a Lie algebra. The generators of this algebra are supposed to be defined as some functions of the phase space variables $\vec{p}$ and $\vec{x}$. It is assumed that the equations of motion for these generators are integrable in terms of some known special functions. Then the canonical quantization scheme, in which quantum Hamiltonian has the same form as the classical one but $\vec{p}$ and $\vec{x}$ are replaced by the operators satisfying canonical commutation relations, does not guarantee that the resulting Schrödinger equation has simple structure of solutions. Loosely speaking, in the symmetry preserving quantization scheme it is necessary to find a quantum Hamiltonian which would obey nonlinear symmetry algebra reducing in the classical limit to the original Poisson algebra [5]. In particular, the simple classical dynamics determined by the latter algebra should correspond to a simple representation theory of the quantum algebra (which in some situations determines spectrum of the Hamiltonian). In the simplest cases it happens that such a procedure generates Hamiltonians which have the same form in the classical and quantum regions. It is this preservation of symmetries in the above sense that plays crucial role in the explanation of the Manning mystery – why certain classical systems remain to be integrable after the canonical quantization [6].

It is evident that the latter quantization procedure is not defined uniquely. At the classical level one can perform arbitrary non-canonical changes of variables and there is no universal criteria for choosing one of the forms of the symmetry algebra in favor of another. There is no such freedom at the quantum level, and, as a result, there are infinitely many quantum partners for one classical system. For the general aspects of relations between classical and quantum nonlinear algebras we refer to [5]. The procedure we use is somewhat analogous to the one appearing in the quantum Toda field theory [7], but we consider a quantum mechanical model. For a similar approach to the q-oscillator systems, see [8] and references therein. The results of this paper have been presented at the Workshop on Finite-Dimensional Integrable Systems (Dubna, Russia, July 1994).

The general class of self-similar potentials analyzed in [2] is related to the following nonlinear symmetry algebra:

$$HA^{\pm} = q^{\pm 2}A^{\pm}H, \tag{1}$$

$$A^+A^- = \prod_{k=1}^{N}(H - E_k), \qquad A^-A^+ = \prod_{k=1}^{N}(q^2 H - E_k),$$

where H is the standard Schrödinger operator for a particle in one-dimensional space,

$$H = -d^2/dx^2 + u(x), \tag{2}$$

and $A^{\pm}$ are some operators containing both the finite-difference and differential parts. In (2) we use the normalization $\hbar = 1$. As it will be shown below, in the self-consistent quantum picture,

3

when $\hbar \neq 0$, the potential depends on $\hbar$: $u = u(x, \hbar)$. For $N = 1$, relations (1) are equivalent to the q-Weyl algebra:

$$A^- A^+ - q^2 A^+ A^- = \omega, \qquad \omega = (q^2 - 1)E_1. \tag{3}$$

For $N = 2$ one gets a q-analogue of the $su(1,1)$ algebra. The $N \geq 2$ cases are not related directly to Lie algebras.

If the parameters q, E_k are real and $0 < q^2 < 1$, $E_k < E_{k+1} < 0$, the algebra (1) has a spectrum generating meaning. Namely, the zero modes of the operator A^- define N vacuum states with the energies E_k. Other bound states are generated by the sequential action of the operator A^+ upon these vacua. Their eigenvalues are accumulating near the zero from below; positive energy states form a continuous spectrum.

Self-similar potentials have interesting properties. In the limit $q \to 0$ they become N-soliton potentials. For $q \to 1$ they are related to the standard Painlevé transcendents [9] and their spectrum consists of a number of arithmetic series. A further simplification of these systems for odd N and some of the even N cases leads to the finite-gap potentials. Special subcases of the finite-gap potentials appear when q is a primitive root of unity [3].

Let us show that the algebra (3) is indeed quantum algebra, that is to say it describes a nontrivial quantum problem such that when Plank's constant $\hbar$ goes to zero, one gets a meaningful classical mechanical system. For simplicity we restrict ourselves to the case $N = 1$ in (1). Denote p and x the classical momentum and coordinate variables obeying the canonical Poisson bracket relations:

$$(p, x) = 1, \qquad (p, p) = (x, x) = 0. \tag{4}$$

We introduce three (commuting) functions of the phase-space coordinates H, A^* and A which satisfy the quadratic Poisson algebra

$$(H, A) = isHA, \qquad (H, A^*) = -isA^*H, \tag{5}$$

$$(A, A^*) = -isH, \tag{6}$$

where $s > 0$ is a free parameter. It is easy to see that $Q = A^*A - H$ lies in the center of this algebra, i.e. it is a Casimir element of it. Fixing Q to be a positive constant c, we write

$$H = A^*A - c. \tag{7}$$

Then the equation (6) can be rewritten as follows

$$(A, A^*) = -is(A^*A - c), \tag{8}$$

which we take as the basic relation. The equations (5) follow from (8) and the definition (7).

Let H be the Hamiltonian of a classical particle in some potential well

$$H = p^2 + u(x), \tag{9}$$

where for convenience we normalized the mass of the particle to be $m = 1/2$. The relation (7) leads naturally to the Ansatz

$$A = (p - if(x))e^{i\phi(p,x)}, \tag{10}$$

where $\phi(p, x)$ is some real function of p and x. The potential $u(x)$ and the function $f(x)$ are related in a simple way: $u(x) = f^2(x) - c$. Substituting (9), (10) into (8), we obtain the following equation upon $\phi(p, x)$:

$$\tfrac{1}{2}s(p^2 + f^2(x) - c) = -f'(x) + p\frac{\partial \phi}{\partial x} - f(x)f'(x)\frac{\partial \phi}{\partial p}. \tag{11}$$

4

This quasilinear partial differential equation has the general solution

$$\phi(p,x) = \tfrac{1}{2}sxp + \int^x \frac{G(y)dy}{\sqrt{p^2 + f^2(x) - f^2(y)}} + F\left(p^2 + f^2(x)\right) \tag{12}$$

$$G(x) = \tfrac{1}{2}s\left(f^2(x) - c + xf(x)f'(x)\right) + f'(x),$$

containing arbitrary function $F(H)$ of the Hamiltonian (9). Note that the latter freedom corresponds to the possibility of making gauge transformations $A \to Ae^{i\theta(H)}$ in (5), (6) without changing the algebra. Discarding this freedom, we see that formally for arbitrary potential $u(x)$ bounded from below (or arbitrary function $f(x)$) there exists a phase function $\phi(p,x)$ such that the algebra (8) is satisfied. This fact reflects a possibility of mapping dynamics of a large variety of systems onto the dynamics described by (5), (6). Despite of this discouraging fact, the quadratic Poisson algebra appears to be useful for derivation of the simplest self-similar potentials from the special quantization procedure.

Consider the simplest solution of (11) linear in the momentum

$$\phi(p,x) = \tfrac{1}{2}sxp, \tag{13}$$

which implies that $G(x) = 0$, or

$$f'(x) = -\frac{s(f^2(x) - c)}{2 + sxf(x)}. \tag{14}$$

This equation determines $f(x)$ as an implicit function of x

$$x = \frac{2\arcsin(f(x)/\sqrt{c}) + \alpha}{s\sqrt{c - f^2(x)}}, \tag{15}$$

where α is an integration constant (we assume that c is positive). It is convenient to work with the symmetric potential $u(x)$ which appears from the condition $f(0) = 0$, or $\alpha = 0$. Under this condition the relation (15) can be rewritten as a transcendental equation

$$\cos(z\sigma(z)) = \sigma(z), \qquad z = \tfrac{1}{2}s\sqrt{c}x, \qquad u(x) \equiv -c\sigma^2\left(\tfrac{1}{2}s\sqrt{c}x\right), \tag{16}$$

which shows clearly that the potential $u(x)$ varies between $u(0) = -c$ and $u(\infty) = 0$. Also, it is easy to see that the only extremal points of the potential are $x = 0$ (the minimum) and $x = \infty$ (the maximum), i.e. there are no oscillations. An interesting feature of (16) is that the function $\sigma(z)$ (and so the potential) is defined uniquely only between the points $z_0 \approx -2.97$ and $z_1 \approx 6.20$ (one has $\sigma(z_0) \approx -0.942$ and $\sigma(z_1) \approx 0.987$). At each edge of this interval a pair of new branches of solutions are appearing which is easy to see by plotting intersections of the graphs $g(\sigma) = \cos z\sigma$ and $g(\sigma) = \sigma$ for various values of the parameter z. Evidently, there are infinitely many such bifurcation points, the next ones close to zero being $z \approx -9.37$, -15.68, and $z \approx 12.53$, 18.82, etc. The solution defined on the whole axis is unique and it is the only physically acceptable one.

Note that in the $x \to 0$ limit one gets the harmonic oscillator potential: $u(x) \approx -c + s^2c^2x^2/4$. The $x \to \infty$ asymptotics

$$u(x) \approx -\frac{(\pi + 2\pi k)^2}{s^2x^2}$$

shows that we have an ordinary short-range potential. The arbitrary integer $k = 0, 1, 2, \ldots$ appeared from the mentioned nonuniqueness of the solution of equation for $f(x)$. Asymptotics of the physical branch of the potential corresponds to the choice $k = 0$.

Classical equations of motion for the variables $A(t)$, $A^*(t)$ determined by the algebra (5) are easily solved

$$\dot{A}(t) = (H, A) = isHA, \qquad A(t) = A(0)e^{isEt}, \tag{17}$$

where E is the energy of the particle. From (17) one can conjecture that the period of bounded motion of the particle depends inversely on the energy

$$T(E) = \frac{2\pi}{s|E|}. \tag{18}$$

It can be checked that the form of the symmetric potential restored from (18) by the known inversion formula [10]

$$x(U) = \frac{1}{2\pi} \int_{-c}^{U} \frac{T(E)dE}{\sqrt{U - E}} \tag{19}$$

coincides with (16). So, for the taken simplest solution of the equation for $\phi(p, x)$ there is a correspondence between the evolution in time of the particle's coordinate and of the dynamical variable $A(p, x)$. This relation holds only in the bounded motion region. When $E > 0$, the variable $A(t)$ continues to be bounded whereas the motion of particle is unbounded. This fact shows that in this case the relation between $x(t)$ and $A(t)$ is not unique.

Consider now quantization of our problem. The simplest possibility consists in defining the Schrödinger operator as the operator arising from (9), (16) after replacement of p by $-i\hbar d/dx$. Then there is no simple symmetry algebra and the energy spectrum is not known. We use another approach when the quantum Hamiltonian is defined from the requirement that the Poisson algebra (8) be transformed into the simple quantum algebra. Indeed, let us replace Poisson brackets by the commutators of operators

$$(A, B) \rightarrow \frac{i}{\hbar}[A, B]. \tag{20}$$

There is some ambiguity in doing this because of the operator ordering problem. We use the following prescription. Let us keep the relation (7) and assume that after the quantization the parameter s is renormalized: $s \rightarrow \tilde{s}(s, \hbar)$ (in principle a similar renormalization should hold for the parameter c, but we keep it fixed). As a result, one gets the following quantum algebra

$$[H, A] = \hbar\tilde{s}(s, \hbar)HA, \qquad [A, A^+] = -\hbar\tilde{s}(s, \hbar)(A^+A - c),$$

or

$$AH = q^2HA, \qquad HA^+ = q^2A^+H, \qquad AA^+ - q^2A^+A = \omega(\hbar), \tag{21}$$

where we denoted

$$q^2 \equiv 1 - \hbar\tilde{s}(s, \hbar), \qquad \omega(\hbar) \equiv c(1 - q^2). \tag{22}$$

This is the q-Weyl, or q-oscillator algebra which we have discussed in the beginning. Formally one may conclude that the discrete spectrum of the Hamiltonian H is determined from the relations (21) and consists of one geometric series:

$$|n\rangle = \frac{(A^+)^n}{\sqrt{\omega^n[n]!}}|0\rangle, \qquad A|0\rangle = 0, \qquad \langle n|m\rangle = \delta_{nm},$$

$$[n]! = [n][n - 1]!, \quad [0]! = 1, \quad [n] = (1 - q^{2n})/(1 - q^2),$$

$$A^+|n\rangle = \omega^{1/2}\sqrt{\frac{1 - q^{2(n+1)}}{1 - q^2}}|n + 1\rangle, \qquad A|n\rangle = \omega^{1/2}\sqrt{\frac{1 - q^{2n}}{1 - q^2}}|n - 1\rangle,$$

6

$$H|n\rangle = E_n|n\rangle, \qquad E_n = -cq^{2n}. \tag{23}$$

However, there is no guarantee that we have a self-consistent picture. It is necessary to check that there exists a quantum analog of the realization (9), (10) satisfying the relations (21).

It is not clear how to define the quantum partner of the general solution $\phi(p,x)$ (12), however for the simplest solution (13) this can be done easily. Indeed, the function $\exp(-isxp/2)$ after the quantization becomes the unitary scaling operator T (the product xp is replaced by the anticommutator of operators $\{x,p\}/2$). In the coordinate representation one has

$$T = \exp\left(-\tfrac{1}{4}s\hbar\{x, \frac{d}{dx}\}\right), \qquad T^+ = T^{-1}, \tag{24}$$

$$T\psi(x) = q^{1/2}\psi(qx), \qquad q \equiv e^{-s\hbar/2}.$$

Comparing with (22) we find the form of the parameter $\tilde{s}$: $\tilde{s}(s,\hbar) = (1 - e^{-s\hbar})/\hbar$, which tends to s when $\hbar \to 0$. As a result, the operators A, A^+ can be represented in the following form

$$A = -iT^{-1}\left(\hbar\frac{d}{dx} + f(x,\hbar)\right), \qquad A^+ = i\left(-\hbar\frac{d}{dx} + f(x,\hbar)\right)T. \tag{25}$$

Substituting these into (21) one gets the following differential-delay equation

$$\hbar\frac{d}{dx}\left(f(x,\hbar) + e^{-s\hbar/2}f(e^{-s\hbar/2}x,\hbar)\right) \tag{26}$$

$$+ f^2(x,\hbar) - e^{-s\hbar}f^2(e^{-s\hbar/2}x,\hbar) = c\left(1 - e^{-s\hbar}\right).$$

This equation corresponds to the simplest self-similar reduction of the dressing chain [1]. Here we have derived it from the completely different point of view – from the symmetry preserving quantization of a simple classical mechanical system. The resulting Hamiltonian has the form

$$H = A^+A - c = -\hbar^2\frac{d^2}{dx^2} + f^2(x,\hbar) - \hbar f'(x,\hbar) - c.$$

Note that the quantum potential is deeper than the classical one: $u(0,\hbar) = -2c/(1+q^2)$.

Consider the $\hbar \to 0$ limit in the equation (26). Expanding $f(x,\hbar)$ into the series in $\hbar$

$$f(x,\hbar) = f(x) + \tfrac{1}{2}s\hbar g(x) + O(\hbar^2) \tag{27}$$

and substituting it into (26) we see that the first term $f(x)$ satisfies the classical motion constraint (14). The quasiclassical approximant $g(x)$ is determined from a more complicated equation

$$(\frac{2}{s} + xf)g' + (xf' + 2f)g = \frac{1}{s}(xf)'' + \tfrac{1}{2}x^2(ff')' + \tfrac{5}{2}xff' + f^2 - c, \tag{28}$$

where the primes denote derivative with respect to x. Note that if in the limit $\hbar \to 0$ the parameter $q = e^{-s\hbar/2}$ is kept fixed, then the q-Weyl algebra relations degenerate (i.e. there will not be nontrivial Poisson algebra) and the admissible potential has the form $u(x) = \alpha/x^2$, where α is an arbitrary constant.

It is interesting to compare these results with those for the well known "exactly solvable" potentials such as the harmonic oscillator, Coulomb, Pöschl-Teller, etc, for which classical and quantum

potentials coincide. This fact is due to the specific choice of the dynamical symmetry algebra (quadratic Jacobi algebra $QJ(3)$ [6]) describing both classical and quantum potentials. Moreover, it appears that the quasiclassical approximation to the discrete part of the spectrum coincides with the exact one. For the potential (16) this is not the case, because classical and quantum symmetry algebras correspond to essentially different potentials. As an illustration, consider the quasiclassical part of the discrete spectrum for the potential (16) using the Bohr correspondence formula $dE_n/dn = 2\pi\hbar/T(E_n)$. Using (18) we get the quasiclassical spectrum $E_n = E_0 \exp(-\hbar s n)$. However, this spectrum does not coincide with the exact one for the "classical" potentilal (16) because as we know already, the exponential spectrum (23) arises for a different, "quantum" potential described by the equation (26).

As a final remark we would like to note that it is possible to repeat the presented above analysis for other region of the parameters s and c, but after the quantization the symmetry algebra will have only a formal meaning [2]. Consideration of the $N \geq 2$ systems (1), which is essentially more complicated than of the $N = 1$ case, lies beyond the scope of this note. The example of exact quantization of the Poisson algebras presented in this paper is quite sophisticated, it would be interesting to understand the general mechanism behind its existence in order to be able to analyze more complicated, e.g., higher-dimensional systems.

It is worth mentioning that one can consider in an analogous manner the self-similar potentials associated with the finite-difference Schrödinger equation

$$H\psi_n \equiv a_{n+1}\psi_{n+1} + a_n\psi_{n-1} + b_n\psi_n = \lambda\psi_n, \qquad (29)$$

which have been described in [11]. In this case the principal difference with respect to the considered above model consists in the definition of a classical Hamiltonian as a nontrivial function of the momentum.

References

[1] A.Shabat, *Inverse Prob.* **8**, 303 (1992).

[2] V.Spiridonov, *Phys.Rev.Lett.* **69**, 398 (1992); Proc. XIXth ICGTMP (CIEMAT/RSEF, 1992) vol.I, p.198; Phys. Rev. **A** (1995), to appear.

[3] S.Skorik and V.Spiridonov, *Lett.Math.Phys.* **28**, 59 (1993).

[4] A.Degasperis and A.Shabat, Theor.Math.Phys. **100**, 970 (1994).

[5] M.V.Karasev and V.P.Maslov, *Nonliear Poisson Brackets. Geometry and Quantization.* (Nauka, 1991) (in Russian).

[6] Ya.I.Granovskii, I.M.Lutzenko, and A.S.Zhedanov, *Ann.Phys. (N.Y.)* **217**, 1 (1992).

[7] A.N.Leznov and M.A.Mukhtarov, *Theor.Math.Phys.* **71**, 370 (1987).

[8] S.Shabanov, J. Phys. **A25**, L1245 (1992); V.I.Man'ko, G.Marmo, S.Solimeno, and F.Zaccaria, *Int.J.Mod.Phys.* **A8**, 3577 (1993).

[9] A.P.Veselov and A.B.Shabat, *Funct.Anal.Appl.* **27**, n2, 1 (1993).

[10] L.D.Landau and E.M.Lifshitz, *Mechanics* (Pergamon Press, 1953).

[11] V.Spiridonov, L.Vinet and A.Zhedanov, Lett. Math. Phys. **29**, 63 (1993).