Quantum fermionic oscillator group from R-matrix method

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Abstract

From unusual R-matrices, we define the fermionic oscillator group and its quantum deformations. We build the associated dual algebras, two of which appear true algebraic deformations of the one-dimensional fermionic algebra.

KEYWORKS: R-matrix, quantum group, fermionic oscillator.

Résumé

À partir de matrices R non standard, on définit le groupe de l'oscillateur fermionique et ses déformations quantiques. On construit les algèbres duales qui leur sont associées. On montre que deux d'entre elles sont de véritables déformations algébriques de l'algèbre fermionique à une dimension.

MOTS CLÉS : matrices R, groupe quantique, oscillateur fermionique.

I Introduction

Let us recall briefly how the R-matrix method of [1] applies to the oscillator group. Starting with the following three-dimensional faithful representation

$$T = \begin{pmatrix} 1 & \alpha & \beta \\ 0 & \eta & \gamma \\ 0 & 0 & 1 \end{pmatrix} \tag{1.1}$$

the R-matrix verifies

$$RT_1T_2 = T_2T_1R, \tag{1.2}$$

where as usually

$$T_1 = T \otimes I, \qquad T_2 = I \otimes T. \tag{1.3}$$

We impose the linear independance of the generators α, β, γ, η and the condition of associativity in such a way that no relations of third order have to be added.

Thus we get a large family of R-matrices in which only a subset is continuously related to the identity matrix. In this case studied in [2], the undeformed initial algebra is a commutative bialgebra with the coproduct:

$$\begin{aligned} \Delta\alpha &= 1 \otimes \alpha + \alpha \otimes \eta, \\ \Delta\gamma &= \eta \otimes \gamma + \gamma \otimes 1, \\ \Delta\beta &= 1 \otimes \beta + \beta \otimes 1 + \alpha \otimes \gamma, \\ \Delta\eta &= \eta \otimes \eta. \end{aligned} \tag{1.4}$$

In the present work, we are interested with another subfamily of R-matrices, continuously related with a diagonal matrix differing from the unit matrix in only one place (see [3] in $SL(2, \mathbb{C})$ case):

$$J = \mathrm{diag}(1, 1, 1, 1, -1, 1, 1, 1, 1). \tag{1.5}$$

Then, denoting the anticommutator by $\{\ \ ,\ \ \}$, the initial bialgebra is given by the following relations:

$$\begin{gathered} \{\alpha, \eta\} = 0, \quad \{\gamma, \eta\} = 0, \quad \alpha^2 = 0, \quad \gamma^2 = 0, \\ [\alpha, \beta] = 0, \quad [\beta, \gamma] = 0, \quad [\alpha, \gamma] = 0, \quad [\beta, \eta] = 0, \end{gathered} \tag{1.6}$$

and the coproduct (1.4).

We want to call it the fermionic oscillator group in so far as its dual algebra, with generators A, B, C, H (see (3.2)) is defined by:

$$\begin{gathered} \{A, C\} = B, \quad A^2 = 0, \quad C^2 = 0, \\ [A, H] = A, \quad [C, H] = -C, \\ [A, B] = [C, B] = [H, B] = 0, \end{gathered}$$

i.e., the usual one-dimensional fermionic algebra.

Therefore the R-matrices in the set under consideration will define quantum fermionic oscillator groups and by duality deformations of the one-dimensional fermionic algebra.

Our paper is organized in the following way: in Section II, we define the three types of quantum fermionic oscillator group we finally get. As a matter of fact, many R-matrices correspond to a given type, among which we define a generic R-matrix as depending of a minimal number of parameters; moreover we characterize those R-matrices verifying the QYBE. In Section III, we build the dual bialgebras associated to each type of quantum groups and, in Section IV, we conclude by a discussion of their properties.

II Quantum fermionic oscillator groups

As a consequence of the conditions imposed on the R-matrices, we get the following algebraic relations:

$$\begin{aligned} &\{\alpha, \eta\} = 0, \quad \{\gamma, \eta\} = 0, \quad \alpha^2 = \frac{q}{2}(1-\eta^2), \quad \gamma^2 = \frac{p}{2}(1-\eta^2), \\ &[\alpha, \beta] = q\gamma\eta + x\alpha, \quad [\alpha, \gamma] = 0, \\ &[\gamma, \beta] = p\alpha\eta + z\gamma, \quad [\beta, \eta] = 0, \end{aligned} \tag{2.1}$$

with the constraints:

$$qx = 0, \qquad pz = 0. \tag{2.2}$$

A priori, we have four cases to consider:

$$\begin{aligned} \text{i)} \quad & x = z = 0; \\ \text{ii)} \quad & q = z = 0; \\ \text{iii)} \quad & p = x = 0; \\ \text{iv)} \quad & p = q = 0. \end{aligned} \tag{2.3}$$

Indeed, ii) and iii) can be identified by exchanging α and γ, so that we finally have three types of quantum groups: type I corresponds to i), type II to ii) and iii) and type III to iv).

As in [2], [4], many R-matrices are associated to a given type of quantum groups. But processing in the same way that in these previous papers, we are able to introduce a generic R-matrix of the following form valid in all cases.

$$R = \begin{pmatrix} 1+y & 0 & 0 & 0 & q & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & y & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & y & 0 & z \\ 0 & y & 0 & 1 & 0 & x & 0 & x & 0 \\ 0 & 0 & 0 & 0 & y-1 & 0 & 0 & 0 & p \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & y & 0 \\ 0 & 0 & y & 0 & 0 & 0 & 1 & 0 & -z \\ 0 & 0 & 0 & 0 & 0 & y & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}. \tag{2.4}$$

This matrix does not verify the QYBE when y is different from zero.

III Dual bialgebras

Since α^2 and γ^2 are always given in terms of η, a complete basis of the quantum fermionic groups is given by

$$\beta^k \eta^\ell \alpha^i \gamma^j, \qquad k, \ell \in \mathbb{N}, \quad i, j = 0, 1. \tag{3.1}$$

Accordingly, we define generators A, B, C, H in the dual bialgebra by the following relations:

$$\begin{aligned} (A, \beta^k \eta^\ell \alpha^i \gamma^j) &= \delta_{k0}\delta_{i1}\delta_{j0}, \\ (B, \beta^k \eta^\ell \alpha^i \gamma^j) &= \delta_{k1}\delta_{i0}\delta_{j0}, \\ (C, \beta^k \eta^\ell \alpha^i \gamma^i) &= \delta_{k0}\delta_{i0}\delta_{j1} \\ (H, \beta^k \eta^\ell \alpha^i \gamma^j) &= \ell\delta_{k0}\delta_{i0}\delta_{j0}. \end{aligned} \tag{3.2}$$

As usual, the algebraic product of two elements in the dual algebra is given by

$$(XY, \beta^k\eta^\ell\alpha^i\gamma^j) = (X \otimes Y, \Delta\beta^k\eta^\ell\alpha^i\gamma^j). \tag{3.3}$$

We have to consider the various cases separately.

Type I

We prove recurrently that $\Delta\beta^k$ has the following form:

$$\begin{aligned}\Delta\beta^k = {} & \Gamma^{k,ij}_{rs;r's'}\beta^r\eta^s\alpha^i\gamma^j \otimes \beta^{r'}\eta^{s'}\alpha^i\gamma^j \\ & + \Delta^{k,ij}_{rs;r's'}\beta^r\eta^s\alpha^i\gamma^j \otimes \beta^{r'}\eta^{s'}\alpha^{i+1}\gamma^{j+1},\end{aligned} \tag{3.4}$$

where summation is over repeated indices and $i+1$, $j+1$ are taken mod 2. From the relation

$$\Delta\beta^{k+1} = \Delta\beta^k\Delta\beta,$$

we get recurrence relation between the various coefficients useful for the computations to follow (see Appendix).

Applying (3.3), we obtain the values of AB, BA, AC, CA, AH, HA, BC, CB, BH, HB, CH, HC, A^2 and C^2 on the basis (3.1) in terms of summations over the coefficients $\Gamma^{k,ij}_{rs;r's'}$, $\Delta^{k,ij}_{rs;r's'}$.

Thus we get:

$$(AB, \beta^k\eta^\ell\alpha^i\gamma^j) = \delta_{i1}\delta_{j0}\sum_{s,s'}\Gamma^{k;00}_{0s;1s'},$$

$$(BA, \beta^k\eta^\ell\alpha^i\gamma^j) = \delta_{i1}\delta_{j0}\sum_{s,s'}\Gamma^{k;00}_{1s;0s'},$$

$$(AC, \beta^k\eta^\ell\alpha^i\gamma^j) = \delta_{i0}\delta_{j0}\sum_{s,s'}\Delta^{k,10}_{0s;0s'} - \delta_{i1}\delta_{j1}\sum_{s,s'}\Gamma^{k;00}_{0s;0s'},$$

$$(CA, \beta^k\eta^\ell\alpha^i\gamma^j) = \delta_{i0}\delta_{j0}\sum_{s,s'}\Delta^{k,01}_{0s;0s'} + \delta_{i1}\delta_{j1}\sum_{s,s'}\Gamma^{k;00}_{0s;0s'},$$

$$(AH, \beta^k\eta^\ell\alpha^k\gamma^j) = p\delta_{i0}\delta_{j1}\sum_{s,s'}\Delta^{k;10}_{0s;0s'} + \delta_{i1}\delta_{j0}\sum_{s,s'}\big((\ell+s')\Gamma^{k,00}_{0s;0s'} - q\Gamma^{k,10}_{0s;0s'}\big),$$

$$(HA, \beta^k\eta^\ell\alpha^k\gamma^j) = -p\delta_{i0}\delta_{j1}\sum_{s,s'}\Delta^{k;01}_{0s;0s'} + \delta_{i1}\delta_{j0}\sum_{s,s'}\big((\ell+s)\Gamma^{k,00}_{0s;0s'} + q\Gamma^{k,10}_{0s;0s'}\big),$$

$$(BC, \beta^k\eta^\ell\alpha^i\gamma^j) = \delta_{i0}\delta_{j1}\sum_{s,s'}\Gamma^{k;00}_{1s;0s'},$$

$$(CB, \beta^k\eta^\ell\alpha^i\gamma^j) = \delta_{i0}\delta_{j1}\sum_{s,s'}\Gamma^{k;00}_{0s;1s'},$$

$$(BH, \beta^k\eta^\ell\alpha^i\gamma^j) = \delta_{i0}\delta_{j0}\sum_{s,s'}(\ell+s')\Gamma^{k,00}_{1s;00},$$

$$(HB, \beta^k\eta^\ell\alpha^i\gamma^j) = \delta_{i0}\delta_{j0}\sum_{s,s'}(\ell+s)\Gamma^{k,00}_{0s;1s'},$$

$$(CH, \beta^k\eta^\ell\alpha^k\gamma^j) = -q\delta_{i1}\delta_{j0}\sum_{s,s'}\Delta^{k;01}_{0s;0s'} + \delta_{i0}\delta_{j1}\sum_{s,s'}\big((\ell+s')\Gamma^{k,00}_{0s;0s'} + p\Gamma^{k,01}_{0s;0s'}\big),$$

$$(HC,\beta^k\eta^\ell\alpha^k\gamma^j) = q\delta_{i1}\delta_{j0}\sum_{s,s'}\Delta^{k;10}_{0s;0s'} + \delta_{i0}\delta_{j1}\sum_{s,s'}\big((\ell+s)\Gamma^{k,00}_{0,s-1;0s'} + p\Gamma^{k,01}_{0s;0s'}\big),$$
$$(A^2,\beta^k\eta^\ell\alpha^i\gamma^j) = \delta_{i0}\delta_{j0}\sum_{s,s'}\Gamma^{k;10}_{0s;0s'},$$
$$(C^2,\beta^k\eta^\ell\alpha^i\gamma^j) = \delta_{i0}\delta_{j0}\sum_{s,s'}\Gamma^{k;01}_{0s;0s'}.$$

The summation in the right members can be performed by using the recurrence relations of the Appendix, where we give some meaningful examples for interested reader. With similar methods, we evaluate B^n, B^nA, B^nC on the basis (3.1). Collecting together all these results and quoting only the nonzero quantities, we have

$$\begin{aligned}
&\{A,C\} = \frac{\sinh 2B\sqrt{pq}}{2\sqrt{pq}}, \quad A^2 = \frac{1-\cosh 2B\sqrt{pq}}{4q}, \quad C^2 = \frac{1-\cosh 2B\sqrt{pq}}{4p} \\
&[A,H] = \frac{1}{2}A(1+\cosh 2B\sqrt{pq}) + p\frac{\sinh 2B\sqrt{pq}}{2\sqrt{pq}}C, \\
&[C,H] = -\frac{1}{2}C(1+\cosh 2B\sqrt{pq}) - q\frac{\sinh 2B\sqrt{pq}}{2\sqrt{pq}}A.
\end{aligned} \tag{3.5}$$

When p and q go to zero, we get the algebraic relations of the usual one-dimensional fermionic algebra.

Let us build now the coproduct. For any element X of the dual algebra, it is define by:

$$(\Delta X,\beta^k\eta^\ell\alpha^i\gamma^j\otimes\beta^{k'}\eta^{\ell'}\alpha^{i'}\gamma^{j'}) = (X,\beta^k\eta^\ell\alpha^i\gamma^j\beta^{k'}\eta^{\ell'}\alpha^{i'}\gamma^{j'}).$$

In reordering the product in the righthand side, we use the following recursively proved relations:

$$\begin{aligned}
\alpha^i\gamma^j\beta^k =& \frac{1}{2}\left[\big(\beta+\eta(i+j)\sqrt{pq}\big)^k + \big(\beta-\eta(i+j)\sqrt{pq}\big)^k\right]\alpha^i\gamma^j \\
&-\frac{1}{2}\left(\frac{q}{p}\right)^{(i-j)/2}\left[\big(\beta+\eta(i+j)\sqrt{pq}\big)^k - \big(\beta-\eta(i+j)\sqrt{pq}\big)^k\right]\alpha^{i+1}\gamma^{j+1},
\end{aligned} \tag{3.6}$$

where $i+j$, $i+1$, $j+1$ are taken mod 2.

Noticing that

$$\begin{aligned}
\big(B^n(-1)^H,\beta^k\eta^\ell\alpha^i\gamma^j\big) &= \delta_{i0}\delta_{j0}\delta_{kn}(-1)^\ell n!, \\
\big(B^n(-1)^HA,\beta^k\eta^\ell\alpha^i\gamma^j\big) &= \delta_{i1}\delta_{j0}\delta_{kn}(-1)^\ell n!, \\
\big(B^n(-1)^HC,\beta^k\eta^\ell\alpha^i\gamma^j\big) &= \delta_{i0}\delta_{j1}\delta_{kn}(-1)^\ell n!,
\end{aligned}$$

we finally get:

$$\begin{aligned}
\Delta A =& 1\otimes A + A\otimes(-1)^H\cosh B\sqrt{pq} - pC\otimes(-1)^H\frac{\sinh B\sqrt{pq}}{\sqrt{pq}}, \\
\Delta B =& 1\otimes B + B\otimes 1, \\
\Delta C =& 1\otimes C + C\otimes(-1)^H\cosh B\sqrt{pq} - qA\otimes(-1)^H\frac{\sinh B\sqrt{pq}}{\sqrt{pq}}, \\
\Delta H =& 1\otimes H + H\otimes 1 - qA\otimes(-1)^H(\cosh B\sqrt{pq})A \\
&-\sqrt{pq}A\otimes(-1)^H(\sinh B\sqrt{pq})C + \sqrt{pq}C\otimes(-1)^H(\sinh B\sqrt{pq})A \\
&+pC\otimes(-1)^H(\cosh B\sqrt{pq})C.
\end{aligned} \tag{3.7}$$

Type II

In this case, we prove the following form of $\Delta\beta^k$:

$$\begin{aligned}\Delta\beta^k = {} & \Gamma^{0,k}_{rs;r's'}\beta^r\eta^s\otimes\beta^{r'}\eta^{s'} + \Gamma^{1,k}_{rs;r's'}\beta^r\eta^s\otimes\beta^{r'}\eta^{s'}\alpha \\ & + \Gamma^{2,k}_{rs;r's'}\beta^r\eta^s\alpha\otimes\beta^{r'}\eta^{s'}\gamma.\end{aligned}$$

From this, we get the evaluations:

$$\begin{aligned}
(AB,\beta^k\eta^\ell\alpha^i\gamma^j) &= \delta_{i1}\delta_{j0}\sum_{s,s'}\Gamma^{0,k}_{0s;1s'} \\
(BA,\beta^k\eta^\ell\alpha^i\gamma^j) &= \delta_{i1}\delta_{j0}\sum_{s,s'}\Gamma^{0,k}_{1s;0s'} \\
(AC,\beta^k\eta^\ell\alpha^i\gamma^j) &= \sum_{s,s'}(\delta_{i0}\delta_{j0}\Gamma^{2,k}_{0s;0s'} - \delta_{i1}\delta_{j1}\Gamma^{0,k}_{0s;0s'}), \\
(CA,\beta^k\eta^\ell\alpha^i\gamma^j) &= \delta_{i1}\delta_{j1}\sum_{s,s'}\Gamma^{0,k}_{0s;0s'} \\
(AH,\beta^k\eta^\ell\alpha^i\gamma^j) &= \sum_{s,s'}\big(\delta_{i1}\delta_{j0}(s'+\ell+1)\Gamma^{0,k}_{0s;0s'} - p\delta_{i0}\delta_{j1}\Gamma^{2,k}_{0s;0s'}\big), \\
(HA,\beta^k\eta^\ell\alpha^i\gamma^j) &= \delta_{i1}\delta_{j0}\sum_{s,s'}(s+\ell),\Gamma^{0,k}_{0s;0s'}, \\
(BC,\beta^k\eta^\ell\alpha^i\gamma^j) &= \delta_{i0}\delta_{j1}\sum_{s,s'}\Gamma^{0,k}_{1s;0s'}, \\
(CB,\beta^k\eta^\ell\alpha^i\gamma^j) &= \delta_{i0}\delta_{j1}\sum_{s,s'}\Gamma^{0,k}_{0s;1s'}, \\
(BH,\beta^k\eta^\ell\alpha^i\gamma^j) &= \delta_{i0}\delta_{j0}\sum_{s,s'}(s'+\ell)\Gamma^{0,k}_{1s;0s'}, \\
(HB,\beta^k\eta^\ell\alpha^i\gamma^j) &= \delta_{i0}\delta_{j0}\sum_{s,s'}(s+\ell)\Gamma^{0,k}_{0s;1s'}, \\
(CH,\beta^k\eta^\ell\alpha^i\gamma^j) &= \delta_{i0}\delta_{j1}\sum_{s,s'}(s'+\ell)\Gamma^{0,k}_{0s;0s'}, \\
(HC,\beta^k\eta^\ell\alpha^i\gamma^j) &= \delta_{i0}\delta_{j1}\sum_{s,s'}(s+\ell+1)\Gamma^{0,k}_{0s;0s'}, \\
(A^2,\beta^k\eta^\ell\alpha^i\gamma^j) &= \delta_{i0}\delta_{j0}\sum_{s,s'}\Gamma^{1,k}_{0s;0s'}, \\
(C^2,\beta^k\eta^\ell\alpha^i\gamma^j) &= 0.
\end{aligned}$$

The right members can be computed from the recurrent relations for $\Gamma^{i,k}_{rs;r's'}$, $i=0,1,2$, deduced from $\Delta\beta^{k+1} = \Delta\beta^k\Delta\beta$. This is much easier to do than above and we leave it to the reader.

We finally get:

$$\begin{gathered}\{A,C\}=\frac{e^{xB}-1}{x},\quad A^2=-\frac{p}{2}\left(\frac{e^{xB}-1}{x}\right)^2,\quad C^2=0,\\ [A,H]=A-p\frac{e^{xB}-1}{x}C,\quad [C,H]=-C,\\ [A,B]=[C,B]=[H,B]=0.\end{gathered}\tag{3.8}$$

The coproduct is easily obtained by using the formulas

$$\begin{gathered}\alpha\beta^k=(\beta+x)^k\alpha,\quad \alpha\gamma\beta^k=(\beta+x)^k\alpha\gamma,\\ \gamma\beta^k=\beta^k\gamma-\frac{p}{x}\big((\beta+x)^k-\beta^k\big)\eta\alpha\end{gathered}\tag{3.9}$$

and we find:

$$\begin{aligned}\Delta A&=1\otimes A+A\otimes e^{xB}(-1)^H+\frac{p}{x}C\otimes(1-e^{xB})(-1)^H,\\ \Delta B&=1\otimes B+B\otimes 1,\\ \Delta C&=1\otimes C+C\otimes(-1)^H,\\ \Delta H&=1\otimes H+H\otimes 1-pC\otimes(-1)^H C.\end{aligned}\tag{3.10}$$

Type III

The form of $\Delta\beta^k$ is yet simpler than previously. Indeed we have

$$\Delta\beta^k=\Gamma^{0,k}_{rs;r's'}\beta^r\eta^s\otimes\beta^{r'}\eta^{s'}+\Gamma^{1,k}_{rs;r's'},\beta^r\eta^s\alpha\otimes\beta^{r'}\eta^{s'}\gamma.$$

As the evaluations are very easy to do, we directly give the final results for both algebraic relations and coproduct. We have

$$\begin{gathered}\{A,C\}=\frac{e^{(x+z)B}-1}{x+z},\quad A^2=C^2=0,\\ [A,H]=A,\quad [C,H]=-C,\\ [A,B]=[C,B]=[H,B]=0.\end{gathered}\tag{3.11}$$

and

$$\begin{aligned}\Delta A&=A\otimes e^{xB}(-1)^H+1\otimes A,\\ \Delta B&=1\otimes B+B\otimes 1,\\ \Delta C&=C\otimes e^{zB}(-1)^H+1\otimes C,\\ \Delta H&=1\otimes H+H\otimes 1.\end{aligned}\tag{3.12}$$

If in (3.11), we introduce:

$$B'=\frac{e^{(x+z)B}-1}{x+z},$$

we get the algebraic relations for the one-dimensional fermionic algebra. The deformation appears only in the coproduct which is now written in terms of B':

$$\begin{aligned}\Delta A&=A\otimes\big(1+(x+z)B'\big)^{x/x+z}(-1)^H+1\otimes A,\\ \Delta B'&=1\otimes B'+B'\otimes 1+(x+z)B'\otimes B',\\ \Delta C&=C\otimes\big(1+(x+z)B'\big)^{x/x+z}(-1)^H+1\otimes C,\\ \Delta H&=1\otimes H+H\otimes 1.\end{aligned}\tag{3.13}$$

IV Conclusion

In this final section, the structure of the bialgebras defined by (3.5), (3.7) and by (3.8), (3.10) respectively is investigated more deeply. We again distinguish the two cases.

Type I

Let us defined new generators A' and C' by the formulas

$$\begin{aligned} A' &= A\cosh B\sqrt{pq} + C\sqrt{\frac{p}{q}}\sinh B\sqrt{pq}, \\ C' &= A\sqrt{\frac{q}{p}}\sinh B\sqrt{pq} + C\cosh B\sqrt{pq}. \end{aligned} \tag{4.1}$$

Then we have

$$\begin{gathered} \{A', C'\} = \frac{\sinh 2B\sqrt{pq}}{2\sqrt{pq}}, \quad A'^2 = \frac{\cosh 2B\sqrt{pq} - 1}{4q}, \quad C'^2 = \frac{\cosh 2B\sqrt{pq} - 1}{4p}, \\ [A', H] = A', \quad [C', H] = -C', \end{gathered} \tag{4.2}$$

all other commutators being zero.

If p, q are not simultaneously equal to zero, it can be shown from (4.2) that it does not exist elements of null square in the algebra having the same commutation relations with H as A' and C'. This proves that our algebra is definitely different from the one-dimensional fermionic algebra and is a true algebraic deformation of it.

Type II

The conclusion is the same as above. Indeed, if we introduce the new generator A' by:

$$A' = A - \frac{p}{2}\frac{e^{xB} - 1}{x},$$

we get the relations:

$$\begin{gathered} \{A', C\} = \frac{e^{xB} - 1}{x}, \quad A'^2 = -p\left(\frac{e^{xB} - 1}{x}\right)^2, \quad C^2 = 0, \\ [A', H] = A', \quad [C, H] = -C, \end{gathered}$$

all other commutators being zero.

A similar argument as above is valid in this case and leads to the same conclusion.

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Appendix A Recurrence relations

By writing $\Delta\beta^{k+1} = \Delta\beta^k \Delta\beta$ and reordering according to algebraic relations (2.1) with $x = z = 0$, we get:

$$\begin{aligned}
\Gamma^{k+1,ij}_{rs;r's'} &= \Gamma^{k,ij}_{rs;r'-1,s'} + \Gamma^{k,ij}_{r-1,s;r's'} + ij\Delta^{k,i+1,j}_{rs;r's'} \\
&\quad + \big(pi(j+1) + qj(i+1)\big)\bigg(\frac{1}{2}(\Delta^{k,i+1,j}_{rs;r's'} - \Delta^{k,i+1,j}_{r,s-2j;r',s'-2i}) \\
&\qquad\qquad - \Delta^{k,ij}_{rs;r',s'-1} - \Delta^{k,i+1,j+1}_{r,s-1;r's'}\bigg) \\
&\quad + (i+1)(j+1)\frac{pq}{4}(\Delta^{k,i+1,j}_{rs;r's'} - \Delta^{k,i+1,j}_{r,s-2;r's'}\Delta^{k,i+1,j}_{rs;r',s'-2} + \Delta^{k,i+1,j}_{r,s-2;r',s'-2}).
\end{aligned} \tag{A.1}$$

$$\begin{aligned}
\Delta^{k+1,ij}_{rs;r's'} &= \Delta^{k,ij}_{rs;r'-1,s'} + \Delta^{k,ij}_{r-1,s;r's'} + i(j+1)\Gamma^{k,i+1,j}_{rs;r's'} \\
&\quad + ij\frac{p}{2}\big(\Gamma^{k,i+1,j}_{rs;r's'} - \Gamma^{k,i+1,j}_{rs;r',s'-2}\big) + (i+1)(j+1)\frac{q}{2}\big(\Gamma^{k,i+1,j}_{rs;r's'} - \Gamma^{k,i+1,j}_{r,s-2;r's'}\big) \\
&\quad - \big((p(i+1)j + qi(j+1)\big)\Gamma^{k,ij}_{rs;r',s'-1} \\
&\quad - \big(pi(j+1) + q(i+1)j\big)\Gamma^{k,i+1,j+1}_{r,s-1;r's'} \\
&\quad + (i+1)j\frac{pq}{4}\big(\Gamma^{k,i+1,j}_{rs;r's'} - \Gamma^{k,i+1,j}_{rs;r',s'-2} - \Gamma^{k,i+1,j}_{r,s-2;r's'} + \Gamma^{k,i+1,j}_{r,s-2;r',s'-2}\big).
\end{aligned} \tag{A.2}$$

By convention, the coefficients with negative lower indices are equal to zero, and $i+1$, $j+1$ are taken mod 2.

Appendix B Some examples of evaluation

(i) $\sum_{s,s'} \Gamma^{k,00}_{0s;0s'}$.
From the recurrence, we get readily

$$\sum_{s,s'} \Gamma^{k+1,00}_{0s;0s'} = 0, \qquad k = 0, 1, \dots,$$

so that we have

$$\sum_{s,s'} \Gamma^{k,00}_{0s;0s'} = \delta_{k0}.$$

(ii) From this result, we deduce

$$\sum_{s,s'} \Gamma^{k,00}_{1s;0s'} = \sum_{s,s'} \Gamma^{k,00}_{0s;1s'} = \delta_{k1},$$

since the recurrence gives:

$$\sum_{s,s'} \Gamma^{k+1,00}_{1s;0s'} = \sum_{s,s'} \Gamma^{k,00}_{0s;0s'} = \sum_{s,s'} \Gamma^{k+1,00}_{0s;1s'}.$$

(iii) A little more involved computation is needed to evaluate the anticommutator $\{A, C\}$. Indeed, we have successively:

$$\begin{aligned} \sum_{s,s'} \Delta^{k+1,10}_{0s;0s'} &= \delta_{k0} - q \sum_{s,s'} \Gamma^{k,10}_{0s,0s'} - p \sum_{s,s'} \Gamma^{k,01}_{0s;0s'}, \\ \sum_{s,s'} \Gamma^{k+1,10}_{0s;0s'} &= -p \sum_{s,s'} \Gamma^{k,10}_{0s,0s'} - p \sum_{s,s'} \Delta^{k,01}_{0s;0s'}, \\ \sum_{s,s'} \Delta^{k+1,01}_{0s;0s'} &= -q \sum_{s,s'} \Gamma^{k,10}_{0s,0s'} - p \sum_{s,s'} \Delta^{k,01}_{0s;0s'}. \end{aligned} \tag{B.1}$$

Let us define

$$X_k = -q \sum_{s,s'} \Gamma^{k,10}_{0s,0s'} - p \sum \Gamma^{k,01}_{0s,0s'}.$$

We have the recurrent relation:

$$X_{k+1} = 2pq\delta_{k1} + 4pqX_{k-1},$$

with the initial values

$$X_0 = X_1 = 1,$$

the solution of which is given by:

$$X_k = \frac{1}{4}(4pq)^{k/2}\big(1 + (-1)^k\big) - \frac{1}{2}\delta_{k0}.$$

From this result we deduce readily from (B.1) the values of

$$\sum_{s,s'} \Delta^{k,01}_{0s;0s'}, \quad \sum_{s,s'} \Gamma^{k,10}_{0s;0s'}, \quad \sum_{s,s'} \Gamma^{k,01}_{0s;0s'}, \text{ and } \sum_{s,s'} \Delta^{k,10}_{0s;0s'}.$$

The other evaluations are left to the reader.

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