Control of systems to sets and their interiors^{*}

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Abstract

For a closed target set $S \subset \mathbb{R}^n$ and a control system (formulated as a differential inclusion and defined near S), the present paper considers a sufficient condition for the property that every point near S can be steered to S in finite time by some trajectory of the system. Estimates are obtained revealing how fast some such trajectory is nearing the target. A strong form of this condition is shown to imply that every trajectory of the system hits the target. With a further assumption on the target set S , we also consider conditions that guarantee that some trajectories enter the interior of S .

Keywords. Weak and strong attainability, control systems, penetrative systems, minimum time function

Résumé

On présente des conditions assurant qu'un système contrôle puisse être dirigé vers un ensemble en temps fini. La question de pénétrer l'intérieur d'un ensemble est également étudiée. L'outil principal est l'analyse non lisse, en particulier les méthodes proximales.

1 Introduction

This paper presents conditions on a control system and a target set S that guarantee that any point near S can be steered to S in finite time by at least one trajectory of the system. We call such a system and target set *weakly attainable*. Similar considerations are made for *strongly attainable* systems, in which all trajectories hit the target. Furthermore, in a similar vein, we consider conditions which allow for arcs to penetrate into the interior of S .

Our results are local in nature, and thus the target set $S \subset \mathbb{R}^n$ is assumed to be compact. We shall formulate the control system as a differential inclusion

$$\begin{aligned}\dot{x}(t) &\in F(x(t)) \text{ a.e. } t \in [0, T] \\ x(0) &= x,\end{aligned}$$

in which $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is a multifunction with compact convex values. A sufficient condition for weak attainability (see Theorem 3.1 below) is that F be continuous and that there exists $\delta > 0$ so that for every $y \in \text{bd } S$ and $\zeta \in N_S^P(y)$, we have

$$\min_{v \in F(y)} \langle \zeta, v \rangle \leq -\delta \|\zeta\|. \quad (1)$$

For strong attainability, the “min” in (1) is replaced by “max”. The notation $N_S^P(y)$ refers to the proximal normal cone at y (see Section 2). Notice that condition (1) (the type of which we refer to as a δ condition) is imposed only at points on the boundary of S , and is nonvacuous at only those points that have a nonzero proximal normal. Condition (1) reduces to the one introduced by Petrov (Ref. 1) (with $S =$ the origin), but which is stated to hold in a neighborhood of the target. More recently, Bardi and Falcone (Ref. 2) and Cannarsa and Sinestrari (Ref. 3) have considered such local assumptions with general closed target sets, but the dynamics require at least Lipschitz state dependence. We show in Theorem 3.4 that our continuity assumption on F can be further weakened to upper semicontinuity if the δ condition (1) is assumed to hold for all y near S and for any $\zeta \in y - \text{proj}_S(y)$, where $\text{proj}_S(y)$ are those elements in S closest to y . Both continuity and upper semicontinuity are new in this context

The proofs of our attainability results are based upon a time discretization algorithm, which can accurately be described as *proximal projection*. This in turn consists of applying the mean value inequality (Ref. 4) in conjunction with new results in (Ref. 5) on properties of the distance function to a closed set. Our essentially algorithmic approach appears to yield the best rate of convergence among the existing ones.

Dynamic programming techniques are used in (Ref. 2) to perform an approximation to the minimum time function, which is defined at a point x as the least time T for which a solution $x(\cdot)$ to (1) exists and has $x(T) \in S$. More specifically, a discretization of the associated Hamilton–Jacobi equation is made in (Ref. 2), and then it is shown that the discrete solutions converge to the so-called viscosity solution. The discretization is in fact an approximation to the minimum time function because the latter is the viscosity solution (which is unique) of the Hamilton–Jacobi equation. A major difference with proximal projection as developed here from a method based on dynamic programming is that the former is moving “forward” in time, while the latter uses a “backward” construction. The appendix in (Ref. 3) also uses a forward time procedure, somewhat similar in nature to ours, but is only applicable for problems formulated with explicit control variables without feedback. For the goals of that paper, further smoothness in the state variable is also required.

After giving some preliminaries in Section 2, we prove the attainability results in Section 3. Under an additional *epi-continuous* assumption on the target set S , we show in Section 4 that there exist trajectories entering the interior of S . Our results appear to be the first on this issue. Examples are provided in Section 5 illustrating the role of the hypotheses.

2 Preliminaries

Let $S \subset \mathbb{R}^n$ be compact, and $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ be a multifunction with compact convex values. Recall that F is said to be continuous of modulus ω_F near S provided

$$\Delta_H(F(x), F(y)) \leq \omega_F(\|x - y\|) \quad \text{for all } x, y \text{ near } S,$$

where Δ_H denotes the Hausdorff metric, and $\omega_F : [0, \infty) \rightarrow [0, \infty)$ is a continuous, nondecreasing function with $\omega_F(0) = 0$. F is upper semicontinuous at x if all sequences $\{y_i\}, \{v_i\}$ satisfying $y_i \rightarrow x$ and $v_i \in F(y_i)$ have the property that every cluster point of $\{v_i\}$ lies in $F(x)$.

We next give the relevant background of proximal analysis featured in this paper. For a more complete treatment with applications to optimization, see Clarke (Ref. 6) and Loewen (Ref. 7). If $y \in \text{bd } S$, a vector $\zeta \in \mathbb{R}^n$ is called a *proximal normal* to S at y provided there exists $r > 0$ so that

$$\{y + r(\zeta + \text{cl } B)\} \cap S = \{y\}.$$

The set of all proximal normals at a point y is a cone, and is denoted by $N_S^P(y)$.

Now suppose $f : \mathbb{R}^n \rightarrow \mathbb{R}^1 \cup \{+\infty\}$ is lower semicontinuous, and $x \in \mathbb{R}^n$ is such that $f(x) < +\infty$. A vector $\zeta \in \mathbb{R}^n$ is called a *proximal subgradient* of f at x provided that $(\zeta, -1) \in N_{\text{epi } f}^P(x, f(x))$, where

$$\text{epi } f := \{(y, \alpha) : \alpha \geq f(y)\}$$

is the epigraph of f . The set of proximal subgradients of f at x is denoted by $\partial_P f(x)$. One can show that $\zeta \in \partial_P f(x)$ if and only if there exists $\sigma > 0$ so that

$$f(y) - \langle \zeta, y - x \rangle + \sigma \|y - x\|^2 \geq f(x)$$

for all y near x .

We shall require the following theorem from the calculus governing proximal subgradients, which is labeled as the mean value inequality (Clarke and Ledyaev (Ref. 4), (Ref. 8)). Suppose Y is compact and convex, and x_0 is such that $f(x_0) < \infty$. We write $[x_0, Y]$ for the convex hull of x_0 and Y .

Theorem 2.1 (Mean value inequality). *Suppose f is lower semicontinuous and bounded below on $[x_0, Y] + kB$ for some $k > 0$. Then for any $\varepsilon > 0$ there exists $z \in [x_0, Y] + \varepsilon B$ and $\zeta \in \partial_P f(z)$ such that*

$$\min_Y f - f(x) < \langle \zeta, y - x_0 \rangle + \varepsilon \text{ for all } y \in Y.$$

Furthermore, z can be chosen to satisfy

$$f(z) < \inf_{[x_0, Y]} f + |\min_Y f - f(x)| + \varepsilon.$$

We shall measure the rate at which a trajectory approaches S by the so-called distance function d_S to S , defined by

$$d_S : \mathbb{R}^n \rightarrow \mathbb{R}^1; \quad d_S(x) := \inf_{y \in S} \|x - y\|.$$

The set of elements in S closest to $x \notin S$ is denoted by $\text{proj}_S(x)$. That is,

$$\text{proj}_S(x) = \{y \in S : d_S(x) = \|x - y\|\}.$$

We have recently carried out a detailed study of the differentiability properties of d_S in (Ref. 5) and (Ref. 9). For the immediate purposes ahead, however, we shall only require the following formula for the proximal subgradient at points off S , which is proven in (Ref. 5).

Theorem 2.2. Suppose $x \notin S$ and $\zeta \in \partial_P d_S(x)$. Then $\text{proj}_S(x)$ consists of a single element y , say, and

$$\zeta = (x - y)/d_S(x).$$

The main concepts in this paper are precisely defined in the following definitions. We suppose that S is compact, and that F has compact convex values and is at least upper semicontinuous.

Definition 2.1. (a) The pair (S, F) is called *weakly attainable* of radius $r > 0$ provided that for every $x \in S + rB$, there exist a time $T > 0$ and a trajectory $x(\cdot)$ of F defined on $[0, T]$ for which $x(0) = x$ and $x(T) \in S$. If in addition there exists a constant $c > 0$ for which

$$d_S(x(t)) \leq d_S(x(s)) - (t - s)c \quad \text{for all } 0 \leq s < t \leq T,$$

then c is called a rate of weak attainability of (S, F) .

(b) The pair (S, F) is called *strongly attainable* of the corresponding radius and rate if the properties in (a) hold for every trajectory of F (but of course for perhaps different T).

We denote the closure of the complement of S by $\overline{\text{comp}} S$. It is convenient for our purposes in Section 4 to state the penetration definitions as originating from a fixed point on the boundary, rather than making a simultaneous declaration at all points of the boundary.

Definition 2.2. (a) The pair (S, F) is *weakly penetrative* at $x \in \text{bd } S$ provided there exists a trajectory $x(\cdot)$ of F for which $x(0) = x$ and $d_{\overline{\text{comp}} S}(x(t)) > 0$ for all small $t > 0$. In this case, a value $r > 0$ so that the trajectory can be chosen on an interval $[0, T]$ with $d_{\overline{\text{comp}} S}(x(T)) \geq r$ is a radius of penetration. If a constant c is also available satisfying

$$x(t) + tcB \subset S \quad \text{for all } 0 \leq t \leq T,$$

then c is called a rate of penetration at x .

(b) The pair (S, F) is called *strongly penetrative* at $x \in \text{bd } S$ of the corresponding size and rate if the properties in (a) hold for every trajectory of F originating from x .

3 Attainable systems

The main result of this section is the following.

Theorem 3.1. Suppose $S \subseteq \mathbb{R}^n$ is nonempty and compact, and $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is a continuous multifunction of modulus ω_F with compact convex values. Suppose there exists $\delta > 0$ so that whenever $y \in S$ and $\zeta \in N_S^P(y)$, there exists $v \in F(y)$ for which

$$\langle \zeta, v \rangle \leq -\delta \|\zeta\|.$$

Then (S, F) is weakly attainable. Moreover, the radius and rate of attainability depend only on δ and ω_F .

Proof. We first fix the constants featured in the estimates to come. Let

$$\begin{aligned} r, c &> 0 \quad \text{so that} \quad \delta/2 - \omega_F(2r) \geq c > 0 \\ T &:= r/c \\ M &:= \sup\{\|v\| : v \in F(x), x \in S + r \text{cl } B\}. \end{aligned}$$

Let $x \in S + rB$. Now let N be a sufficiently large positive integer so that $h := T/N$ satisfies

$$h < \min\{r/(2(M + \delta)), (r - d_S(x))/M\}.$$

We shall proceed by constructing a sequence of length no more than N for which the last element will lie within a distance of $(h\delta)/2$ of S .

The Mean Value Inequality applied with data $f := d_S$, $Y := x_0 + hF(x_0)$ and $\varepsilon := (h\delta)/2$ provides a point z and a vector ζ in such a manner that

$$\zeta \in \partial_P d_S(z), \quad (2)$$

$$z \in [x_0, x_0 + hF(x_0)] + \varepsilon B, \quad (3)$$

$$d_S(z) \leq d_S(x_0) + hM + \varepsilon, \quad (4)$$

and that the following inequality holds for all $w \in F(x_0)$:

$$\min_{v \in F(x_0)} d_S(x_0 + hv) - d_S(x_0) < \langle \zeta, hw \rangle + \varepsilon. \quad (5)$$

Our construction will terminate when z lands in S , so let us suppose for the moment that $z \notin S$. By Theorem 2.2, we have from (2) that

$$\zeta = (z - y)/d_S(z),$$

where $\{y\} = \text{proj}_S(z)$, which immediately implies that

$$\zeta \in N_S^P(y) \quad \text{and} \quad \|\zeta\| = 1. \quad (6)$$

From the δ assumption in the theorem applied to ζ as in (6), there exists $v \in F(y)$ so that

$$\langle \zeta, v \rangle \leq -\delta. \quad (7)$$

The continuity hypothesis on F implies the existence of some $w \in F(x_0)$ so that

$$\|w - v\| \leq \omega_F(\|y - x_0\|). \quad (8)$$

Hence from (5), we see that

$$\begin{aligned} \min_{v' \in F(x_0)} d_S(x_0 + hv') - d_S(x_0) &< h\langle \zeta, w \rangle + \varepsilon \\ &\leq h\langle \zeta, v \rangle + h\langle \zeta, w - v \rangle + \varepsilon \\ &\leq -\delta h + h\omega_F(\|y - x_0\|) + \varepsilon, \end{aligned} \quad (9)$$

where we have used (6), (7), and (8) in deducing the last inequality. We see from (3) and (4) that

$$\begin{aligned} \|y - x_0\| &\leq \|y - z\| + \|z - x_0\| \\ &\leq d_S(z) + hM + \varepsilon \\ &\leq d_S(x_0) + 2(hM + \varepsilon) \\ &\leq r + 2h(M + \delta) \\ &\leq 2r. \end{aligned} \quad (10)$$

The last inequality follows from the choice of N . Since modulus functions are nondecreasing, we immediately have from (10) that

$$\omega_F(\|y - x_0\|) \leq \omega_F(2r) \quad (11)$$

Inserting (11) into (9), we deduce from (9) and our choices of ε and c that there exists $v_0 \in F(x_0)$ so that

$$\begin{aligned} d_S(x_0 + hv_0) - d_S(x_0) &< \varepsilon - \delta h + h\omega_F(2r) \\ &\leq h(\omega_F(2r) - \delta/2) \\ &\leq -hc. \end{aligned} \quad (12)$$

Now let $x_1 = x_0 + hv_0$, and apply the above procedure initialized at the point x_1 in place of x_0 . Continuing in this manner, a sequence $x_0, x_1, x_2, \dots$ is obtained for which

$$d_S(x_{i+1}) - d_S(x_i) \leq -hc \quad (13)$$

holds for each i (as (12) holds in the case $i = 0$), and for which the choice of z does not lie in S . When $z \in S$, the process is terminated. Let us next observe that the termination must occur at some step i_N that cannot be larger than N . Indeed, at each step before termination, (13) holds, and with i applications yields that

$$\begin{aligned} 0 \leq d_S(x_i) &\leq d_S(x_{i-1}) - ch \\ &\leq d_S(x_0) - (icT)/N \\ &\leq (1 - i/N)r. \end{aligned} \quad (14)$$

Hence we must have $i_N \leq N$.

In the terminating step i_N in which $z \in S$, we have

$$z \in \left\{ \left[x_{i_N-1}, x_{i_N-1} + hF(x_{i_N-1}) \right] + \varepsilon \right\} \cap S,$$

and choose

$$v_{i_N-1} \in F(x_{i_N-1}) \quad \text{and} \quad h' \in [0, h] \quad (15)$$

so that $x_{i_N} := x_{i_N-1} + h'v_{i_N-1}$ satisfies $\|z - x_{i_N}\| < \varepsilon$. Consequently the last element x_{i_N} in the sequence satisfies

$$d_S(x_{i_N}) < \varepsilon \leq (h\delta)/2. \quad (16)$$

The entire sequence $x_1, x_2, \dots, x_{i_N}$ of course depends on N . Let h'_N be the h' value found in (15), and let $T_N := Ti_N - 1/N + h'_N$. Define an arc $x^N(\cdot)$ on $[0, T_n]$ as the linear interpolation of these points, which means precisely that

$$x^N(t) := \begin{cases} x_i + \left(t - (iT)/N \right) (x_{i+1} - x_i) (1/h) & \text{if } t \in [(i-1)T/N, (i)T/N] \\ & \text{and } 1 \leq i < i_N \\ x_{i_N-1} + \left(t - (i_N-1)T/N \right) (x_{i_N} - x_{i_N-1}) (1/h) & \text{if } t \in [(i_N-1)T/N, T_N] \end{cases}$$

We pass to a subsequence $\{N_j\}$ so that $T_{N_j} \rightarrow T'$ as $j \rightarrow \infty$ for some $T' \in (0, T]$, and so that $x^{N_j}(t) \rightarrow x(t)$ uniformly over $t \in [0, \min\{T_{N_j}, T'\}]$, where $x(\cdot)$ is a trajectory for F on $[0, T']$. The convergence of the arcs is assured by the compactness of trajectories theorem (see (Ref. 10), Theorem 3.1.7). From (16), we see that

$$d_S(x(T')) = \lim_{j \rightarrow \infty} d_S(x^{N_j}(T_{N_j})) = 0,$$

which says that $x(T') \in S$.

The radius of attainability has thus been shown to be r , which depends only on δ and ω_F . To verify the rate assertion in the theorem, let $0 \leq s < t \leq T'$. Let s_j and t_j be grid elements of the form kT/N_j , and so that $s_j \rightarrow s$ and $t_j \rightarrow t$ as $j \rightarrow \infty$. We have by (13) that

$$d_S(x_{N_j}(t_j)) - d_S(x_{N_j}(s_j)) \leq -(t_j - s_j)c. \quad (17)$$

Letting $j \rightarrow \infty$ in (17) yields that

$$d_S(x(t)) \leq d_S(x(s)) - (t - s)c,$$

which is the rate assertion in the theorem. $\square$

Remark 3.1. Suppose the pair (S, F) is given and $x \notin S$. The *minimum time* $T_S(x)$ that it takes to reach S from x is given by

$$T_S(x) := \min\{T : \text{there exists a trajectory } x(\cdot) \text{ for } F \text{ on } [0, T] \\ \text{so that } x(0) = x \text{ and } x(T) \in S\}$$

It is immediate from the definitions that if (S, F) has a radius of attainability of r and a rate of attainability of c , then

$$T_S(x) \leq d_S(x)/c \quad (18)$$

for all $x \in S + rB$. We record this fact in the next corollary as it pertains to the data in Theorem 3.1.

Corollary 3.1. *Suppose the hypotheses of Theorem 3.1 hold, and $x \in S + rB$. Then (18) holds.*

We now turn to sufficient conditions for strong attainability.

Theorem 3.2. *Suppose $S \subseteq \mathbb{R}^n$ is compact and $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is a continuous multifunction of modulus ω_F with compact convex values. Suppose there exists $\delta > 0$ so that for every $y \in S$, $\zeta \in N_S^P(y)$, and $v \in F(y)$ we have*

$$\langle \zeta, v \rangle \leq -\delta \|\zeta\|.$$

Then (S, F) is strongly attainable. Moreover, the radius and rate of attainability depend only on δ and ω_F .

Proof. We shall show that for $t \in [0, \frac{r}{c}]$, the inequality

$$d_S(x(t)) - d_S(x) \leq -ct \quad (19)$$

holds for every trajectory $x(\cdot)$ of F on $[0, r/c]$ that begins at $x \in S + rB$, where the constants c and r are chosen as in the proof of Theorem 3.1. This suffices to prove the theorem.

Let $t < r/c$, $x \in S + rB$, and $x(\cdot)$ a trajectory for F that begins at x . We apply a simple instance of the Mean Value Inequality, the data taken as $f := d_S$, $x_0 := x$, $Y := \{x(t)\}$, and $\varepsilon := t\delta/2$. We so obtain z and ζ satisfying

$$\begin{aligned}\zeta &\in \partial_P d_S(z) \\ z &\in [x, x(t)] + \varepsilon B \\ d_S(z) &\leq d_S(x_0) + hM + \varepsilon\end{aligned}$$

and such that

$$d_S(x(t)) - d_S(x) < \langle \zeta, x(t) - x \rangle + \varepsilon. \quad (20)$$

Let y be such that $\{y\} = \text{proj}_S(z)$, and let $v(\cdot)$ be chosen so that for almost all $s \in [0, t]$, we have $v(s) \in F(y)$ and

$$d_{F(y)}(\dot{x}(s)) = \|v(s) - \dot{x}(s)\|.$$

Then $v(\cdot)$ is measurable and bounded, and the convexity of $F(y)$ assures that

$$v := (1/t) \int_0^t v(s) ds$$

belongs to $F(y)$. Hence by assumption, we have

$$\langle v, \zeta \rangle \leq -\delta. \quad (21)$$

We also have that

$$\begin{aligned}\left\| v - \frac{x(t) - x}{t} \right\| &\leq \frac{1}{t} \int_0^t \|v(s) - \dot{x}(s)\| ds \\ &\leq \frac{1}{t} \int_0^t \omega_F(\|y - x(s)\|) ds \\ &\leq \omega_F(2r),\end{aligned} \quad (22)$$

where the last inequality is obtained as in (10). We now have that

$$\begin{aligned}d_S(x(t)) - d_S(x) &\leq \langle \zeta, x(t) - x \rangle + \varepsilon && \text{(by (20))} \\ &\leq t\langle \zeta, v \rangle + t\omega_F(2r) + t\frac{\delta}{2} && \text{(by (22), } \|\zeta\| = 1) \\ &\leq t(\omega_F(2r) - \frac{\delta}{2}) && \text{(by (21))} \\ &\leq -ct,\end{aligned}$$

which finishes the proof of the theorem. $\square$

The above sufficient conditions for attainability have contained a constant δ that must be independent of the particular $y \in \text{bd } S$. As we see below in Example 5.2, such uniformity in the choice of δ is required. However, in the next result, we give a condition which must only be verified pointwise, although it is now nonvacuous on each boundary point of S . To state the result, we need a few more concepts from nonsmooth analysis.

Let $y \in S$. The limiting normal cone $N_S^L(y)$ of S at y is defined as

$$N_S^L(y) := \left\{ \zeta : \text{there exist sequences } \{y_i\}, \{\zeta_i\} \text{ so that } y_i \rightarrow y, \zeta_i \rightarrow \zeta, \right. \\ \left. \text{and } \zeta_i \in N_S^P(y_i) \text{ for each } i \right\}.$$

The (Clarke, complete, or convexified) normal cone $N_L^C(y)$ is defined as the closed convex hull of $N_S^L(y)$. Finally, the tangent cone $T_S^C(y)$ of S at $y \in S$ is the polar of $N_L^C(y)$. That is,

$$T_S^C(y) := \left\{ v : \langle \zeta, v \rangle \leq 0 \text{ for every } \zeta \in N_L^C(y) \right\}.$$

See (Ref. 10) for additional properties and characterizations of T_S^C .

The next result gives sufficient conditions for attainability in terms of the tangent cone. The interior of the tangent cone is denoted by $\text{int } T_S^C(y)$.

Theorem 3.3. *Suppose for each $y \in S$, we have*

$$F(y) \cap \text{int } T_S^C(y) \neq \phi.$$

Then (S, F) is weakly attainable. If we have

$$F(y) \subseteq \text{int } T_S^C(y)$$

for each $y \in S$, then (S, F) is strongly attainable.

Proof. Fix $y \in \text{bd } S$. By assumption, we can choose $v_y \in F(y)$ and $\delta_y > 0$ so that

$$v_y + \delta_y B \subset T_S^C(y).$$

Thus for any $\zeta \in N_S^C(y)$, we have that

$$\begin{aligned} \min_{v \in F(y)} \langle \zeta, v \rangle &\leq \langle \zeta, v_y \rangle \\ &\leq \min_{b \in B} \left\{ -\langle \zeta, \delta_y b \rangle + \langle \zeta, v_y + \delta_y b \rangle \right\} \\ &\leq -\delta_y \|\zeta\|. \end{aligned} \tag{23}$$

Let $\rho_y > 0$ be chosen so that $\omega_F(\rho_y) < \delta_y/2$. Notice that (23) and the continuity assumption on F implies that for any $z \in y + \rho_y B$, we have

$$\begin{aligned} \min_{w \in F(z)} \langle \zeta, w \rangle &\leq \min_{\substack{w \in F(z) \\ v \in F(y)}} \left\{ \langle \zeta, w - v \rangle + \langle \zeta, v \rangle \right\} \\ &\leq \left(\omega_F(\|z - y\|) - \delta_y \right) \|\zeta\| \\ &\leq -(\delta_y/2) \|\zeta\| \end{aligned} \tag{24}$$

Now suppose $\{z_i\}$ is any sequence with $z_i \rightarrow y$ and for which there exists $0 \neq \zeta_i \in N_S^P(z_i)$. Passing to a subsequence if necessary, we assume there exists ζ so that $\zeta_i/\|\zeta_i\| \rightarrow \zeta \in N_S^C(y)$. By (23) and (24), we have

$$\min_{w \in F(z_i)} \langle \zeta_i, w \rangle \leq -(\delta_y/4) \|\zeta_i\| \tag{25}$$

for all i sufficiently large. Now (25) holds for arbitrary sequences $\{z_i\}$ and $\{\zeta_i\}$, and thus by shrinking ρ_y if necessary, we may conclude that

$$\min_{w \in F(z)} \langle \zeta, w \rangle \leq -(\delta_y/4)\|\zeta\| \quad (26)$$

for all $z \in y + \rho_y B$ and $\zeta \in N_S^P(z)$. We next observe that $\{y + \rho_y B\}_{y \in \text{bd } S}$ is an open cover of $\text{bd } S$. A standard compactness argument yields that the hypotheses of Theorem 3.1 hold, the conclusion of which says that (S, F) is weakly attainable.

The proof of the strong attainable assertion is similar. $\square$

The next attainability result states that the continuity assumption on F can be weakened to upper semicontinuity provided a δ condition holds in a fixed neighborhood of S .

Theorem 3.4. *Suppose $S \subseteq \mathbb{R}^n$ is compact and $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is a multifunction with compact convex values and is upper semicontinuous on $S + rB$, where $r > 0$. Suppose there exists $\delta > 0$ so that whenever $y \notin S$ with $d_S(y) < r$, and $\zeta \in y - \text{proj}_S(y)$, there exists $v \in F(y)$ for which*

$$\langle \zeta, v \rangle \leq -\delta\|\zeta\|.$$

Then (S, F) is weakly attainable of radius r and with rate δ . If this inequality holds for all $v \in F(y)$, then (S, F) is strongly attainable of radius r and rate δ .

Proof. Let us first assume that F is Lipschitz continuous of rank λ_F , and we refer back to the proof of Theorem 3.1. The proximal projection algorithm can in fact be made simpler here because the δ assumption can be applied directly to the point z obtained from the mean value inequality. One must however initially take potentially smaller values of h and ε . We introduce another small parameter η with $0 < \eta < \delta$, and choose $h > 0$ and $\varepsilon > 0$ small enough to satisfy

$$2\lambda_F(hM + \varepsilon) < \eta/2, \quad \varepsilon < (h\eta)/2, \quad \text{and} \quad [x_0, x_0 + hF(x_0)] + \varepsilon B \subset S + rB.$$

Suppose z and ζ are obtained to satisfy (2)–(5). Under the present hypothesis, which is active in the neighborhood $S + rB$ containing z , we are assured the existence of $v \in F(z)$ so that

$$\langle \zeta, v \rangle \leq -\delta. \quad (27)$$

Let $w_0 \in F(x_0)$ be so that $\|w_0 - v\| \leq \lambda_F\|x_0 - z\| \leq \lambda_F(hM + \varepsilon)$. Now by (5), (27), and the choices of h and ε , we have

$$\begin{aligned} \min_{v' \in F(x_0)} d_S(x_0 + hv') - d_S(x_0) &\leq h\langle \zeta, w_0 \rangle + \varepsilon \\ &= h\langle \zeta, w_0 - v \rangle + h\langle \zeta, v \rangle + \varepsilon \\ &\leq -h\delta + h\lambda_F(hM + \varepsilon) + \varepsilon \\ &\leq -h(\delta - \eta). \end{aligned}$$

Hence there exists $v_0 \in F(x_0)$ so that

$$d_S(x_0 + hv_0) - d_S(x_0) \leq -h(\delta - \eta), \quad (28)$$

and the proof can now proceed as before. We note that (28) does not depend on the Lipschitz constant, provided that h is chosen sufficiently small. After passing to the limit as $h \downarrow 0$, therefore,

we conclude that (S, F) has an absorption rate $\delta - \eta$. By letting $\eta \downarrow 0$, we deduce that (S, F) has an absorption rate δ .

In the case of a general upper semicontinuous multifunction F , we apply a technique to approximate F by Lipschitz multifunctions. See Deimling (Ref. 11). There exists a sequence of Lipschitz multifunctions F_j that have compact convex values and satisfy

$$F(x) \subseteq F_{j+1}(x) \subseteq F_j(x) \subseteq F(\{x + (1/j)B\} \cap \{S + rB\})$$

for $j = 1, 2, \dots$. It is clear that the δ hypothesis assumed by F is inherited by each F_j . The above procedure thus provides a trajectory $x_j(\cdot)$ of F_j approaching S at a rate δ . Letting $j \rightarrow \infty$, we obtain an arc $x(\cdot)$ as the uniform limit of some subsequence of $x_j(\cdot)$. We have that $x(\cdot)$ is also a trajectory of F , and is being absorbed into S at the rate δ .

Similar considerations are operational for strong attainability. $\square$

Remark 3.2. It is interesting to consider the δ condition as hypothesized in Theorem 3.1, but in what could be considered the limiting case of $\delta = 0$. It has recently been shown in (Ref. 12) that such a condition (and with the multifunction F upper semicontinuous) is equivalent to the notion of weak invariance. Recall that a pair (S, F) is weakly invariant if for every $x \in S$, there exists a trajectory for F emanating from x of arbitrarily long duration and which remains in S . The following corollary illustrates that this invariance criterion in (Ref. 12) can also be deduced from the attainability result Theorem 3.4. See also Haddad (Ref. 13) and Aubin (Ref. 14) for further developments regarding invariance.

Corollary 3.2. *Suppose $S \subset \mathbb{R}^n$ is compact, and $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is a multifunction with compact convex values and is upper semicontinuous on S . Furthermore, suppose that*

$$\inf_{v \in F(x)} \langle \xi, v \rangle \leq 0 \quad (29)$$

for all $\xi \in N_S^P(x)$. Then (S, F) is weakly invariant.

Proof. Let $T > 0$ and suppose $x \in S$. We must show there exists a trajectory $x(\cdot)$ for F on $[0, T]$ satisfying $x(t) \in S$ for all $t \in [0, T]$. Let $r > 0$, and extend F to $S + rB$ by letting

$$F(y) = \overline{\text{co}} \bigcup \left\{ v : v \in F(z), z \in \text{proj}_S(y) \right\}.$$

It can easily be checked that F has compact convex values and is upper semicontinuous on $S + rB$. Let $\delta > 0$, and define the multifunction F_δ on $S + rB$ by setting $F_\delta(y) := F(y) + \delta \text{cl } B$ for $y \in S + rB$. It follows readily that (S, F_δ) satisfies the hypotheses of Theorem 3.4.

Now choose M as in the proof of Theorem 3.1, and let

$$0 < h < (\min\{T, r\})/(M + \delta).$$

Let $x_1(\cdot)$ be any solution of F_δ defined on $[0, h]$, and note that $d_S(x_1(h)) \leq h(M + \delta) < r$. By Theorem 3.4, the arc $x_1(\cdot)$ can be extended to be defined on an interval $[0, \tau_1]$ with $x(\tau_1) \in S$, and (by Remark 3.2) to satisfy $\tau_1 \leq h(1 + (M + \delta)/\delta)$. We proceed in a similar manner with x replaced by $x_1(\tau_1)$. In this way, a trajectory $x(\cdot)$ of F_δ is constructed on $[0, T]$ with the property that $x(\tau_j) \in S$ for a collection of τ_j . Furthermore, the $\{\tau_j\}$ are dispersed throughout the interval $[0, T]$ with consecutive elements being no more than $h(1 + (M + \delta)/\delta)$ apart. Taking a limit as $h \downarrow 0$ produces a trajectory of F_δ that stays in S , and taking a limit as $\delta \downarrow 0$ produces a trajectory of F that stays in S . $\square$

4 Penetrative Systems

In the previous section, we explored conditions that were used to drive solutions toward a target set. We now consider conditions for which solutions can be driven into the interior of a set. As examples illustrate (see Example 5.3 below), the penetrative set must be somewhat special besides merely having nonempty interior.

Definition 4.1. A set $S \subseteq \mathbb{R}^n$ is said to be *epi-continuous* near $x \in \text{bd } S$ if there exists a unitary linear transformation $A : \mathbb{R}^n \rightarrow \mathbb{R}^n$, a neighborhood $U \subset \mathbb{R}^n$ of x , and a continuous function $g : \mathbb{R}^{n-1} \rightarrow \mathbb{R}^1$ so that $Ax = (x^0, g(x^0))$ for some $x^0 \in \mathbb{R}^{n-1}$, and

$$U \cap S = U \cap A^{-1}(\text{epi } g).$$

If g can in addition be chosen Lipschitz, then S is said to be *epi-Lipschitz* near x .

Requiring that A be unitary rather than merely invertible in the above definition is not critical, but has the advantage of preserving the geometry. In particular, one can easily verify that if A is unitary and $\zeta \in N_S^P(y)$, then $A\zeta \in N_{AS}^P(Ay)$. The main result of this section is the following.

Theorem 4.1. *Suppose S is epi-continuous near $x \in \text{bd } S$, and (S, F) satisfies the hypotheses of Theorem 3.1. Then (S, F) is weakly penetrative near x of a size depending on δ , ω_F , U , and the modulus of continuity of the function g , where U and g are as in Definition 4.1. If S is in addition epi-Lipschitz, then (S, F) has a rate of penetration depending on δ , ω_F , U , and the Lipschitz rank of g .*

Proof. Let $x \in \text{bd } S$, and let A and U be given as in Definition 4.1. The constants r , c , and M are chosen as in the proof of Theorem 3.1 with the additional stipulation that $x + r \text{cl } B \subset U$. Elements in the range of A are represented as $Ay = (y^0, y^1) \in \mathbb{R}^{n-1} \times \mathbb{R}^1$, where $y \in U$. Without loss of generality, we assume further that $AU = \{x^0 + r^0 B\} \times \{g(x^0) + r^1 B\}$, where the constants $r^0, r^1 > 0$ satisfy $g(x^0 + r^0 B) \subset g(x^0) + r^1/2 B$. For $\alpha > 0$, we define

$$S_\alpha := A^{-1} \left\{ (y^0, y^1) \in \{x^0 + r^0 B\} \times \{g(x^0) + r^1 B\} : y^1 \geq g(y^0) + \alpha \right\}$$

Note that $S_0 \cap U = S \cap U$. Also note that if $y \in U$ so that $Ay = (y^0, g(y^0))$ and $\zeta \in N_S^P(y)$, then $\zeta \in N_S^P \left(A^{-1}(y^0, g(y^0) + \alpha) \right)$ for all $\alpha < r^1/2$.

Next, let $\bar{\alpha} > 0$ be chosen so that $S_{\bar{\alpha}} \cap U \subset x + rB$, and so that the following holds:

$$\min_{v \in F \left(A^{-1}(y^0, y^1 + \alpha) \right)} \langle v, \zeta \rangle \leq -\delta \|\zeta\|/2 \quad (30)$$

whenever $y = (y^0, y^1) \in (\text{bd } S) \cap U$ with $y^0 \in x^0 + r^0 B$, $\zeta \in N_S^P(y)$, and $\alpha \in [0, \bar{\alpha}]$. Such a choice of $\bar{\alpha}$ is possible in view of the δ assumption and the continuity of F . We shall want trajectories of F that originate near x to stay in $x + rB$, and so the requirement $\bar{\alpha} < rc/M$ is also imposed.

For $\alpha \in [0, \bar{\alpha}]$, the distance between S and S_α is denoted by $\Delta(\alpha)$, which means that

$$\Delta(\alpha) := \inf \left\{ \|y - z\| : y, z \in U, \text{ and } y \in S, z \in S_\alpha \right\}.$$

It is clear that $\Delta(\cdot)$ is continuous, strictly increasing, and $\Delta(0) = 0$. Obviously one also has $\alpha \geq \Delta(\alpha)$. A lower bound of $\Delta(\alpha)$ can be obtained as follows. Let $\gamma > 0$ be chosen so that

$\omega_g(\gamma) < \alpha/2$, where ω_g is the modulus of continuity of g . Then we have that

$$\begin{aligned}\Delta(\alpha) &\geq \inf \left\{ \left\{ \|y^0 - z^0\|^2 + |g(y^0) - g(z^0) - \alpha|^2 \right\}^{1/2} : y^0, z^0 \in x^0 + r^0 B \right\} \\ &\geq \begin{cases} \alpha/2 & \text{if } \|y^0 - z^0\| < \gamma \\ \gamma & \text{if } \|y^0 - z^0\| \geq \gamma. \end{cases}\end{aligned}$$

By the nature of the sets S_α , it can easily be seen for any $0 \leq \alpha < \beta \leq \bar{\alpha}$, that whenever $y \in (\overline{\text{comp}} S_\alpha) \cap U$, $z \in S_\beta$, we have

$$\|y - z\| \geq \Delta(\beta - \alpha) \quad (31)$$

Note also that if g is Lipschitz of rank $\lambda_g \geq 1$, then ω_g can be chosen so that $\omega_g(\rho) = \lambda_g \rho$, and consequently the estimate

$$\Delta(\alpha) \geq \alpha/(2\lambda_g) \quad (32)$$

holds in this case.

We now proceed to construct a trajectory that will penetrate the interior of S . Let N be a positive integer, and $\alpha_j := j\bar{\alpha}/N$, for $j = 1, 2, \dots, N$. To shorten the notation, we write S_j for S_{α_j} . We shall successively apply (the local version of) Theorem 3.1 to the sets S_j . The hypotheses of that theorem are satisfied in view of (30). There exists a trajectory $x_1(\cdot)$ for F on an interval $[0, \tau_1]$ with $x_1(0) = x$ and $x_1(\tau_1) \in S_1$, and so that (by Corollary 3.1) $\tau_1 := T_{S_1}(x) \leq d_{S_1}(x)/c \leq \Delta(\alpha_1)/c \leq \bar{\alpha}/(cN)$. Inductively, there exists a trajectory $x_{j+1}(\cdot)$ of F on an interval $[\tau_j, \tau_{j+1}]$ so that $x_{j+1}(\tau_j) = x_j(\tau_j)$ and $x_{j+1}(\tau_{j+1}) \in S_{j+1}$, and with $\tau_{j+1} := \tau_j + T_{S_{j+1}}(x_j(\tau_j)) \leq (j\bar{\alpha})/(cN)$.

Let $T_N := \tau_N \leq \bar{\alpha}/c$, and define $x^N(\cdot)$ on $[0, T_N]$ by setting $x^N(t) = x_j(t)$ whenever $t \in [\tau_{j-1}, \tau_j]$ ($\tau_0 := 0$). Then $x^N(\cdot)$ is a trajectory of F which stays in $S + rB$ (cf. (Ref. 15), Lemma 5.1).

Now let $N \rightarrow \infty$. We obtain another trajectory $x(\cdot)$ of F defined on an interval $[0, T]$ so that $x(\cdot)$ is the uniform limit of a subsequence of the arcs $x^N(\cdot)$, and T is the limit of T_N . We show that $x(\cdot)$ penetrates the interior of S .

Define $\alpha : [0, T] \rightarrow [0, \bar{\alpha}]$ by

$$\alpha(t) := \sup \left\{ \alpha \in [0, \bar{\alpha}] : x(t) \in S_\alpha \right\}.$$

We claim that

$$\alpha(t) - \alpha(s) \geq c(t - s) \quad (33)$$

whenever $0 \leq s < t \leq T$. Indeed, for fixed N , if τ_i and τ_j ($i < j$) are grid points associated with the construction of $x^N(\cdot)$, then we have that

$$\tau_j - \tau_i \leq (j - i)\bar{\alpha}/(cN). \quad (34)$$

There exists a subsequence $\{N_k\}$ so that

$$x^{N_k}(\cdot) \rightarrow x(\cdot) \quad \text{uniformly on } [0, \min\{T, T^{N_k}\}],$$

and there exist grid points $\{\tau_{i_k}\}, \{\tau_{j_k}\}$ associated with N_k such that

$$\tau_{i_k} \rightarrow s \quad \text{and} \quad \tau_{j_k} \rightarrow t \quad (35)$$

as $k \rightarrow \infty$. We have $x^{N_k}(\tau_{i_k}) \rightarrow x(s)$ and $x^{N_k}(\tau_{j_k}) \rightarrow x(t)$, and by the nature of the construction of $x^{N_k}(\cdot)$, we conclude that

$$\alpha(s) = \lim_{k \rightarrow \infty} i_k \bar{\alpha} / N_k \quad \text{and} \quad \alpha(t) = \lim_{k \rightarrow \infty} j_k \bar{\alpha} / N_k,$$

where in particular the limits exist. Therefore by (34), we have

$$\alpha(t) - \alpha(s) = \lim_{k \rightarrow \infty} (j_k - i_k) \bar{\alpha} / (N_k) \geq c(\tau_{j_k} - \tau_{i_k}) = c(t - s),$$

which is the claim (33).

We now define $\rho : [0, T] \rightarrow [0, \bar{\alpha}]$ by setting $\rho(t) := \Delta(\alpha(t))$. We have that $\rho(0) = 0$, and it is immediate from the claim (33) and the definition of Δ that ρ is strictly increasing. Furthermore, in the epi-Lipschitz case, from (32) and (33), it follows that

$$\rho(t) \geq \frac{c}{2\lambda_g} t.$$

Thus to finish the proof of the theorem, it is only necessary to show that

$$x(t) + \rho(t) \text{ cl } B \subset S. \tag{36}$$

But (36) is an immediate consequence of the definitions. Indeed, if $\|y - x(t)\| \leq \rho(t)$, then $y \in S_0 = S$ by (31). $\square$

Remark 4.1. (a) We do not go into the details here, but one can formulate and prove similar weak penetration results akin to Theorems 3.3 and 3.4. Furthermore, strong penetration results can likewise be derived.

(b) As pointed out by L. D. Berkovitz, the convexity of the values of the multifunction F can be dropped in Theorem 4.1 when F is Lipschitz, since then original trajectories are dense in relaxed ones (see Ref. 10, Chapter 3).

(c) We also mention that Rockafellar (Ref. 16) has characterized epi-Lipschitz sets in $\mathbb{R}^n$ as those sets S that satisfy $\text{int } T_S^C(x) \neq \emptyset$ for $x \in \text{bd } S$. Hence the following is an immediate corollary to this result and Theorem 4.1.

Corollary 4.1. *Suppose (S, F) satisfies the hypotheses of Theorem 3.1 and*

$$\text{int } T_S^C(x) \neq \emptyset$$

for each $x \in \text{bd } S$. Then (S, F) is weakly penetrative.

5 Examples

In this section, we provide some examples to justify the hypotheses in the above results.

Example 5.1. In this first example, we give a simple instance in which F is upper semicontinuous and the δ condition in Theorem 3.1 is satisfied, but for which weak attainability fails. Thus continuity is required in the theorem.

We consider $n = 1$, $S := [-1, 1]$, and $F : \mathbb{R}^1 \rightrightarrows \mathbb{R}^1$ given by:

$$F(x) = \begin{cases} \{-1 - x\} & \text{if } x < -1 \\ [x, 0] & \text{if } -1 \leq x \leq 0 \\ [0, x] & \text{if } 0 \leq x \leq 1 \\ \{1 - x\} & \text{if } 1 < x. \end{cases}$$

Then the trajectories of F starting outside S are $x(t) = 1 + (x - 1)e^{-t}$ if $x > 1$, and $x(t) = -1 + (x + 1)e^{-t}$ if $x < -1$, which do not reach S in finite time.

Example 5.2. This example shows that a uniform δ in Theorem 3.1 is necessary. Note also that while the δ hypothesis of Theorem 3.3 is not satisfied, it fails at only a single point.

Consider $n = 2$ and $S := \text{cl } B \cap \{(x, y) : y \geq -|x|\}$. We are only interested in the multifunction $F : \mathbb{R}^2 \rightrightarrows \mathbb{R}^2$ near the origin defined by $F(x, y) := \{(0, -|y|)\}$. Any trajectory of F beginning at $(0, y)$ with $y < 0$ fails to reach S in finite time, although for each boundary point with a proximal normal, there exists a $\delta > 0$ (depending on the point) that satisfies the δ condition in Theorem 3.1

Example 5.3. This example shows that if (S, F) satisfies all the hypotheses of Theorem 4.1, but S fails to be epi-continuous, then (S, F) may fail to be penetrative. Note that this S coincides with the closure of its interior, a hypothesis one may at first think could replace epi-continuity.

Let $n = 2$,

$$S := \text{cl } B \cap \{(x, y) : x \leq |y|^2\} \cap \{(x, y) : x \geq -|y|^2\},$$

and F be given by $F(x, y) = [-1, 1] \times \{0\}$. Again we are only interested in the behavior of the system near the origin, in which it is clear that no trajectory beginning at a point $(x, 0)$ can enter the interior of S . All of the hypotheses of Theorem 4.1 are satisfied except for epi-continuity.

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