A Proximal Characterization of the Reachable Set

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Abstract

We show that the graph of the reachable set of a control system given by a differential inclusion is uniquely characterized by a Hamilton-Jacobi equation involving proximal normals.

Résumé

On démontre que le graphe de l'ensemble des points accessibles d'un système de contrôle décrit par une inclusion différentielle est caractérisé par une équation Hamilton-Jacobi impliquant les normales proximales.

We study a control system defined via a differential inclusion

$$\dot{x}(t) \in F\big(t, x(t)\big) \text{ a. e.} \tag{1}$$

As usual, a *trajectory* of (1) refers to an absolutely continuous function $x(\cdot)$ satisfying (1) on a given interval $[a, b]$. The equivalence of (1) to a classical control system $\dot{x} = f(t, x, u)$, $u \in U$ is well-understood; we shall not dwell upon it. For a given choice of initial time t_0 and nonempty compact subset A of $\mathbb{R}^n$, we consider the set $\mathcal{R}$ defined as follows:

$$\mathcal{R} = \Big\{ \big(\sqcup, \S(\sqcup)\big) \colon \sqcup \geq \sqcup_{\prime}, \S(\cdot) \text{ is a trajectory on } [\sqcup_{\prime}, \sqcup], \S(\sqcup_{\prime}) \in \mathcal{A} \Big\}.$$

The assumptions on the multifunction F are as follows:

(H1) For each $(t, x) \in [t_0, \infty) \times \mathbb{R}^n$, the set $F(t, x)$ is a nonempty, convex, compact subset of $\mathbb{R}^n$.

(H2) For some constants γ and c, and for all (t, x) in $[t_0, \infty) \times \mathbb{R}^n$, one has

$$v \in F(t, x) \Rightarrow |v| \leq \gamma |x| + c.$$

(H3) F is locally Lipschitz on $[t_0, \infty) \times \mathbb{R}^n$; i.e., for any bounded subset S of $[t_0, \infty) \times \mathbb{R}^n$ there is a constant K such that, for all $(t_i, x_i) \in S$ $(i = 1, 2)$, we have

$$F(t_2, x_2) \subseteq F(t_1, x_1) + K\big|(t_2 - t_1, x_2 - x_1)\big|\overline{B},$$

where B denotes the closed unit ball in $\mathbb{R}^n$.

It is a well-known fact that under these hypotheses the set $\mathcal{R}$ is closed, and that its "slice" at time T, the *reachable set* $\mathcal{R}_{\mathcal{T}} := \big\{\S \colon (\mathcal{T}, \S) \in \mathcal{R}\big\}$ is compact and nonempty for each $T \geq t_0$.

A *proximal normal* [3] ζ to a closed set S at a point $x \in S$ is a vector ζ such that, for some $\sigma \geq 0$, one has

$$\langle \zeta, x' - x \rangle \leq \sigma |x' - x|^2 \quad \forall x' \in S.$$

The set of proximal normals to S at x is a cone; we denote it $\partial_P S(x)$. Note that $0 \in \partial_P S(x) \quad \forall x \in S$, and that $\partial_P S(x)$ is undefined when $x \notin S$. It is known that $\partial_P S(x)$ reduces to the set of normals in the usual sense when S is a smooth manifold (with or without boundary), or when S is a convex set. The set of points x for which $\partial_P S(x)$ is nontrivial (i.e., $\neq \{0\}$) can be "small", but is always dense in the boundary of S.

The (upper) Hamiltonian $H \colon \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ is the function defined by

$$H(t, x, p) := \max\big\{\langle p, v \rangle \colon v \in F(t, x)\big\}.$$

Theorem 1. *$\mathcal{R}$ is the unique closed subset S of $[t_0, \infty) \times \mathbb{R}^n$ satisfying:*

(i) $\theta + H(t, x, \zeta) = 0 \quad \forall (\theta, \zeta) \in \partial_P S(t, x), \forall (t, x) \in (t_0, \infty) \times \mathbb{R}^n$,

(ii) $\lim_{T \downarrow 0} S_T = A$.

Remark 1. (a) Since $\partial_P S(t, x)$ is only defined when (t, x) lies in S, the "proximal Hamilton-Jacobi equation" in (i) is in issue only at such points. Since $H(t, x, 0) = 0$, it holds automatically at any point $(t, x) \in S$ for which $\partial_P S(t, x)$ is trivial.

(b) The initial condition (ii) is to be understood in the Hausdorff metric ρ; that is, for any $\varepsilon > 0$ there exists $\delta > 0$ such that

$$T \in [t_0, t_0 + \delta) \Rightarrow \rho(S_T, A) < \varepsilon.$$

It follows in particular that $S_{t_0} = A$. It is equivalent to (ii) to require this last equality together with the uniform boundedness of S_T for T near t_0.

Proof of the Theorem. Let us verify first that $\mathcal{R}$ satisfies (i). Given any point (τ, α) in $\mathcal{R}$, let x be any trajectory on $[\tau, \infty)$ with $x(\tau) = \alpha$. We claim that $\big(t, x(t)\big) \in \mathcal{R}$ for all $t > \tau$. If $\tau = t_0$, then $\alpha \in A$ necessarily, and so $\big(t, x(t)\big) \in \mathcal{R}$ by the very definition of $\mathcal{R}$. If $\tau > t_0$, there is a trajectory y on $[t_0, \tau]$ with $y(t_0) \in A$, $y(\tau) = \alpha$. But then "y followed by x" is a trajectory on $[t_0, \infty)$ beginning in A, whence $\big(t, x(t)\big) \in \mathcal{R} \quad \forall \sqcup > \tau$ as claimed.

This argument implies that the set $\mathcal{R}$ is strongly invariant [1] [5] relative to the trajectories of $\widetilde{F}$, where $\widetilde{F}$ is the familiar augmented multifunction [5] [6]

$$\widetilde{F}(t,x) := \big\{(1,v) \colon v \in F(t,x)\big\}.$$

This property is characterized [5] by the condition

$$\theta + H(t,x,\zeta) \leq 0 \quad \forall(\theta,\zeta) \in \partial_P \mathcal{R}(\sqcup, \S), \quad \forall(\sqcup, \S) \in \mathcal{R}. \tag{2}$$

We now proceed to observe another invariance property of $\mathcal{R}$. Let $(\tau,\alpha) \in \mathcal{R}$, $\tau > t_0$. Then there is at least one trajectory $x(\cdot)$ on $[t_0,\tau]$ such that $x(\tau) = \alpha$, $x(t_0) \in A$. Evidently, we have $\big(t,x(t)\big) \in \mathcal{R}$ for all $t \in [t_0,\tau]$. Thus $\mathcal{R}$ is weakly preinvariant for $\widetilde{F}$ [5] for $\tau > t_0$, which is a property characterized by the condition

$$\theta + H(t,x,\zeta) \geq 0 \quad \forall(\theta,\zeta) \in \partial_P \mathcal{R}(\sqcup, \S), \quad \forall(\sqcup, \S) \in \mathcal{R}, \quad \sqcup > \sqcup_\prime. \tag{3}$$

Combining (2) and (3) gives (i) of the theorem, for $S = \mathcal{R}$. That $\mathcal{R}$ satisfies property (ii) is an easy consequence of the fact that F is bounded on compact sets; we omit the details.

Now let S be another closed subset of $[t_0,\infty) \times \mathbb{R}^n$ satisfying (i) (ii). We first establish that $S \supseteq \mathcal{R}$. Let $(\tau,\alpha) \in \mathcal{R}$ for $\tau > t_0$. Then there is a trajectory $x(\cdot)$ on $[t_0,\tau]$ with $x(t_0) =: a \in A$, $x(\tau) = \alpha$. For any $\varepsilon > 0$, we can find (by (ii)) $T \in [t_0, t_0 + \varepsilon]$ and $a' \in S_T$ such that $|a' - a| < \varepsilon$. By continuous dependence of attainable sets, we can also suppose that there is a trajectory x' for F on $[T,\tau]$ with $x'(T) = a'$ and $\big|x'(\tau) - \alpha\big| < \varepsilon$. But S is known to satisfy (2) (with $\mathcal{R}$ replaced by S, for $t > t_0$), which characterizes strong invariance relative to $\widetilde{F}$. Then, since $(T,a') \in S$, we have $\big(\tau, x'(\tau)\big) \in S$ as well. Since ε is arbitrary, it follows that $(\tau,\alpha) \in S$. This confirms the inclusion $S \supseteq \mathcal{R}$.

Now let (τ,α) lie in S, $\tau > t_0$. Since S satisfies (3) (with $\mathcal{R}$ replaced by S), S is weakly preinvariant for $\widetilde{F}$ and for $t > t_0$. So for any $\varepsilon > 0$, there is a trajectory $x_\varepsilon(\cdot)$ for F on $[t_0 + \varepsilon, \tau]$ such that $x_\varepsilon(\tau) = \alpha$, and such that $\big(t, x_\varepsilon(t)\big)$ lies in S for all $t \in [t_0 + \varepsilon, \tau]$. In particular, $x_\varepsilon(t_0 + \varepsilon)$ lies in $S_{t_0+\varepsilon}$, which converges to A as $\varepsilon \downarrow 0$. It follows from sequential compactness of trajectories [2, Theorem 3.1.7] that along some sequence of ε_i decreasing to 0, there is convergence of the corresponding x_{ε_i} to a trajectory $x(\cdot)$ for F on $[t_0,\tau]$ such that $x(t_0) \in A$ and $x(\tau) = \alpha$. Then $(\tau,\alpha) \in \mathcal{R}$. Thus $S \subseteq \mathcal{R}$. $\square$

Remark 2. If the Lipschitz condition (H3) is weakened to mere upper semicontinuity, it still follows that $\mathcal{R}$ is the maximal set satisfying (3) together with (ii). Thus $\mathcal{R}$ is a maximal "semisolution". This "comparison theorem" is another respect in which strong analogies exist with the various theories of generalized solutions of the Hamilton-Jacobi equation (proximal, minimax, viscosity: see [4] [5]). The evident analogy has a value function in the role of the set $\mathcal{R}$.

Remark 3. A connection between controllability and certain smooth approximations of the Hamilton-Jacobi equation has been obtained by Vinter [9]. The set $\mathcal{R}$ is of interest in connection with propogating fronts; we refer the reader to the interesting articles of Soravia [7] and Subbotin [8] for a functional approach to this issue.

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