VaR and Low Interest Rates

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By Louis Doray (U de M)
Frédéric Edoukou (U de M)
Rim Labdi (HEC Montréal)
Zichun Ye (UBC)

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Plan of the talk

1. Introduction + Financial Mathematics 101
2. The data
3. The CDPQ model
4. Scenario generation: the CEV model
   - Estimation and testing
   - Weekly VaR
5. Annualization
6. Further work

Introduction

- Ideal team composed of
  - Financial engineer (or actuary)
  - Probabilist
  - Computer expert
  - Statistician

Financial Mathematics 101

- Let Y_t= value of asset at time t;
- Return $r_t = (Y_{t+1}-Y_t)/Y_t$; Log-return $\ln r_t = \ln(Y_{t+1}-Y_t)$;
- If r_t is small, $\ln(1+r_t) \approx r_t$ ($r_t > 0$).
- Assuming $\ln r_t \sim N(0, \sigma^2)$, this implies $r_t \sim$ Lognormal $LN(0, \sigma^2)$ and ensures r_t is never negative.
- The log-returns over a period (week) is the sum of the log-returns over sub-periods (days).

$$Z_t = \ln \frac{Y_{t+5}}{Y_t} = \ln\left(\frac{Y_{t+5}}{Y_{t+4}} \times \frac{Y_{t+4}}{Y_{t+3}} \times \frac{Y_{t+3}}{Y_{t+2}} \times \frac{Y_{t+2}}{Y_{t+1}} \times \frac{Y_{t+1}}{Y_t}\right)$$

$$= \ln(r_{t+4} \times r_{t+3} \times r_{t+2} \times r_{t+1} \times r_t)$$

$$= \sum_{k=0}^{4} \ln r_{t+k}$$

Financial Mathematics 101

- Assuming $\ln r_t$ are i.i.d. $\sim N(0, \sigma^2)$;
- This implies $Z_t = \ln Y_{t+5}/Y_t \sim N(0, 5\sigma^2)$.
- Under this model, it can be shown that correlation between (Monday to Monday) and (Tuesday to Tuesday) log-return is 0.8, because of the overlapping intervals.

$$\rho_{Z_{t+1},Z_t} = 0.8$$

$$\rho_{Z_{t+2},Z_t} = 0.6$$

$$\rho_{Z_{t+3},Z_t} = 0.4$$

$$\rho_{Z_{t+4},Z_t} = 0.2$$

$$\rho_{Z_{t+k},Z_t} = 0, \text{ for } k \geq 5.$$

Data: Interest Rate 2000-2016

- Start : January 3 2000
- End : April 29 2016
- Maturity : 20
- Observations by maturity : 4260

nb. of days	Maturity
1	Day (d)
7	Week (w)
30	Month (m)
90	Quarter(q)
…	…
10955	30 years (30y)

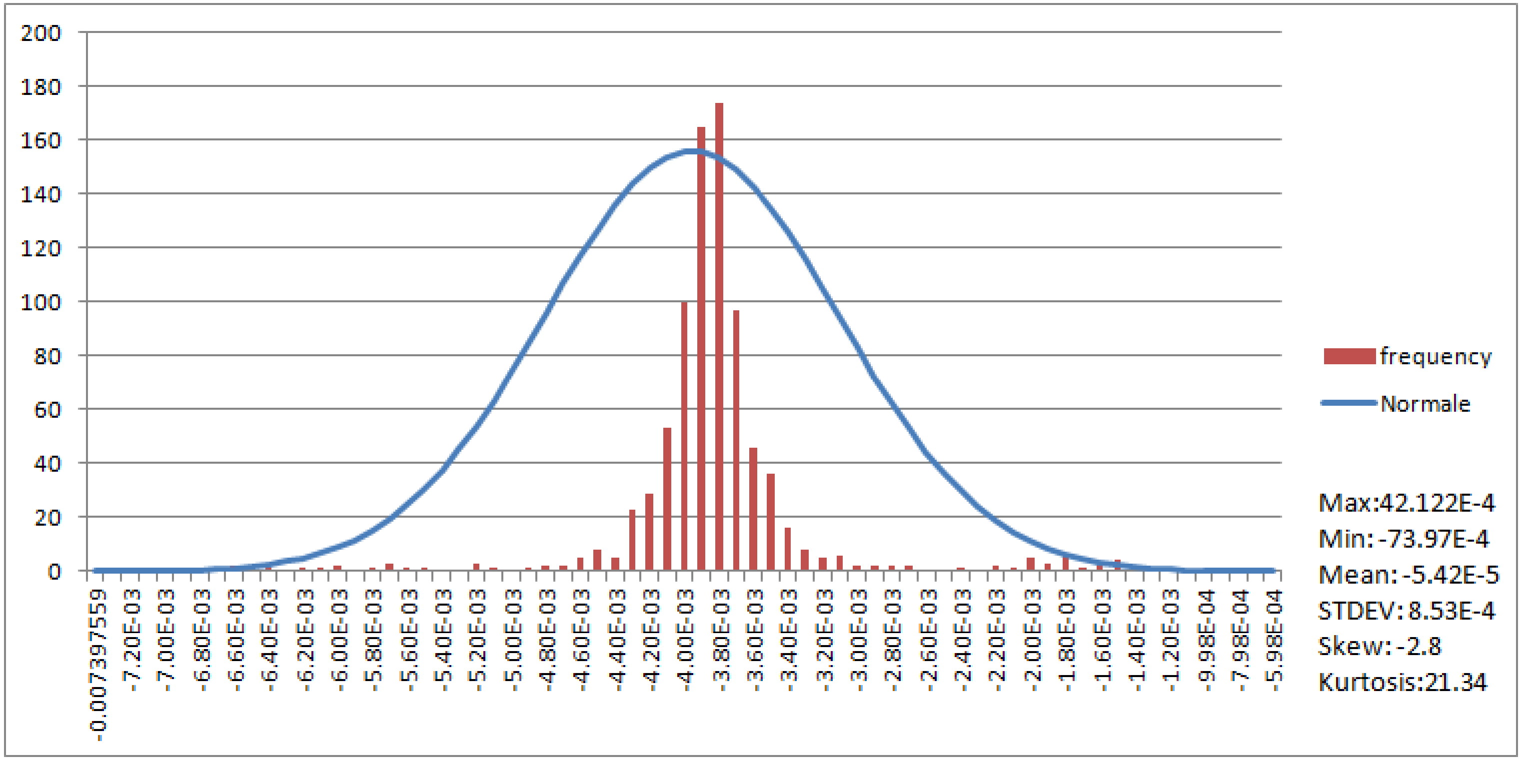
Data

- A summary of numbers of iid maturity data.

30y 25y 20y 15y 12y 10y 9y	8y 7y 6y	5y	4y	3y	2y	1y	s	q	m	w	d
1	2	3	4	5	8	16	32	49	196	852	4260

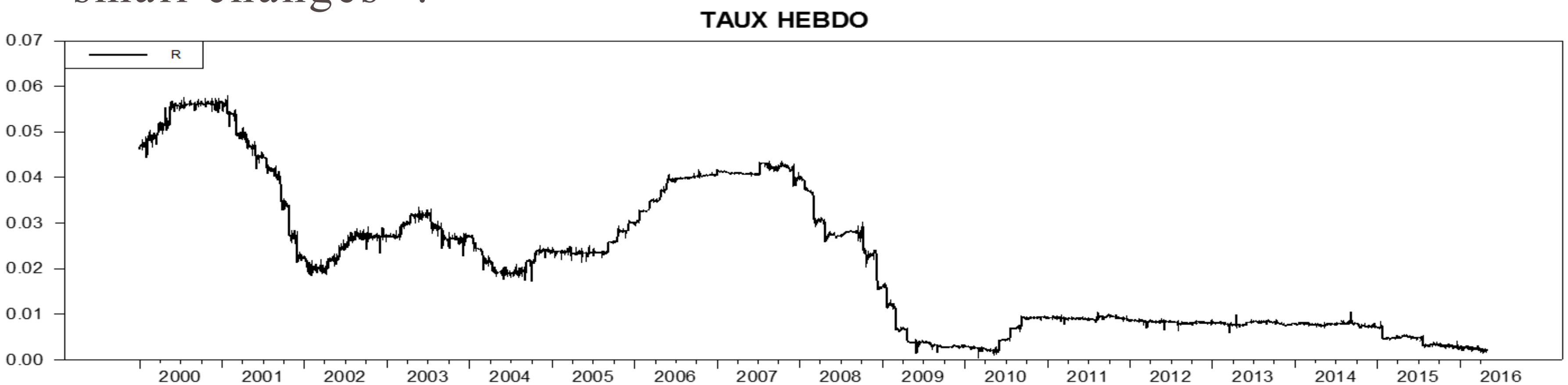
Data

- Histogram for maturity w



Data

- non stationary process: mean, variance and covariance change over time.(first difference is stationary)
- nonlinear process: the existence of different states of the world (or regimes).
- volatility clustering : "large changes tend to be followed by large changes, of either sign, and small changes tend to be followed by small changes’’.



The CDPQ model

- For CDPQ, simulations of interest rate are based on the assumption that variations are independent of the level of interest rates.

$$\tilde{r}_{m,t_0,t} = r_{m,t_0} + r_{m,t} - r_{m,t-w}$$

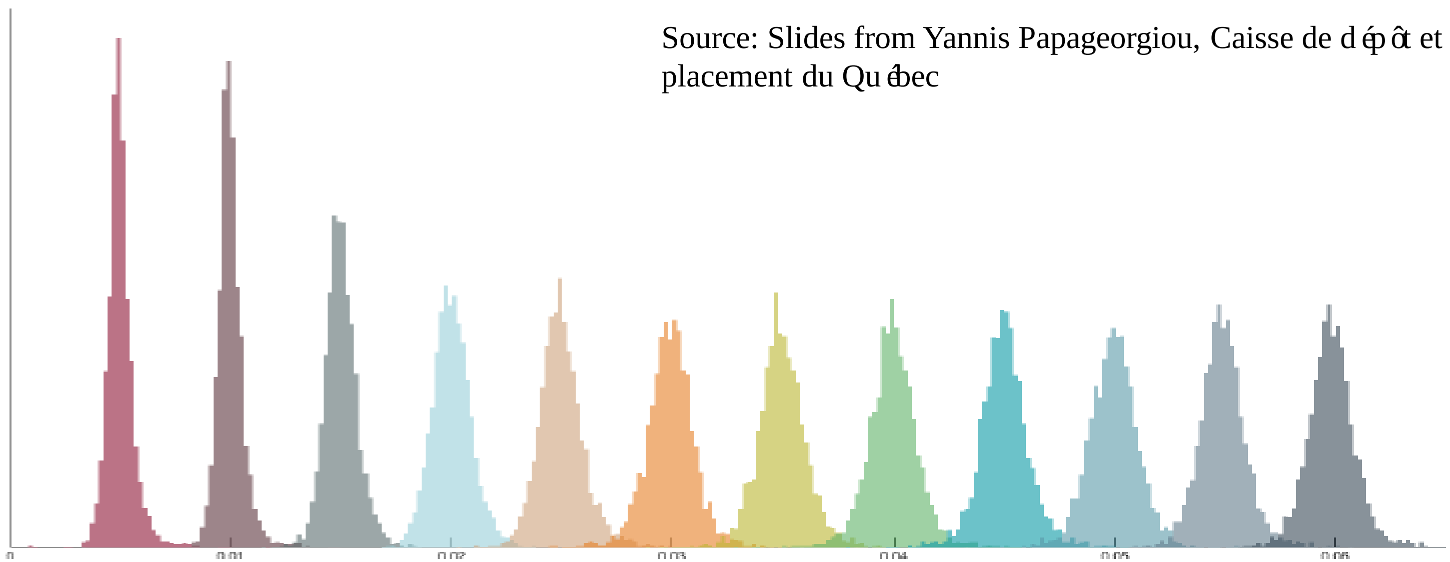
- In order to simulate the value of a bond, re-value the bond under the simulated interest rates and then calculate corresponding VaR (weekly).

$$\tilde{V}_{t_0,t} = \sum_{k=0}^{n} \frac{C}{(1+\tilde{r}_{k,t_0,t})^k} + \frac{P}{(1+\tilde{r}_{k,t_0,t})^n}$$

- In practice, there are two problems associated with this method.

The CDPQ model

- Scenario generation: the way CDPQ simulates interest movements is by simply adding the historical variation to current values.
- However, the distribution of interest rate variations seems to depend on the level.

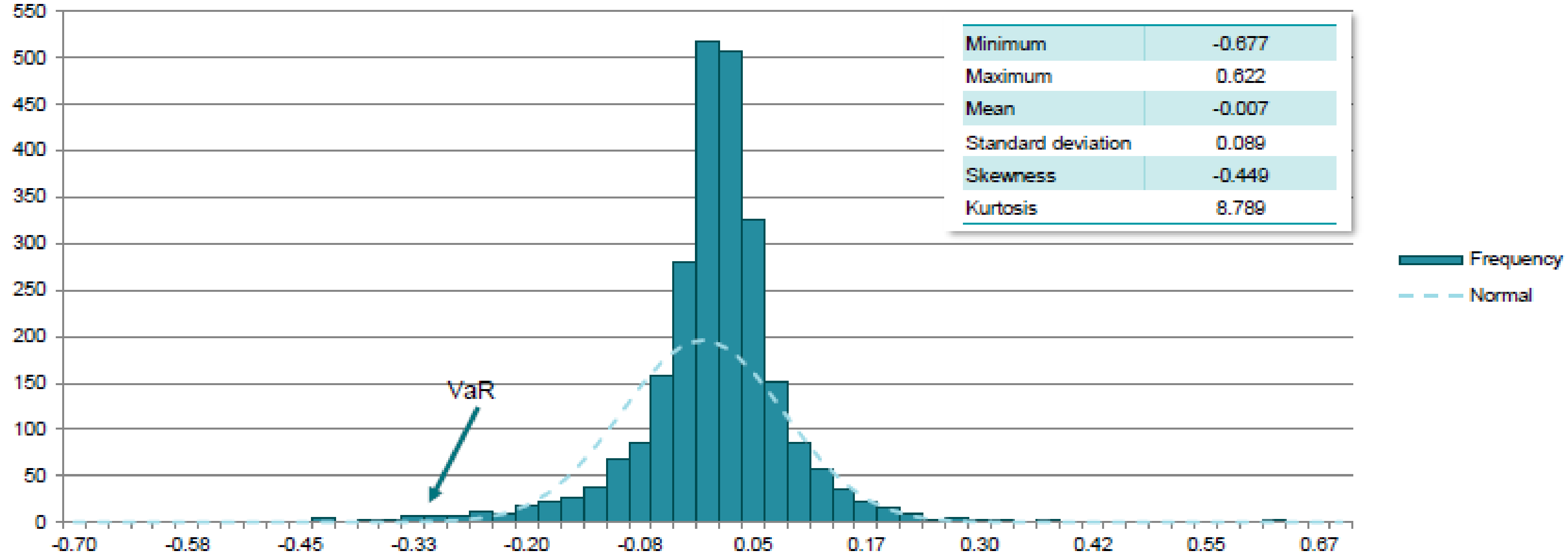


Source: Slides from Yannis Papageorgiou, Caisse de dépôt et placement du Québec

The CDPQ model

- Annualization: in order to transform VaR from the time scale of the measurement frequency to that of the investment horizon, we need our observations to be i.i.d. and normal.
- the following graphic illustrates a heavy-tail distribution of weekly profits and losses.

Source: Slides from Yannis Papageorgiou, Caisse de dépôt et placement du Québec



Minimum	-0.677
Maximum	0.622
Mean	-0.007
Standard deviation	0.089
Skewness	-0.449
Kurtosis	8.789

Scenario generation: Model

- To evaluate the influence of interest rate level to its distribution, we apply CEV model (a special case of SABR model).
- Continuous version of CEV model (Wt is the standard BM)

$$\mathrm{d}r_t = \sigma r_t^{\beta} \mathrm{d}W_t$$

- Discrete version (by proper scaling of sigma, Z ~ N(0,1)):

$$\Delta r_t = \sigma r_t^{\beta} Z$$

- Given the interest rate level, the change in interest rate follows a normal distribution with variance depending on current level.

$$\Delta r_t \sim N(0, \sigma^2 r_t^{2\beta})$$

Scenario generation: Model

- Based on above CEV model, the formula of the evolution of interest rate becomes

$$\tilde{r}_{m,t,t_0} = r_{m,t_0} + \frac{r_{m,t_0}^{\beta}}{r_{m,t-w}^{\beta}}(r_{m,t} - r_{m,t-w})$$

- The change of interest rate are re-scaled by the level of interest rate.
- When $\beta = 0$, the additive model (without scale).
  When $\beta = 1$, the multiplicative model.

Scenario generation: MLE

- Recall that given the interest rate level,

$$\Delta r_t \sim N(0, \sigma^2 r_t^{2\beta}) \quad \text{independent.}$$

- Likelihood function $L(\sigma, \beta)$
- MLE $(\hat{\sigma}, \hat{\beta})$ maximize $L(\sigma, \beta)$ or equivalently the log-likelihood function $l(\sigma, \beta) = \ln L(\sigma, \beta)$.
- MLE $(\hat{\sigma}, \hat{\beta})$ solutions of system of equations
- Solution:
  1. Find $\hat{\sigma}$ numerically as the root of an equation;
  2. Calculate $\hat{\beta} = g(\hat{\sigma})$.

Scenario generation: MLE

- The observed information matrix can be obtained from the inverse of minus the matrix of the second partial derivatives of the log-likelihood function with respect to the 2 parameters σ and β.

- Asymptotic distribution of MLE $(\hat{\sigma}, \hat{\beta})$ is normal;
- MLE: asymptotically unbiased; efficient estimator.

- The distribution of MLE permits the calculation of confidence intervals for σ and β and to do hypothesis testing.

Scenario generation

- For example, to test the simple hypothesis H_0: $\beta = 0$ (CDPQ model), use test statistic $z_0 = \widehat{\beta} / s.e.(\widehat{\beta})$. If $|z_0| > 1.96$, reject H_0 at 5% level (CDPQ model).

- For example, with data of interest rate with 1 year maturity (852 iid observations from weekly non-overlapping intervals), estimated value $\widehat{\beta} = 0.43337$.

- Conclusion: reject H_0: $\beta = 0$ (CDPQ model).

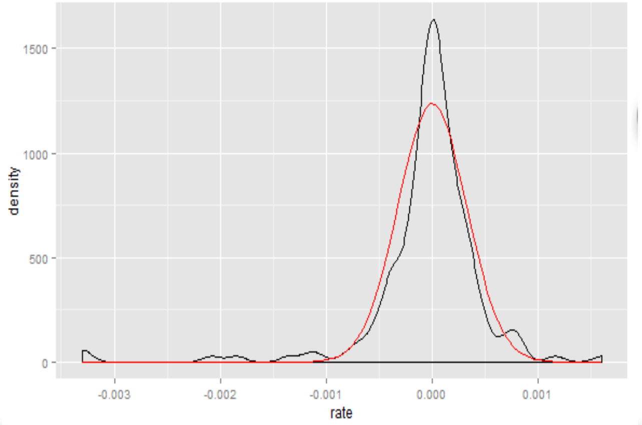
Scenario generation: VaR

- With above method of maximum likelihood estimation and non-overlapping data, Here is an example of estimations for two maturity.

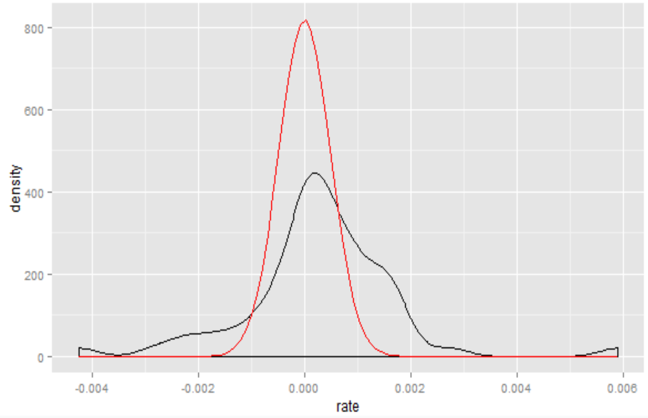
Maturity	180 (Semi annual)	365 (Annual)
β	0.4981587	0.43337

- Comparison of empirical (black) and theoretical (red) density.

level = 0.01



level = 0.026



Scenario generation: VaR

- As a test, we calculate the VaR of a short position in following two kinds of bonds:
  - Zero coupon bond with one year maturity;
  - 4% coupon bond with one year maturity and semi-annual coupon payment.
- Time horizon: 1 week.
- Confidence level: 99%.
- For each case, we generate 500 scenarios.

Bond	4% Coupon	Zero Coupon
VaR	-0.18%	-0.22%

Annualization: Self-Similar Process

- To fit the heavy tail distribution while keeping the form of square-root-of-time rule, we generalize BM to self-similar process {X(t)} satisfying:

$$X(kt) \overset{d}{=} k^{H} X(t)$$

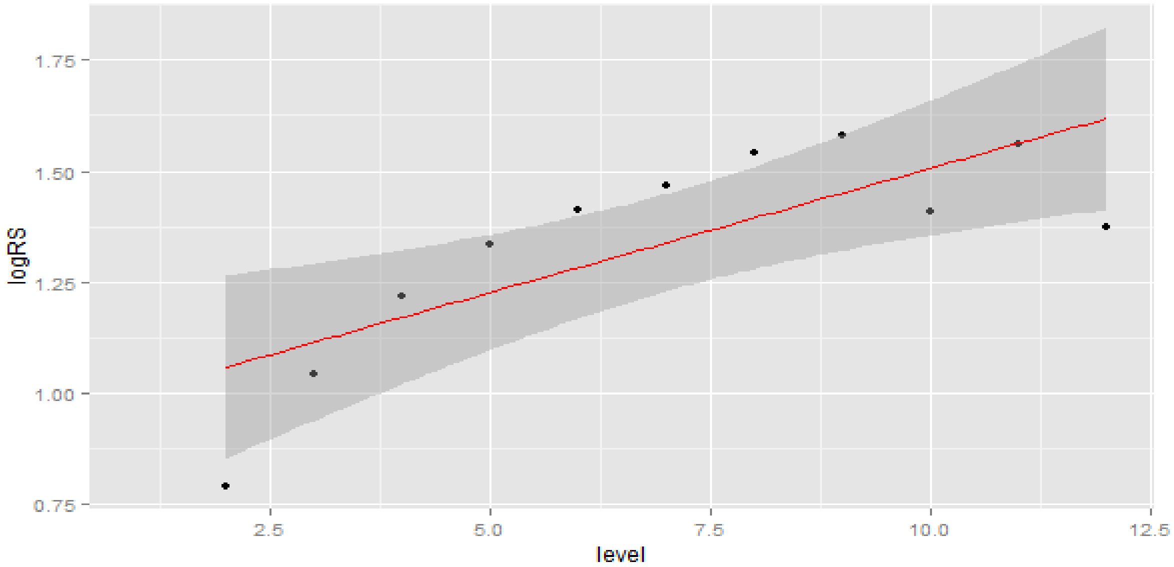
- H: Hurst coefficient.
- Examples of self-similar process:
  - Brownian motion: H = 0.5;
  - α-stable process: H = 1/ α.
- Heavy tail when H < 0.5!
- If the underlying prices is a self-similar process with Hurst coefficient H, then

$$\mathrm{VaR}(kt) = k^{h} \mathrm{VaR}(t)$$

Annualization: Estimation

- The estimation of Hurst coefficient H is based on the the Rescaled Range (R/S) Calculation.
- Regression: log(Range/S.d) ~ log(Scale).

Log(R/S) v.s. Log(Scale)



Data: interest rates corresponding to one year maturity, 2000-2016

Annualization: Estimation

- As an example, we run the regression for the data of interest rates corresponding to one year maturity, 2000-2016.

```
Coefficients:
            Estimate Std. Error t value Pr(>|t|)    
(Intercept)  0.94757    0.11810   8.024 2.16e-05 ***
level        0.05599    0.01537   3.642  0.00539 ** 
---
Signif. codes:  0 ‘***’ 0.001 ‘**’ 0.01 ‘*’ 0.05 ‘.’ 0.1 ‘ ’ 1
```

- As the weekly VaR for one year zero coupon is -0.22%, then the annualized VaR is -0.22%*52^0.056 = -0.274%.

Further work

- There are still lots of models to be studied for interest rate process
  1. ARCH-GARCH models;
  2. Comparison of VaR from independent vs dependent observations;
  3. GNL (Brownian-Laplace motion);
  4. HMM;

  …

Further work: GARCH

- GARCH (Generalized Auto-Regressive Conditional Heteroskedasticity) models volatility clustering.
- In this model :the volatility process is time varying and is modeled to be dependent upon both the past volatility and past innovations.

$$
\begin{aligned}
r_t &= \mu + \delta\, h_t + \epsilon_t \\
h_t &= \gamma_1 + \gamma_2 h_{t-1} + \gamma_3 \epsilon_{t-1}^2
\end{aligned}
$$

Further work:MS-GARCH

- Markov-switching GARCH model (MS-GARCH) is an extension of GARCH.
- The conditional mean and variance switch in time from one GARCH process to another.
- The conditional variance may originate from structural changes in the variance process which are not accounted for by standard GARCH models.
- Estimate a model that permits regime switching in the parameters.

$$y_t = \mu_{s_t} + \sigma_t u_t$$

$$\sigma_t^2 = \omega_{s_t} + \alpha_{s_t}\epsilon_{t-1}^2 + \beta_{s_t}\sigma_{t-1}^2$$

- S: states or r égimes.
- Transition probablilities: $\{\eta_{ij} = \mathrm{P}(s_t = i|s_{t-1} = j)\}$

Question Time

Thank you for your listening