Finite element analysis of ultralight metallic lattices

Sixth Montreal Industrial Problem Solving Workshop

Problem submitted by
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- Context : Additive manufacturing (3D printing)

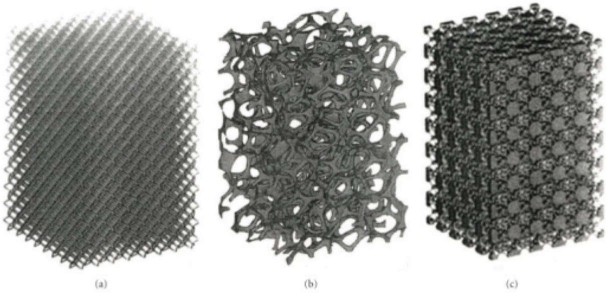

FIGURE: Source : http://openi.nlm.nih.gov/

- Goal : use ultralight lattice structures to reduce weight of pieces of machinery while preserving or improving their characteristics.
- Problem : Because of the small size of the cells ($\leq$ 1mm), number of Finite Elements required for the computational analysis is huge.

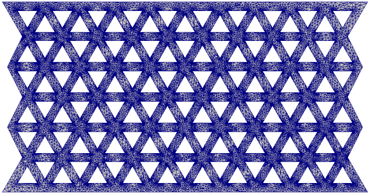

FIGURE: Adapted mesh for a 2D plate lattice

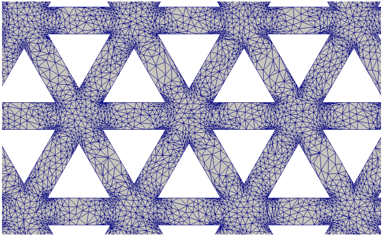

FIGURE: Zoom in

- Goal of the project : Find a model (equations for the displacement, temperature) that can be discretized and solved numerically at a reasonable cost and within a reasonable time

- Research avenues :
  1. use **beam models** (but struts are irregular and struts diameters are not small compared to cells sizes)
  2. use directly a **discrete model** of interacting nodes dictated by the lattice structures (but we need to find the laws of interaction between the nodes)
  3. find a **replacement homogeneous solid** with equivalent mechanical properties to those of the original piece

We followed the 3rd avenue, using homogenization theory.

Specificities of the applications

The engine parts that PWC has in mind are very thin (with shell like shape), thus with only a very small number of cell layers (between 4 and 10).

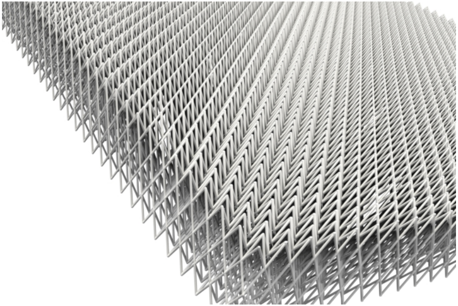

FIGURE: 3D plate lattice (source : http://architectedmaterials.com/)

A model diffusion problem on a plate

Ω=horizontal plate in the (x_1, x_2, x_3) coordinate system, with periodic inhomogeneities.
ε = characteristic length of a cell

$$\begin{cases} -\mathrm{div}\left(A\left(\frac{\boldsymbol{x}}{\varepsilon}\right)\nabla u_\varepsilon(\boldsymbol{x})\right) = f & \text{in} \quad \Omega, \\ A\nabla u_\varepsilon \cdot n = 0 & \text{on} \quad \partial\Omega. \end{cases} \tag{0.1}$$

Two-scale method (see G. Allaire [2007]) :

$$u_\varepsilon(x) = u_0(x,y) + \varepsilon u_1(x,y) + \varepsilon^2 u_2(x,y) + ...$$

where $y = x/\varepsilon \in Y = [0,1]^3$ (scaled unit cell) is a **microscale** variable.
Inserting this expansion into the PDE we obtain equations for u_0, u_1, u_2, ...beginning with the lowest order (in ε) terms.

After a few manipulations :

$$u_0(\boldsymbol{x}, \boldsymbol{y}) = u(\boldsymbol{x}).$$

$$u_1(\boldsymbol{x}, \boldsymbol{y}) = \sum_{i=1}^{3} \frac{\partial u(\boldsymbol{x})}{\partial x_i} w_i(\boldsymbol{y}).$$

where w_i is the solution of

$$\left\{ \begin{array}{rcl} \operatorname{div}_{\mathbf{y}} \left(A(\boldsymbol{y})(e_i + \nabla_{\mathbf{y}} w_i(\boldsymbol{y})) \right) = 0 & \text{in} & Y, \\ w_i(\boldsymbol{y}) \quad \text{periodic} & \text{on} & \partial Y. \end{array} \right. \tag{0.2}$$

$$-\operatorname{div}_{\mathbf{y}} \left(A \nabla_{\mathbf{y}} u_2 \right) = \operatorname{div}_{\mathbf{y}} \left(A \nabla_{\mathbf{x}} u_1 \right) + \operatorname{div}_{\mathbf{x}} \left(A \left(\nabla_{\mathbf{x}} u + \nabla_{\mathbf{y}} u_1 \right) \right) + f.$$

and, last but not least, the (very nice) homogeneous problem

$$-\operatorname{div}_{\mathbf{x}} \left(A^* \nabla_{\mathbf{x}} u \right) = f(\boldsymbol{x}) \qquad \text{in} \quad \Omega, \tag{0.3}$$

where A^* is called the homogenized tensor and can be computed at the cell level :

$$A^*_{ij} := \int_Y A(\boldsymbol{y}) \left(e_j + \nabla_{\mathbf{y}} w_j \right) \cdot e_i \, d\boldsymbol{y}.$$

But we are not comfortable with applying this to our situation :
- the plate has only a small number K of layers of cells and this technique disregards the boundary layers
- The plate thickness converges to 0 as $\varepsilon \to 0$.

These weaknesses can be overcome by :
- replacing the unit cell with a cell column : $Y = [0,\, 1]^2 \times [0,\, K]$;
- looking for a plate model (2D model) with solution $u(x_1, x_2)$.
We thus modify the expansion :

$$u_\varepsilon(x) = u_0(\overline{x}, y) + \varepsilon u_1(x, y) + \varepsilon^2 u_2(x, y) + \ldots$$

where $\overline{x} = (x_1, x_2)$, $\overline{y} = \overline{x}/\varepsilon$. Again, this gives that $u_0(\overline{x}, y) = u(\overline{x}) = u(x_1, x_2)$ and that u is solution of

$$\left\{ \; -\mathrm{div}_{\mathbf{x}} \left(A^* \nabla_{\mathbf{x}} u \right) = Kf \qquad \text{in} \quad \Omega^*, \right. \tag{0.4}$$

where $\Omega = \Omega^* \times [0, K]$ and A^* has a form similar to the previous one.

- Remarks :
  In the case of a domain with holes (a lattice !), the homogenized problem takes the form

$$\left\{ \; -\mathrm{div}_{\mathbf{x}} \left(A^{*} \nabla_{\mathbf{x}} u \right) + \theta u = \theta K f \qquad \text{in} \quad \Omega^{*}, \right. \tag{0.5}$$

  where θ is the volumic fraction occupied by the solid material.
- What could be done next :
  - extend the present study to shell-like lattice structure and to the elasticty system ;
  - litterature review on the modelling of plate-like and shell-like lattice structures.
  - Compare numerically the proposed homogenized plate and shell models with the starting models.