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The interval in graph theory

Generally, to decompose a mathematical structure defined on a set S, we search for a partition P of S which is *compatible* with the structure, that is to say, the value of the structure does not depend on the choice of an element inside a subset of E which belongs to P. This compatibility then allows by contraction, by projection or by decomposition to define the *quotient* of the structure by P.

For combinatorical structures as simple as graphs, the passage to the quotient is easily clarified by the notion of interval. Recall that a *graph* G consists of a finite *vertex* set V and an *edge* set E, where an edge is an unordered pair of distinct vertices. For instance, given a set V, the graph $(V, \emptyset)$ is the *empty* graph defined on V whereas $(V, \binom{V}{2})$ is the *complete* graph. Let $G = (V, E)$ be a graph. A subset X of V is an *interval* (or a *clan* or a *homogeneous* set or a *module*) of G provided that for any $x, y \in X$ and $u \in V - X$, we have: $\{u, x\} \in E$ if and only if $\{u, y\} \in E$. For disjoint intervals X and Y of G, we obtain that either for any $x \in X$ and $y \in Y$, $\{x, y\} \in E$ or for any $x \in X$ and $y \in Y$, $\{x, y\} \notin E$. Thus, the partitions compatible with G are the partitions of V constituted by intervals of G which are called *interval* partitions of G. The elements of an interval partition P of G may then be considered as the vertices of a new graph, the *quotient* of G by P which is denoted by G/P and defined on P as follows: given distinct elements X and Y of P, $\{X, Y\}$ is an edge of G/P if $\{x, y\} \in A$, where $x \in X$ and $y \in Y$.

A graph possesses several interval partitions. For example, each partition of a set V realizes an interval partition of the empty graph defined on V. On the other hand, there exist graphs $G = (V, E)$, the only interval partitions of which are $\{V\}$ and $\{\{x\}; x \in V\}$. Such graphs are called *indecomposable* because they admit only two quotients: one reduced to a vertex, the other isomorphic to the initial graph.

The uniform decomposition problem is to specify an interval partition for each graph and to characterize the corresponding quotient. By strengthening the notion of interval, Gallai (1967) succeeded in intrinsically associating an interval partition with each graph which induces a complete, empty or indecomposable quotient. The *decomposition tree* of a graph is then obtained by reiterating this decomposition. It permits the characterization of important classes of graphs.

According to the decomposition theorem, the difficulty lies in the structural examination of the indecomposable graphs. The first conclusive results appeared at the beginning of the nineties. Some of them yield simple recognition algorithms of indecomposable graphs.