EXERCISES

SCHOOL : COMBINATORICS AND STATISTICAL MECHANICS

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1) Let A and a respectively denote the species of trees and of rooted trees.

a) Prove, by making appropriate drawings, the following combinatorial equation

$$X^2 a'' = (E_2(A))^{\bullet}.$$

b) Deduce from it an identity involving the family of numbers n^{n-1}.

2) Let L_+ denote the species of non-empty lists and C be the species of oriented cycles. Define the species Oct by the combinatorial equation $\mathrm{Oct} = C(L_+)$.

a) Explain why an Oct-structure should be called an *octopus*.

b) Prove, by making appropriate drawings, the following combinatorial equations

$$\mathrm{Oct}(X) + C(X) = C(2X), \qquad \mathrm{Oct}'(X) = L(X)L(2X),$$

where $L = 1 + L_+$ is the species of (possibly empty) lists.

c) Deduce a closed formula for the number of n-octopuses from any one of these equations.

d) Prove that for all $n \geq 0$, the n-fold composition $L_+ \circ L_+ \circ \cdots \circ L_+$ satisfies

$$L_+ \circ L_+ \circ \cdots \circ L_+ = XL(nX).$$

3) It is well-known that a permutation of $[n] = \{1, 2, ..., n\}$ is called *even* if can be expressed as a product of an even number of transpositions. But this definition uses the underlying total order of the set $[n]$ and cannot be extended to a permutation of an arbitrary (non-ordered) finite set.

a) Define an « order-free» notion of even permutation which is applicable to any finite U (hint : use of the cyclic decomposition of arbitrary permutations).

b) Show that the corresponding species S_{alt} of even permutations satisfies the combinatorial equation $S_{alt} = E(C_1 + C_3 + C_5 + \cdots)E_{even}(C_2 + C_4 + C_6 + \cdots)$ where, E_{even} denotes the species of sets having an even number of elements.

c) It is well-known that the species End of endofunctions satisfies End = $S(A)$. By analogy, define the species End_{alt} of *even endofunctions* by the formula $\mathrm{End}_{alt} = S_{alt}(A)$ Find a closed formula for the number of even endofunctions on an n-element set.

4) (Counting enriched trees and rooted trees) The *sequence of binomial type* associated to a species F is, by definition, the sequence of functions $(f_n(\lambda))_{n\geq 0}$ such that

$$(F(x))^{\lambda} = \sum_{n\geq 0} f_n(\lambda)\frac{x^n}{n!}.$$

We use the notation $F \lozenge (f_n(\lambda))_{n\geq 0}$ to express this fact.

a) Show that if $F \lozenge (f_n(\lambda))_{n\geq 0}$, then the following family of «binomial identities» hold

$$f_n(\lambda+\mu) = \sum_{k=0}^{n}\binom{n}{k} f_k(\lambda) f_{n-k}(\mu).$$

b) It has been combinatorially proven that if $R \lozenge (r_n(\lambda))_{n\geq 0}$ and $R' \lozenge (\rho_n(\lambda))_{n\geq 0}$ then

$r_{n-1}(n)$ = the number of R-enriched rooted trees on n elements, for $n \geq 1$;

$\rho_{n-2}(n)$ = the number of R-enriched trees on n elements, for $n \geq 2$; (= $R(0)$, for n = 1).

Apply these formulas in the following contexts :

(*i*) $R = E$; (*ii*) $R = 1+C$; (*iii*) $R = L$; (*iv*) $R = 1+X$; (*v*) $R = 1+X+E_k$.

c) (Generalized Abel identities) The classical identities of Abel are stated as

$$(u+v+n)^n = \sum_{i+j=n}\binom{n}{i} u\cdot(u+i)^{i-1}(v+j)^j,$$

$$(u+v)(u+v+n)^{n-1} = \sum_{i+j=n}\binom{n}{i} u\cdot(u+i)^{i-1} v\cdot(v+j)^{j-1}.$$

Their R-enriched versions are as follows : if $R \lozenge (r_n(\lambda))_{n\geq 0}$, then

$$r_n(u+v+n) = \sum_{i+j=n}\binom{n}{i} u\cdot\frac{r_i(u+i)}{u+i} r_j(v+j),$$

$$(u+v)\frac{r_n(u+v+n)}{u+v+n} = \sum_{i+j=n}\binom{n}{i} u\cdot\frac{r_i(u+i)}{u+i} v\cdot\frac{r_j(v+j)}{v+j}.$$

The classical identities correspond to the case $R = E$. Prove these identities by considering the combinatorial identities $A_R^{[u+v]} = A_R^u \cdot A_R^{[v]}$ and $A_R^{u+v} = A_R^u \cdot A_R^v$ where $A_R^{[k]}$ denotes the species of « hedges » formed from k R-enriched rooted trees where the last rooted tree is pointed. Hint : note that $vA_R^{[v]}(x) = (A^v)^{\bullet}(x)$.

5) Compute, from scratch, the cycle index series $Z_{E_2}(x_1,x_2,...)$ and the asymmetry index series $\Gamma_{E_2}(x_1,x_2,...)$ of the species E_2 of 2-element sets. Deduce that the combinatorial equation $X^2 = 2E_2$ is *false* (although the species X^2 and $2E_2$ are equipotent).

6) Prove that

$$Z_E(x_1,x_2,x_3,...) = \exp(x_1 + x_2/2 + x_3/3 + ...), \qquad \Gamma_E(x_1,x_2,x_3,...) = \exp(x_1 - x_2/2 + x_3/3 - ...),$$

$$Z_C(x_1,x_2,x_3,...) = -\sum_{k\geq 1}\frac{\phi(k)}{k}\log(1-x_k), \qquad \Gamma_C(x_1,x_2,x_3,...) = -\sum_{k\geq 1}\frac{\mu(k)}{k}\log(1-x_k).$$

7) Prove the following enriched version of the dissymmetry theorem for the species $\mathfrak{a}_R$ of *R*-enriched trees and the species A_R of *R*-enriched rooted trees :

$$XR(A_{R'}) + E_2(A_{R'}) = \mathfrak{a}_R + (A_{R'})^2,$$

and write the corresponding combinatorial equations in the cases

(*i*) $R = E$; (*ii*) $R = 1 + C$; (*iii*) $R = L$; (*iv*) $R = 1 + X$; (*v*) $R = 1 + X + E_k$.

8) Let *S* be the species of permutations. Assuming that $\Gamma_S = (1-x_2)/(1-x_1)$, show that the number $\overline{a_n}$ of asymmetric *S*-enriched rooted trees satisfy the recurrence

$$\overline{a_0} = 0, \quad \overline{a_1} = 1, \quad \overline{a_{n+1}} = (\overline{a_1}\,\overline{a_n} + \overline{a_2}\,\overline{a_{n-1}} + \cdots + \overline{a_n}\,\overline{a_1}) - \chi(n \text{ even})\overline{a_{n/2}}.$$

9) Let Gr denote the species of simple graphs. It can be shown that

$$\operatorname{fix} Gr[\sigma] = 2^{1/2\sum_{i,j\geq 1}\gcd(i,j)\sigma_i\sigma_j - 1/2\sum_{k\geq 1}(k \bmod 2)\sigma_k}.$$

Devise a recursive method to count *unlabelled* connected graphs on *n* points, $n = 0,1,2, ...$.

10) By making appropriate drawings, write the first terms, up to degree 4, of the molecular decomposition of the species Gr_w of graphs weighted according to connected components, that is, $w(g) = t^{\text{number of connected components of } g}$, for any graph *g*.

11) Let δ be the species of *triangular cacti* (that is, connected graphs whose blocks are triangles) and $\Delta = \delta^{\bullet}$. Prove that

$$\Delta = XE(E_2(\Delta)) \quad \text{and} \quad \delta + \Delta E_2(\Delta) = \Delta + E_3(\Delta).$$

12) Show that if $f : [0,\infty) \to \mathrm{R}$ is measurable and bounded then, for $\varepsilon > 0$,

$$|f(r)| = \mathrm{O}(1/r^{d+2\varepsilon}) \quad \Rightarrow \quad \int_0^\infty r^{d+\varepsilon-1}|f(r)|dr < \infty.$$

13) Compute the coefficients $\gamma_2(T)$ and $\gamma_3(T)$ of the virial expansion

$$\frac{P}{kT} = \frac{\overline{N}}{V} + \gamma_2(T)\left(\frac{\overline{N}}{V}\right)^2 + \gamma_3(T)\left(\frac{\overline{N}}{V}\right)^3 + \cdots$$

in the case of a gaussian interaction.

14) It can be shown that the second Mayer's weight $w(K_N)$ of the complete graph K_N for the hard-core continuum gas in one dimension is given by

$$w(K_N) = (-1)^{n(n-1)/2} N \,.$$

Verify this, for N=2 and N=3, by evaluating the corresponding integrals ($f(r) = \chi(r < 1)$).