An introduction to asymptotic enumeration

Part 1. Why are asymptotic results interesting?

Part 2. Asymptotics of sequences $a(n)$

Part 3. Limit laws

From exact to asymptotic enumeration

Assumption: we have an exact description of the sequence $a(n)$, like

- recurrence relation
- formula
- expression of the ordinary or exponential g.f. $A(t)$ or $\tilde{A}(t)$
- functional equation defining $A(t)$ or $\tilde{A}(t)$...

Wanted

$$a(n) \sim s(n) \cdots \quad \text{or} \quad a(n) = s(n) + O(r(n))$$

Overview

Three rather general techniques:

- when $a(n)$ is given explicitly as a sum
- when $A(t)$ has dominant singularities of an algebraic-logarithmic type
- when $A(t)$ is entire or diverges violently near its dominant singularity(-ies)

What you (really) need to remember from these techniques:

- their existence
- their range of applicability (how they work)
- where to find them

Three rather general techniques

- Bare-hand asymptotics on sums

$$a(n) = \sum_{j=1}^{n} \frac{3^j\,(n-1)!\;n!\;(2j)!(24+18\,n+(5+13\,n)\,j)}{(j-1)!\;j!\;(j+1)!\;(j+4)!\;(n-j)!\;(n-j+2)!}$$

- Singularity analysis: when $A(t)$ has dominant singularities of an algebraic-logarithmic type

$$A(t) = \frac{1}{2}\left(1 - \sqrt{1 - 4\log\frac{1}{1-\log\frac{1}{1-t}}}\right)$$

- Saddle-point asymptotics: when $A(t)$ is entire or diverges violently near its dominant singularity(-ies)

$$A(t) = \exp\left(\frac{t}{1-t}\right)$$

Three rather general techniques... and their applications

- Software: When $A(t)$ is explicitly given

$$A(t) = \frac{1}{2}\left(1 - \sqrt{1 - 4\log\frac{1}{1-\log\frac{1}{1-t}}}\right), \qquad A(t) = \exp\left(\frac{t}{1-t}\right)$$

- Almost automatic asymptotic analysis for $A(t)$ defined implicitly by
  - an algebraic equation

$$t^3A(t)^4 + t^2\,(3+4\,t)\,A(t)^3 + t\left(3 - 29\,t + 6\,t^2\right)A(t)^2 + \left(1 - 7\,t + 29\,t^2 + 4\,t^3\right)A(t) - (1-t)^3 = 0$$

  - a linear differential equation

$$2 + 2\,(-1+6\,t)\,A(t) + 4\,t\,(-1+12\,t)\,\frac{d}{dt}A(t) + t^2\,(16\,t-1)\,\frac{d^2}{dt^2}A(t) = 0$$

First method: Asymptotics of sums

Bare-hand asymptotics on sums

Input

$$a(n) = \sum_k b(n,k)$$

Output

$$a(n) \sim s(n) \cdots \quad \text{or} \quad a(n) = s(n) + O(r(n))$$

Bare-hand asymptotics on sums

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Output

$$a(n) \sim s(n) \cdots \quad \text{or} \quad a(n) = s(n) + 0(r(n))$$

Example

$$a(n) = \sum_{j=1}^{n} \frac{4(n-1)!\, n!\, (2j)!}{(j-1)!\, j!\, (j+1)!\, (j+4)!\, (n-j)!\, (n-j+2)!} P(j,n)$$

with

$$P(j,n) = 24 + 18\,n + (5 - 13\,n)\,j + (11\,n + 20)\,j^2 + (10\,n - 2)\,j^3 + (4\,n - 11)\,j^4 - 6\,j^5.$$

$$a(n) \sim \frac{3^9 5}{2^5} \frac{\sqrt{3}}{\pi} \frac{9^n}{n^7}$$

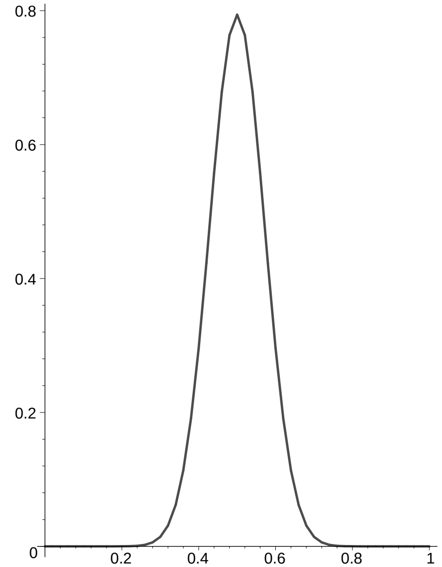
The method on a toy example

Input

$$a(n) = \sum_{k=0}^{n} b(n,k) \qquad \text{with} \qquad b(n,k) = \binom{n}{k}$$

Output

$$a(n) \sim 2^n$$



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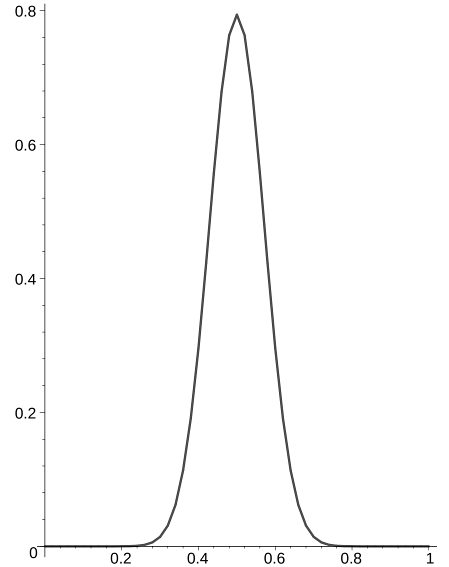


Output

$$a(n) \sim 2^n$$

The method:

1. Locate and estimate the largest summand(s) ($k \sim n/2$)
2. Recenter the problem in the vicinity of the "heavy" k's ($k = n/2 + r$, with r small).
   Estimate $b(n,k)$ for those k's (Stirling)
3. Determine a window on which to focus ($|k - n/2| < n^{2/3}$). Prove that the other summands are negligible
4. Replace the sum in the window by an integral

The method on a toy example

Input

$$a(n) = \sum_{k=0}^{n} b(n,k) \qquad \text{with} \qquad b(n,k) = \binom{n}{k}$$

1. Locate and estimate the largest summand(s)

   The sequence $b(n,k)$ is unimodal, with max. at $k = \lfloor n/2 \rfloor$. At this point,

$$b(n,k) = \binom{n}{\lfloor n/2 \rfloor} = \sqrt{2}\, \frac{2^n}{\sqrt{\pi n}}\, (1 + O(1/n)).$$

The method on a toy example

Input

$$a(n) = \sum_{k=0}^{n} b(n,k) \qquad \text{with} \qquad b(n,k) = \binom{n}{k}$$

1. Locate and estimate the largest summand(s)
   The sequence $b(n,k)$ is unimodal, with max. at $k = \lfloor n/2 \rfloor$. At this point,
   $$b(n,k) = \binom{n}{\lfloor n/2 \rfloor} = \sqrt{2}\,\frac{2^n}{\sqrt{\pi n}}\,(1 + O(1/n)).$$

2. Recenter the problem in the vicinity of the "heavy" k's.
   Estimate $b(n,k)$ for those k's (Stirling).
   Let $k = n/2 + r$, with $r = o(n^{3/4})$. Then
   $$b(n, n/2 + r) = \sqrt{2}\,\frac{2^n}{\sqrt{\pi n}}\,\exp\left(-\frac{2r^2}{n}\right)\left(1 + O(1/n) + O(r^4/n^3)\right).$$
   $\Rightarrow$ The significant summands are found among the terms $b(n, n/2 + r)$ for $r = O(n^{1/2+\epsilon})$

3. Determine a window on which to focus.
Prove that the other summands are negligible.
For $|r| \geq n^{2/3}$,

$$b(n, n/2 + r) \leq b(n, n/2 + n^{2/3}) = O\left(\frac{2^n}{\sqrt{n}} \exp(-2n^{1/3})\right)$$

$$\Longrightarrow \sum_{|r| \geq n^{2/3}} b(n, n/2 + r) = O\left(2^n \sqrt{n} \exp(-2n^{1/3})\right) = o(2^n/\sqrt{n}).$$

4. Replace the sum in the window by an integral [Riemann]
For $|r| < n^{2/3}$, the following holds uniformly in r:

$$b(n, n/2 + r) = \sqrt{2}\, \frac{2^n}{\sqrt{\pi n}} \exp\left(-\frac{2r^2}{n}\right) \left(1 + O(n^{-1/3})\right).$$

Hence

$$\begin{aligned}
\sum_{r=-n^{2/3}}^{n^{2/3}} b(n, n/2 + r) &= \sqrt{2}\, \frac{2^n}{\sqrt{\pi n}} \left(\sum_{r=-n^{2/3}}^{n^{2/3}} \exp\left(-\frac{2r^2}{n}\right)\right) \left(1 + O(n^{-1/3})\right) \\
&= \sqrt{2}\, \frac{2^n}{\sqrt{\pi n}} \sqrt{n} \left(\int_{\mathbb{R}} \exp(-2x^2) dx\right) (1 + o(1)) \\
&= 2^n (1 + o(1)).
\end{aligned}$$

References: Asymptotics of sums

- Bender, Edward A. Asymptotic methods in enumeration. SIAM Rev. 16 (1974), 485–515.

- Odlyzko, A. M. Asymptotic enumeration methods. Handbook of combinatorics, Vol. 1, 2, 1063–1229, Elsevier, Amsterdam, 1995.

Asymptotics and generating functions: general principles

• Ph. Flajolet and R. Sedgewick, Analytic Combinatorics
http://algo.inria.fr/flajolet/Publications/books.html
Analytic Combinatorics (Chapters I,II,III,IV,V,VI,VII,VIII,IX*). 743p.+x.
Version of October 23, 2006, "Oktoberfest", with Chapters I-VII in quasi-semi-final form.

An elementary example: Rational generating functions

Let

$$A(t) = \frac{1}{(1-\mu t)^k}$$

with $k \in \mathbb{N}$ and $\mu \in \mathbb{C}$. Then

$$A(t) = \sum_{n \geq 0} \binom{n+k-1}{k-1} \mu^n t^n = \sum_{n \geq 0} \frac{(n+1)(n+2)\cdots(n+k-1)}{(k-1)!} \mu^n t^n$$

Then

$$a(n) = \frac{n^{k-1}}{(k-1)!} \mu^n \left(1 + O(1/n)\right).$$

(Important) remark: As a function of the complex variable z, the function $A(z)$ has radius of convergence $1/|\mu|$, and a pole of multiplicity k at $z_c = 1/\mu$.

An elementary example: Rational generating functions

Let

$$A(t) = \frac{N(t)}{D(t)} \quad \text{with} \quad D(t) = \prod_{i=1}^{d} (1 - \mu_i t)^{k_i}$$

- Partial fraction expansion:

$$A(t) = P(t) + \sum_{i=1}^{d} \sum_{k=1}^{k_i} \frac{c_{i,k}}{(1 - \mu_i t)^k}$$

- If $\mu = |\mu_1| = \cdots = |\mu_j| > |\mu_{j+1}| \geq \cdots \geq |\mu_d|$ and $e = \max(k_1, k_2, \cdots, k_j)$,

$$a(n) = O(\mu^n n^{e-1})$$

If moreover $e = k_1 > k_2 \geq \ldots \geq k_j$,

$$a(n) = c_{1,1} \frac{n^{e-1}}{(e-1)!} \mu_1^n \left(1 + O(1/n)\right).$$

Rational generating functions: some observations

Let $A(t) = \frac{N(t)}{D(t)}$ with $D(t) = \prod_{i=1}^{d}(1 - \mu_i t)^{k_i}$

If $\mu = |\mu_1| = \cdots = |\mu_j| > |\mu_{j+1}| \geq \cdots \geq |\mu_d|$ and $e = \max(k_1, k_2, \cdots, k_j)$,

$$a(n) = O(\mu^n n^{e-1})$$

As a function of a complex variable z, the function $A(z)$ has radius $\rho = 1/\mu$, and a pole of order e at one of the μ_i's, $1 \leq i \leq j$.

$\triangleleft \; \triangleleft \; \diamond \; \triangleright \; \triangleright$

"The location and nature of the *dominant singularities* of a function $A(z) = \sum a(n)z^n$, analytic around 0, are reflected in the asymptotic behaviour of its coefficients $a(n)$."

Analytic functions

Defn. The function $A(z)$ is analytic near 0 if it is the sum of a convergent power series: for $|z| < \rho$, with $\rho > 0$,

$$A(z) = \sum_{n \geq 0} a(n) t^n$$

The radius of convergence of $A(z)$ is $\sup\{r > 0 : \sum a(n) r^n \text{ converges }\}$

Radius of cv. and exponential growth of coefficients

If the radius is ρ, then for all $\varepsilon > 0$

$$|a(n)| \leq (\epsilon + 1/\rho)^n \quad \text{for } n \text{ large enough}$$

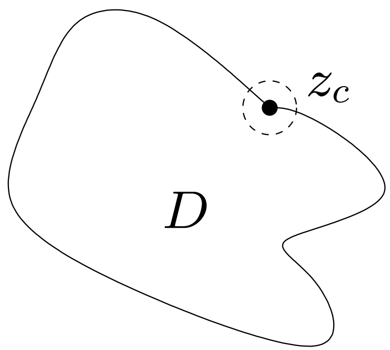
$$(-\epsilon + 1/\rho)^n \leq |a(n)| \quad \text{infinitely often}$$

"The location of the dominant singularity (the radius) gives the exponential growth rate."

Singularities

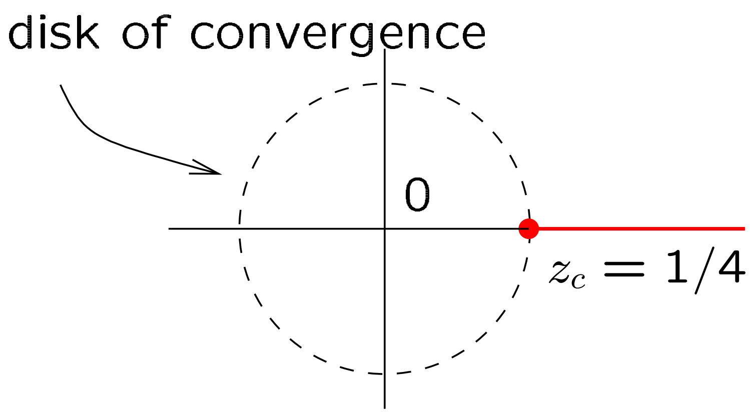
Let $A(z)$ be analytic in a domain D. A point z_c of the border of D is a singularity of A if no analytic continuation is possible at z_c. That is, we cannot write

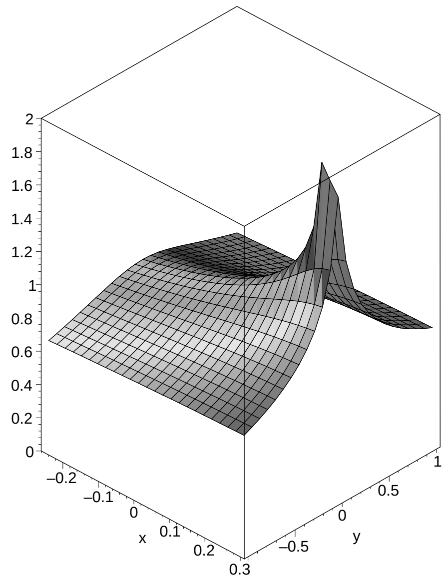
$$A(z) = \sum b(n)(z - z_c)^n \qquad \text{for } |z - z_c| \text{ small .}$$



Examples

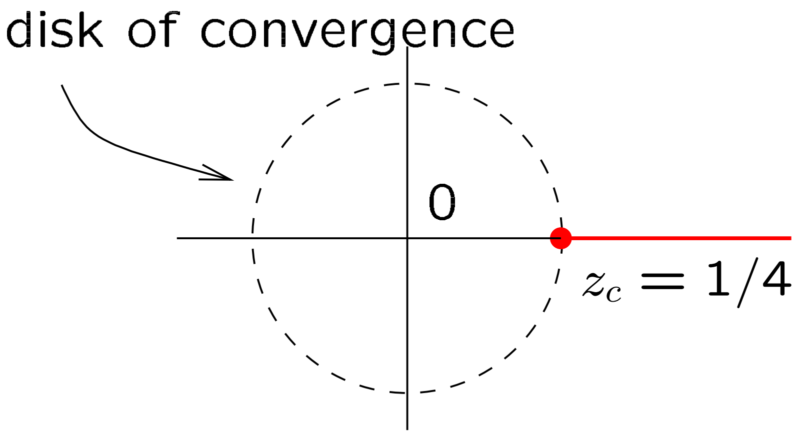
- $\log z$ and $\sqrt{z}$ are analytic in $D = \mathbb{C} \setminus (-\infty, 0]$, with a unique singularity at 0.
- The Catalan g.f. $A(z) = (1 - \sqrt{1 - 4z})/(2z)$ is analytic in $\mathbb{C} \setminus [1/4, \infty)$, with a unique singularity at $1/4$

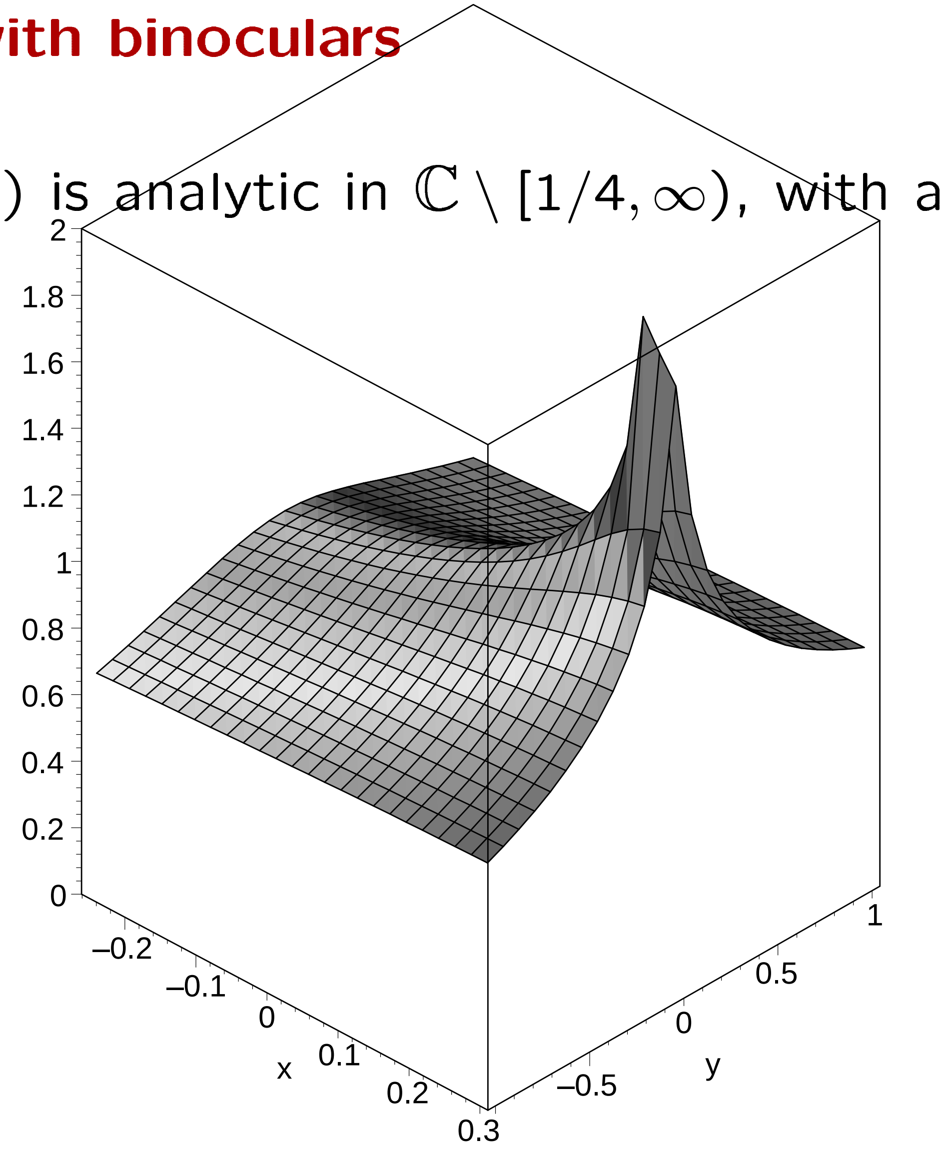




The same example with binoculars

The Catalan g.f. $A(z) = (1 - \sqrt{1-4z})/(2z)$ is analytic in $\mathbb{C} \setminus [1/4, \infty)$, with a unique singularity at $1/4$





Dominant singularities of power series

Let $A(z) = \sum a(n)z^n$ be a series with positive radius of cv ρ.

• Then $A(z)$ has at least one singularity on the circle $|z| = \rho$. Such singularities are called dominant.

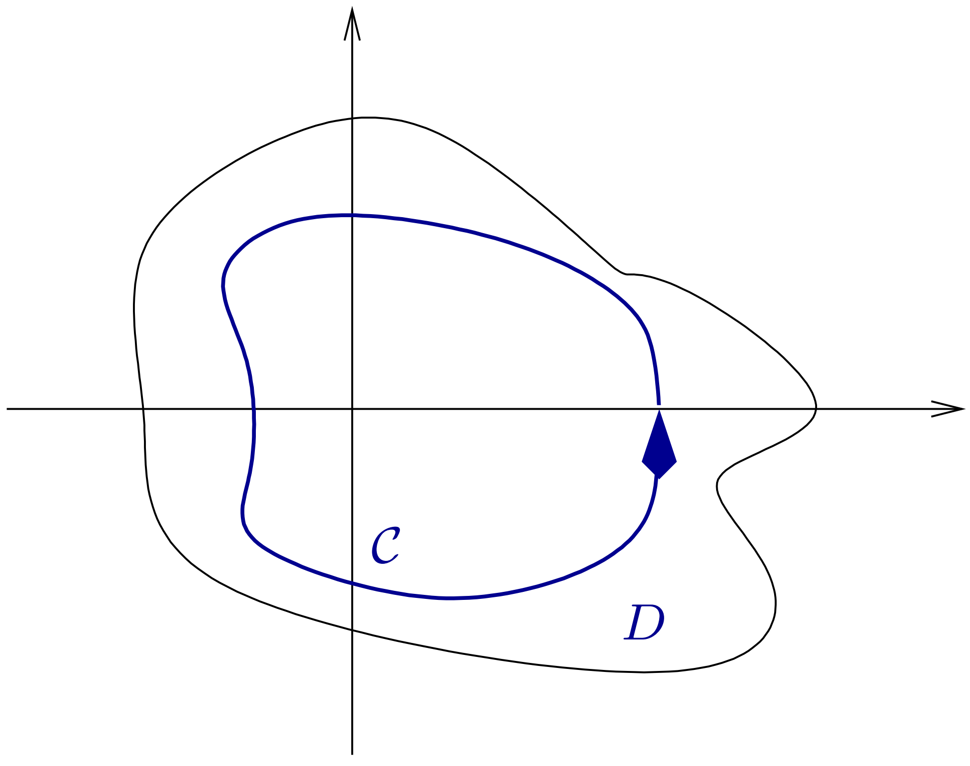
• If $a(n) \geq 0$, one of the dominant singularities is ρ itself [Pringsheim].

Extraction of coefficients: Cauchy's formula

Let $A(z) = \sum a(n)z^n$ be a series with positive radius of cv, and analytic in a domain D. Then

$$a(n) = \frac{1}{2i\pi}\int_{\mathcal{C}} A(z)\frac{dz}{z^{n+1}}$$

for every contour $\mathcal{C} \subset D$ encircling positively the origin.



Second method: singularity analysis

- Ph. Flajolet and R. Sedgewick, Analytic Combinatorics
http://algo.inria.fr/flajolet/Publications/books.html
Chapters VI,VII

- Ph. Flajolet and A. Odlyzko, Singularity analysis of generating functions, SIAM J. on Algebraic and Discrete Maths, 1990.

Singularity analysis

• Basic series: We know the asymptotic behaviour of the coefficients of certain simple (algebraic-logarithmic) functions, typically

$$S(z) = \frac{1}{(1-t)^{\alpha}} \left(\frac{1}{t} \log \frac{1}{1-t} \right)^{\beta}$$

$$\Rightarrow s(n) = \frac{n^{\alpha-1}}{\Gamma(\alpha)} (\log n)^{\beta} \ \ (1 + O(1/\log n))$$

• Transfer theorems: Under certain hypotheses, if, in the neighborhood of its dominant singularity z_c (supposed to be unique),

$$A(z) = S(z) + O(R(z)) \qquad \text{as } z \to z_c,$$

where $R(z)$ is simple, this estimate can be transfered to coefficients:

$$a(n) = s(n) + O(r(n)) \qquad \text{as } n \to \infty.$$

• Similar statement with $o(R(z))$ and $o(r(n))$ instead of $O(R(z))$ and $O(r(n))$.

Singularity analysis: blind application

- GF of Motzkin numbers:

$$A(z) = \frac{1 - z - \sqrt{(1+z)(1-3z)}}{2z^2}.$$

- Dominant singularity at $z_c = 1/3$. As $z \to z_c$,

$$\begin{aligned} A(z) &= 3 - 3^{3/2}\sqrt{1-3z} + 15/2(1-3z) \quad + O((1-3z)^{3/2}) \\ &= S(z) \quad + O(R(z)) \end{aligned}$$

- "Hence", for n large,

$$\begin{aligned} a(n) &= [z^n]\left(-3^{3/2}\sqrt{1-3z}\right) + O\left([z^n](1-3z)^{3/2}\right) \\ &= -\frac{3^{3/2}}{\Gamma(-1/2)}\, 3^n\, n^{-3/2} + O(3^n n^{-5/2}) \end{aligned}$$

which can also be painfully obtained by analysing the sum

$$a(n) = \sum_k \frac{n!}{k!(k+1)!(n-2k)!}.$$

Singularity analysis: blind application

- GF of column-convex polygons counted by (half-)perimeter:

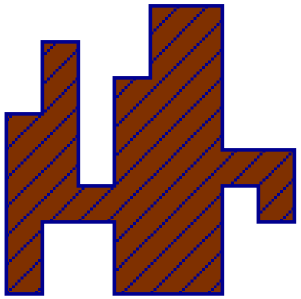
$$A(z) = (1-z)\left(1 - \frac{2\sqrt{2}}{3\sqrt{2} - \sqrt{(1+z)/(1-z)(1-z+\sqrt{1-6z+z^2})}}\right)$$

- Dominant singularity at $z_c = 3-2\sqrt{2}$. As $z \to z_c$,

$$A(z) = c_0 + c_1(1-z/z_c)^{1/2} + o((1-z/z_c)^{1/2})$$

- "Hence"

$$a_n = \frac{c_1}{\Gamma(-1/2)}(3+2\sqrt{2})^n n^{-3/2} + o(z_c^{-n} n^{-3/2})$$



Applying singularity analysis to Gilbert L.

- Permutations with k cycles (k fixed):

$$S_k(z) = \frac{1}{k!}\left(\log\frac{1}{1-z}\right)^k$$

This is a simple algebraic-logarithmic function:

$$\frac{s_k(n)}{n!} \sim \frac{1}{(k-1)!}\; n^{-1}\;\; (\log n)^{k-1}$$

- Derangements:

$$D(z) = \frac{e^{-z}}{1-z} \sim \frac{e^{-1}}{1-z} \qquad \Rightarrow \frac{d(n)}{n!} \sim e^{-1}$$

1. Coefficients of simple algebraic-logarithmic series

The basic series:

$$S(z) = \frac{1}{(1-t)^{\alpha}} \left(\frac{1}{t} \log \frac{1}{1-t} \right)^{\beta}$$

$$\Longrightarrow s(n) = \frac{n^{\alpha-1}}{\Gamma(\alpha)} (\log n)^{\beta} \left(1 + O(1/\log n)\right)$$

Remarks

- if $\alpha = 0, -1, -2, -3, \ldots$, then $1/\Gamma(\alpha) = 0$ and this should be read

$$s(n) = O\left(n^{\alpha-1} (\log n)^{\beta-1}\right)$$

Actually

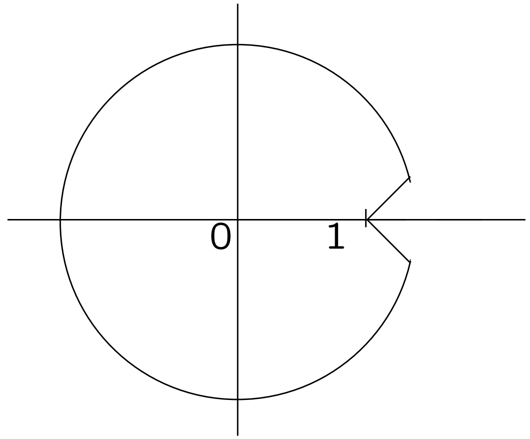
$$[z^n](1-z)^k \left(\frac{1}{t} \log \frac{1}{1-t} \right)^{\beta} = (-1)^k \beta k! n^{-k-1} (\log n)^{\beta-1} \left(1 + O(1/\log n)\right)$$

- The numbers $s(n)$ can always be expanded further.
- Maple does this for you!

2. Transfer theorems: Δ-domains

- If $A(z)$ is analytic in a Δ-domain, and

$$A(z) = O(R(z)) \quad \text{as } z \to 1,$$



where $R(z)$ is a simple alg-log function, then

$$a(n) = O(r(n)) \quad \text{as } n \to \infty.$$

- Similarly:

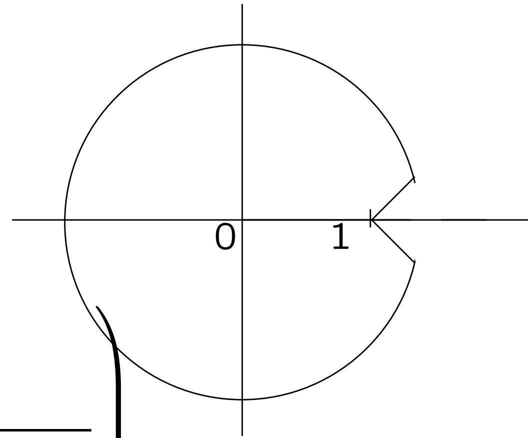
$$A(z) = o(R(z)) \Rightarrow a(n) = o(r(n))$$

$$A(z) \sim R(z) \Rightarrow a(n) \sim r(n)$$

[Flajolet-Odlyzko 90]

Examples of series analytic in a Δ-domain

$$D(z) = \frac{e^{-z}}{1-z}$$



$$C(z) = (1-z)\left(1 - \frac{2\sqrt{2}}{3\sqrt{2} - \sqrt{(1+z)/(1-z)(1-z+\sqrt{1-6z+z^2})}}\right)$$

Counter-examples:

$$A(z) = \frac{e^{z}}{\sqrt{1-z^2}}$$ two dominant singularities

$$P(z) = \frac{1}{(1-z)(1-z^2)(1-z^3)\cdots}$$ natural boundary

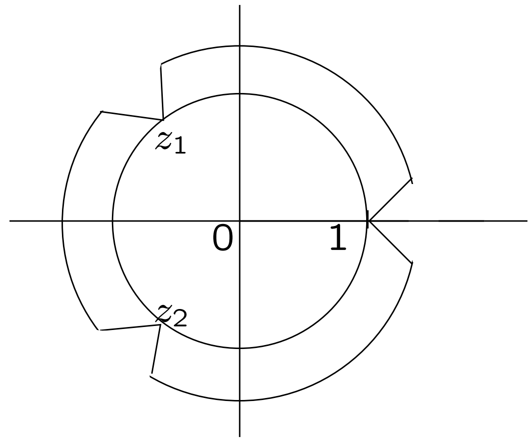
Several dominant singularities

• If $A(z)$ is analytic in a generalized Δ-domain, and, in the neighborhood of each dominant singularity z_i,

$$A(z) = H_i(z) + O((1 - z/z_i)^{\alpha}) \qquad \text{as } z \to z_i,$$

where H_i is analytic around z_i, then

$$a(n) = O(n^{\alpha - 1}) \qquad \text{as } n \to \infty.$$



The contributions of the dominant singularities add up

Several dominant singularities: examples

Let

$$A(z) = \frac{e^z}{\sqrt{1-z^2}}$$

There are two dominant singularities, ± 1.

$$A(z) = \frac{e}{\sqrt{2(1-z)}} + O(\sqrt{1-z}) \qquad \text{as } z \to 1,$$

$$A(z) = \frac{e^{-1}}{\sqrt{2(1+z)}} + O(\sqrt{1+z}) \qquad \text{as } z \to -1.$$

Hence

$$a(n) = \frac{e}{\sqrt{2\pi n}} + \frac{(-1)^n e^{-1}}{\sqrt{2\pi n}} + O(n^{-3/2}).$$

Maple does this for you! [Algolib] http://algo.inria.fr/libraries/

Implicit generating functions

The generating function of labelled rooted trees is given by

$$A(z) = z\,\mathrm{exp}(A(z))$$

- The only dominant singularity is at $z_c = 1/e$, and $A(z_c) = 1$.
- $A(z)$ is analytically defined in a Δ-domain
- Singular behaviour of $A(z)$.

$$A(z) = 1 - \sqrt{2}\sqrt{1-ze} + o(\sqrt{1-ze}) \qquad \text{as } z \to 1/e$$

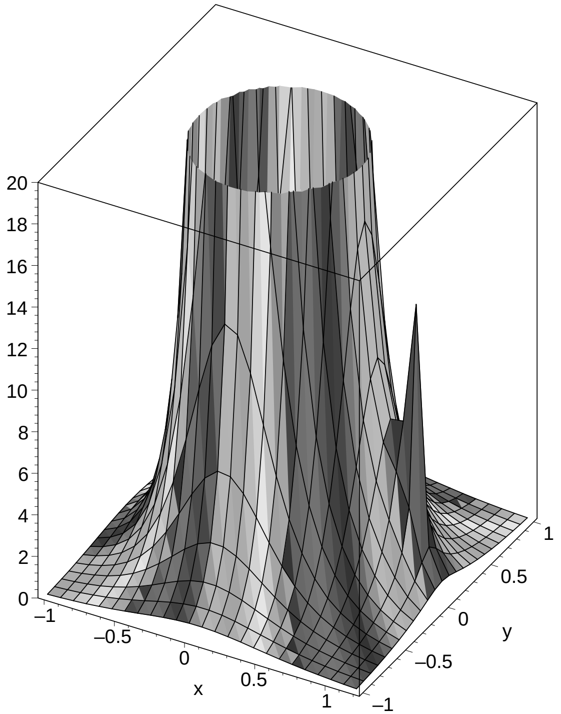
Hence

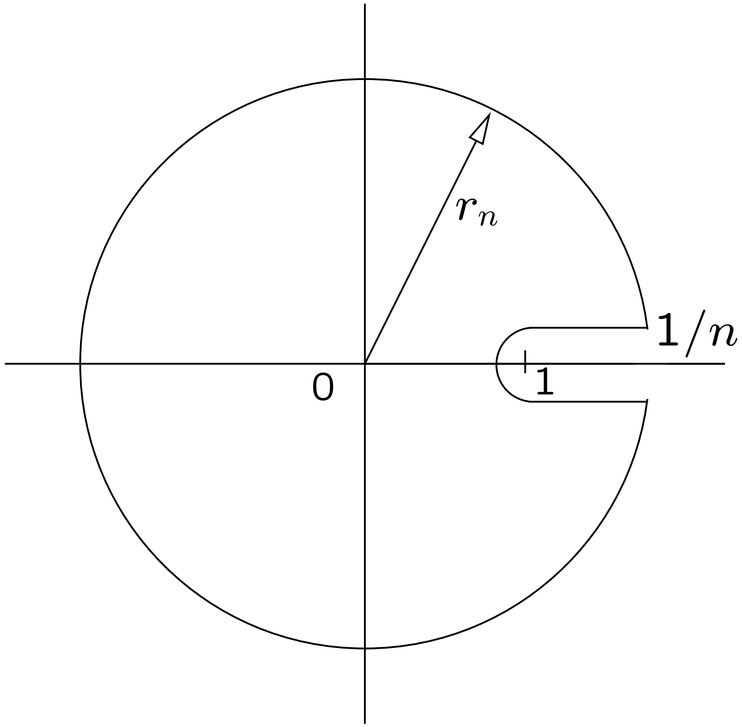
$$\frac{a(n)}{n!} \sim \frac{-\sqrt{2}}{\Gamma(-1/2)} e^n n^{-3/2} \sim \frac{e^n n^{-3/2}}{\sqrt{2\pi}}$$

Proofs of the asymptotics of basic series: The "right" integration contour

The basic series:

$$S(z) = \frac{1}{(1-t)^\alpha}\left(\frac{1}{t}\log\frac{1}{1-t}\right)^\beta \qquad \rightarrow \qquad s(n) = \frac{1}{2i\pi}\int_{\mathcal{C}} S(z)\frac{dz}{z^{n+1}}$$



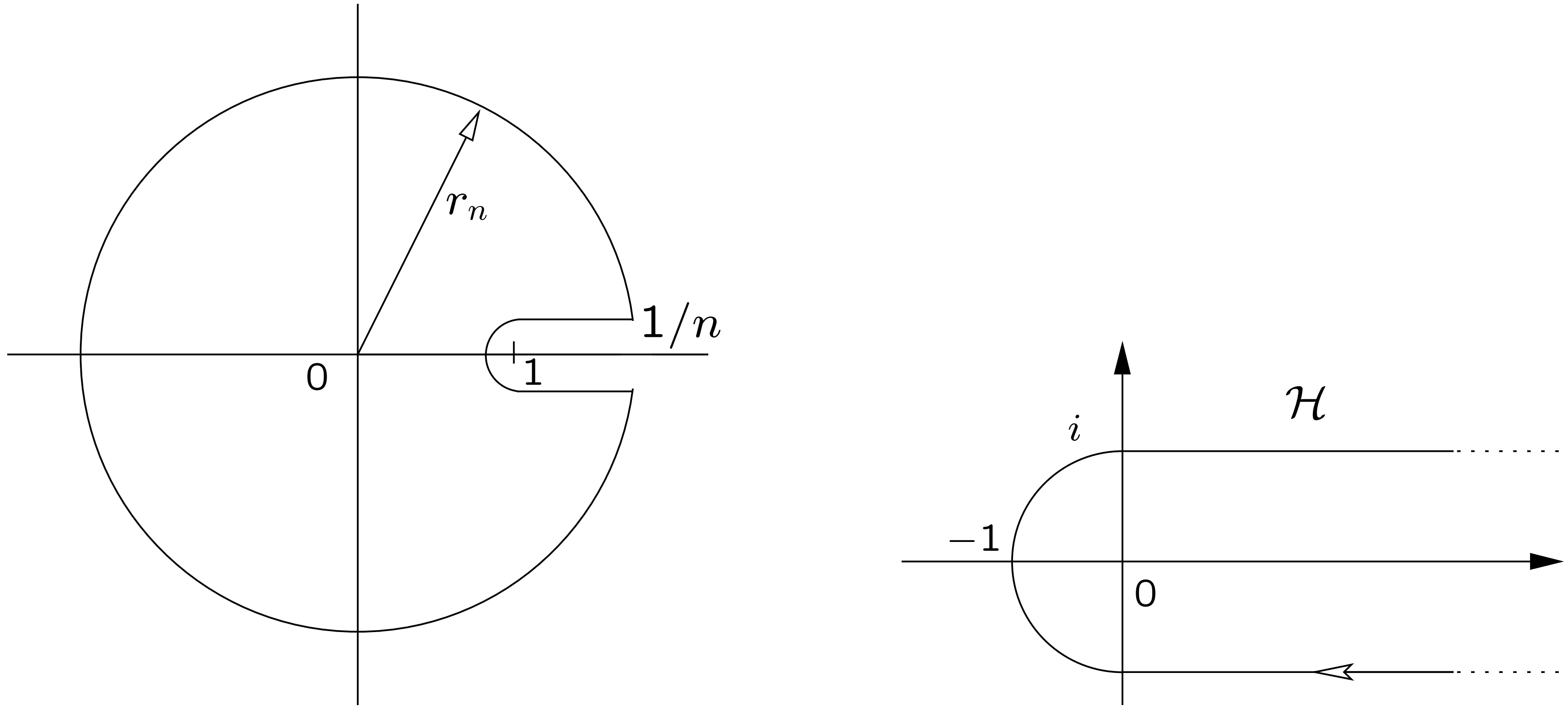


$S(z) = (1-z)^{-2/3}$ and $n = 4$ $\qquad\qquad$ $r_n = 1 + \log^2 n/n$

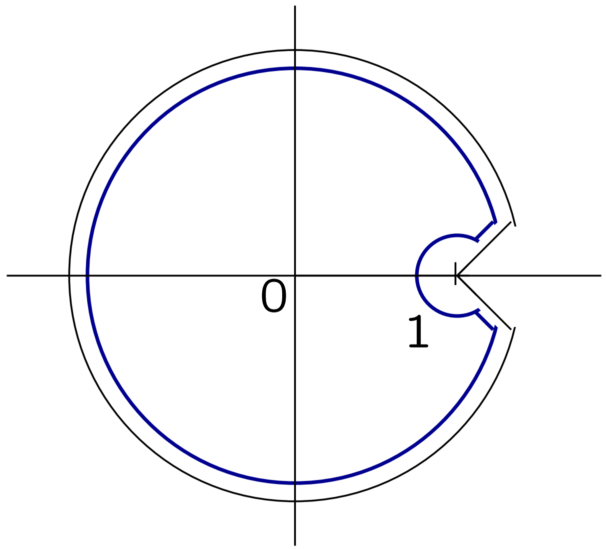
Why the Gamma function?

Hankel's expression of the reciprocal of the Gamma function:

$$\frac{1}{\Gamma(s)} = \frac{1}{2i\pi} \int_{\mathcal{H}} (-z)^{-s} e^{-z} dz$$



Proof of the transfer theorems: The “right” integration contour



Third method: Saddle point asymptotics

• Ph. Flajolet and R. Sedgewick, Analytic Combinatorics
http://algo.inria.fr/flajolet/Publications/books.html
Chapter VIII

Saddle point bounds

- Let $A(z) = \sum_n a(n) z^n$ be a power series with non-negative coefficients and radius of cv. ρ. Then

$$a(n) \leq \frac{A(x)}{x^n} \qquad \text{for all } 0 < x < \rho$$

- Let $r = x$ be chosen so as to minimize $A(x)/x^n$:

$$\frac{rA'(r)}{A(r)} = n \qquad \text{(saddle-point equation)}$$

Then

$$a(n) \leq \frac{A(r)}{r^n}.$$

Saddle point bounds

- Let $r = x$ be chosen so as to minimize $A(x)/x^n$:

$$\frac{rA'(r)}{A(r)} = n \quad \text{(saddle-point equation)}$$

Then

$$a(n) \le \frac{A(r)}{r^n}.$$

Example: the exponential

Take $A(z) = e^z = \sum z^n/n!$. Then $A'(z) = e^z = A(z)$ and $r = n$

$$\frac{1}{n!} \le \left(\frac{e}{n}\right)^n$$

Cf. Stirling's formula

$$\frac{1}{n!} = \left(\frac{e}{n}\right)^n \frac{1}{\sqrt{2\pi n}} (1 + O(1/n))$$

Saddle point estimates

We have seen

$$a(n) \le \frac{A(r)}{r^n} \quad \text{with} \quad \frac{rA'(r)}{A(r)} = n.$$

Theorem. Under suitable conditions (...)

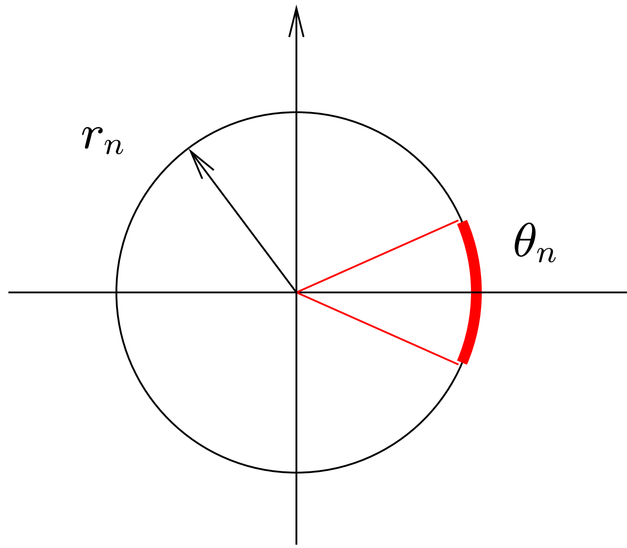
$$a(n) \sim \frac{A(r)}{r^n\sqrt{2r^2\pi G''(r)}}$$

with $G(z) = \log \frac{A(z)}{z^n}$

Proof: integration on a circle

$$\frac{A(z)}{z^n} = \exp\left(G(z)\right)$$

$$a(n) = \frac{1}{2i\pi}\int_{\mathcal{C}} \frac{A(z)}{z^n}\frac{dz}{z} \quad = \frac{1}{2\pi}\int_{-\pi}^{\pi} \exp(G(re^{i\theta}))d\theta$$



What are the suitable conditions?

- The part of integral before $-\theta_n$ and after θ_n is negligible
- For $z = r_n e^{i\theta}$, with $\theta \in [-\theta_n, \theta_n]$, the quadratic approximation holds:

$$G(z) = G(r_n) - \frac{1}{2}\, r_n^2\, \theta^2\, G''(r_n) + o(1)$$

- θ is large enough to complete the integral to a Gaussian integral

$$r_n^2\, \theta_n^2\, G''(r_n) \to \infty$$

Examples

- Involutions

$$A(z) = \exp(z + z^2/2)$$

- Bell numbers (set partitions)

$$A(z) = \exp(e^z - 1)$$

- A fast growing singular function

$$A(z) = \exp\left(\frac{z}{1-z}\right)$$

- Integer partitions

$$A(z) = \frac{1}{(1-z)(1-z^2)(1-z^3)\cdots}$$

Maple does this for you... [Algolib]

Applications of singularity analysis: Automated asymptotics for algebraic generating functions

Def. The series $A(t)$ is algebraic if there exists a non-trivial polynomial $P(\cdot,\cdot)$ such that $P(t, A(t)) = 0$

Example: Dyck paths: $A(t) = 1 + t^2 A(t)^2$

• Ph. Flajolet and R. Sedgewick, Analytic Combinatorics, Chapter VII http://algo.inria.fr/flajolet/Publications/books.html

Algebraic series via singularity analysis

Let $A(t)$ be a solution of $P(A(t)) = 0$.

• The singularities of $A(t)$ are found among:
– the roots of the dominant coefficient of P
– the roots of the discriminant of P

• In the neighborhood of a singularity z_c, $A(z)$ admits a local expansion of the form

$$A(z) = \sum_{k \geq k_0} b_k (1 - z/z_c)^{k/d}$$

with $k_0 \in \mathbb{Z}$ and $d \in \{1, 2, \ldots\}$

• Singularity analysis:

$$a(n) \sim b_{k_0} z_c^{-n} \frac{n^{k_0/d-1}}{\Gamma(k_0/n_d)}$$

Applications: (almost) automated asymptotics for D-finite generating functions

Def. The series $A(t)$ is D-finite if its satisfies a linear differential equation with polynomial coefficients:

$$P_e(t)A^{(e)}(t) + \cdots + P_1(t)A'(t) + P_0(t)A(t) = 0$$

Equivalently, the sequence $a(n)$ satisfies a linear recurrence relation with polynomial coefficients

$$Q_0(n)a(n) + Q_1(n)a(n-1) + \cdots + Q_d(n)a(n-d) = 0$$

• Ince, E. L. Ordinary Differential Equations. Dover Publications, New York, 1944.

• Wimp, J.; Zeilberger, D. Resurrecting the asymptotics of linear recurrences. J. Math. Anal. Appl. 111 (1985), no. 1, 162–176.

Singular behaviour of D-finite series

Let $A(t)$ be D-finite:

$$P_e(t)A^{(e)}(t) + \cdots + P_1(t)A'(t) + P_0(t)A(t) = 0$$

• The singularities of A are found among the roots of $P_e(t)$

• In the neighborhood of a regular root z_c of P_e, the solutions of the ODE have a regular local expansion formed of terms

$$(1 - z/z_c)^\sigma \log\left(\frac{1}{1 - z/z_c}\right)^i$$

• In the neighborhood of a irregular root z_c of P_e, the expansions may involve terms of the form

$$\exp(P(1/w))R(w)$$

where R is regular, P is a polynomial and $w = (1 - z/z_c)^{1/d}$