Exercises on enumeration and asymptotics

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1 1D walks

1. Use generating functions to prove that the number of n-step walks with steps ± 1 that start from 0 and always remain non-negative is

$$a(n) = \binom{n}{\lfloor n/2 \rfloor}.$$

Give a bijective proof of this result, given that $a(n)$ also counts walks ending at 0 (if n is even) and 1 (of n is odd).

2. We recall Stirling's formula:

$$n! \left(\frac{n}{e}\right)^n \sqrt{2\pi n}\,(1 + O(1/n))$$

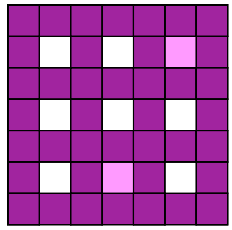
Derive from this estimate that

- the probability that an n-step random walk remains non-negative is asymptotic to $\sqrt{2/(\pi n)}$,
- the probability that a $2n$-step random walk ends at 0 is asymptotic to $1/\sqrt{\pi n}$,
- the probability that a $2n$-step random walk remains non-negative and ends at 0 is asymptotic to $1/\sqrt{\pi n^3}$.

2 Polyominoes by perimeter

Let $P(t) = \sum_n p(n) t^n$ denote the GF of polyominoes, counted by (half-)perimeter. By starring at the Figure below, prove that the radius of convergence of $P(t)$ is 0. Conclude that for all $r > 0$, and n large enough,

$$p(n) > r^n.$$



3 Bivariate generating functions and probabilistic distributions

Let $A(t,u)$ be the ordinary generating function of a class $\mathcal{A}$ of objects, counted by their size $|\cdot|$ (variable t) and another statistic s (variable u). Let S_n denote the random value of $s(a)$, when a is taken uniformly among objects of $\mathcal{A}$ of size n. Work out the links between $A(t,u)$ and: the moments of S_n, its probability generating function, Laplace transform, Fourier transform (or characteristic function), respectively defined by

$$L(x) = \mathbb{E}(e^{xS_n}) \quad \text{and} \quad \Phi(x) = \mathbb{E}(e^{ixS_n}).$$

Same questions when we start from the exponential generating function

$$\tilde{A}(t,u) = \sum_{a\in\mathcal{A}} \frac{t^{|a|}}{|a|!} u^{s(a)}.$$

4 A phase transition in rectangles

Consider a random rectangle of perimeter $2n$, taken with probability

$$\text{prob}(p) = \frac{1}{Z_n(u)} u^{a(p)}$$

where $a(p)$ is the area of the rectangle p and

$$Z_n(u) = \sum_{|p|=2n} u^{a(p)} = \sum_{i=1}^{n-1} u^{i(n-i)}$$

counts rectangles of perimeter $2n$ by their area. Let A_n denote the average area of these rectangles.

1. Express the average area $\mathbb{E}(A_n)$ in terms of Z_n.
2. For $u = 1$, what are $Z_n(u)$ and $\mathbb{E}(A_n)$?
3. With elementary arguments, give an equivalent of $Z_n(u)$ for $u < 1$.
4. Same question when $u > 1$
5. Give an equivalent of $\mathbb{E}(A_n)$ for $u < 1$ and $u > 1$.

5 Generating non-negative walks by the rejection method

The principle of this generation algorithm is to generate a random walk by tossing an unbiased coin n times, and to keep the resulting n-step random walk only if it is non-negative. Moreover, we decide to reject the walk as soon as it reaches level -1. Study the average number of tosses required to find a non-negative walk.

6 Asymptotic analysis of sums

Consider the Motzkin numbers, defined by

$$M_n = \sum_k \frac{n!}{k!(k+1)!(n-2k)!}.$$

Prove that

$$M_n \sim \frac{3^{3/2}}{2\sqrt{\pi}} \, 3^n \, n^{-3/2}.$$

7 2D lattice walks

• **General walks.** We consider n-step walks on the square lattice, starting from the origin. Of course, the total number of such walks is 4^n.

Prove that the number of such walks ending at the origin is $\binom{n}{n/2}^2$. Such walks only exist when n is even. (Hint: project the walk on the diagonal axes.) Asymptotics?

• **Diagonal half-plane.** Prove that the number of n-step walks that always stay on or above the diagonal $x+y=0$ is $2^n\binom{n}{\lfloor n/2 \rfloor}$. (Hint: use the projection on diagonals, and the exercise on 1D walks. Alternatively, use generating functions.) Asymptotics?

For n even, (prove that...) the number of such walks that end at the origin is $\binom{n}{n/2}C_{n/2}$, where C_k is the kth Catalan number. Asymptotics?

• **Upper half-plane.** The number of n-step walks that always stay on or above the line $y=0$ is $\binom{2n+1}{n}$. Compare the asymptotics with what you got for the diagonal half-plane, and "observe universality"...

For n even, the number of such walks ending at the origin is $C_{n/2}\binom{n+1}{n/2}$. Compare the asymptotics with what you got for the diagonal half-plane, and "observe universality"...

• **Tilted quarter plane.** The number of n-step walks that always stay on or above the line $x+y=0$ and on or below the line $x=y$ is $\binom{n}{\lfloor n/2 \rfloor}^2$. Asymptotics?

For n even, the number of such walks ending at the origin is $C_{n/2}^2$. Asymptotics?

• **Ordinary quarter plane.** The number of n-step walks that always stay on or above the line $y = 0$ and on or to the right of the line $x = 0$ is $\binom{n}{\lfloor n/2 \rfloor}\binom{n+1}{\lfloor (n+1)/2 \rfloor}$. Compare the asymptotics with what you got for the tilted quarter plane, and "observe universality"...

For n even, the number of such walks ending at the origin is $C_{n/2}C_{n/2+1}$. Compare the asymptotics with what you got for the tilted quarter plane, and "observe universality"...

8 An explicit expansion

For $k \in \mathbb{N}$ and n large enough,

$$[t^n](1-t)^k \log \frac{1}{1-t} = \frac{(-1)^k k!}{n(n-1)\cdots(n-k)}.$$

9 Walks on the slit plane

The length generating function of 2D walks that start from $(0,0)$, but then never hit the half-line $y = 0, x \leq 0$, is

$$A(z) = \frac{(1+\sqrt{1+4z})^{1/2}(1+\sqrt{1-4z})^{1/2}}{2(1-4z)^{3/4}}.$$

It is not asked to prove this (but you are welcome to try!). Using analysis of singularities, determine the asymptotic behaviour of the number of n-step walks of this type.

10 A discrete limit law for Cayley trees

Recall that the exponential generating function of rooted labelled trees (Cayley trees), counted by the number of vertices, satisfies $T(z) = z\exp(T(z))$. As a complex function of z, the series $T(z)$ has a unique singularity, at $z = 1/e$, with a local expansion

$$T(z) = 1 - \sqrt{2}\sqrt{1-ze} + O(1-ze).$$

Express the series $T(z,u)$ that counts these trees by the number of vertices and the degree of the root. Using analyis of singularities, and the continuity theorem for probability generating functions, prove that the (random) degree of the root in a tree of size n converges in distribution to the discrete law given by

$$\mathbb{P}(S = k) = \frac{1}{e(k-1)!}.$$

11 Contacts in bilateral Dyck paths

The generating function of bilateral Dyck paths (1D walks ending at 0) counted by their half-length and number of visits at 0 is

$$A(t,u) = \frac{u}{1-2uP(t)} \quad \text{with} \quad P(t) = \frac{1-\sqrt{1-4t}}{2}.$$

Let C_n denote the (random) number of contacts of a $2n$-step bilateral Dyck path. For $j \geq 0$, give an explicit expression of the generating function of the sequence

$$\binom{2n}{n}\mathbb{E}(C_n(C_n-1)\dots(C_n-j+1)).$$

Using analysis of singularities, derive from this expression that, as $n \to \infty$,

$$\mathbb{E}\left(\left(\frac{C_n}{\sqrt{n}}\right)^j\right) \to j!\,\frac{\Gamma(1/2)}{\Gamma(j+1)/2}.$$

Show that m_j is the jth moment of the Rayleigh law of density $x/2\exp(-x^2/4)$ for $x > 0$.

12 Saddle point analysis of Bell numbers

A standard! The Bell numbers are defined by

$$\exp(e^t - 1) = \sum_{n \geq 0} B_n \frac{t^n}{n!}.$$

Show that

$$\frac{B_n}{n!} \sim \frac{\exp(n/r)}{er^n\sqrt{2\pi rn}}$$

where $r \equiv r(n)$ is given by $re^r = n$.