Deux Problèmes
Ecole Combinatoire
et mécanique statistique

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arbres, chemins et polyominos:

un problème avec 4^n

remarque:
nécessite la notion de factorisation de Catalan
et de mots de Catalan

Using the two following bijections:

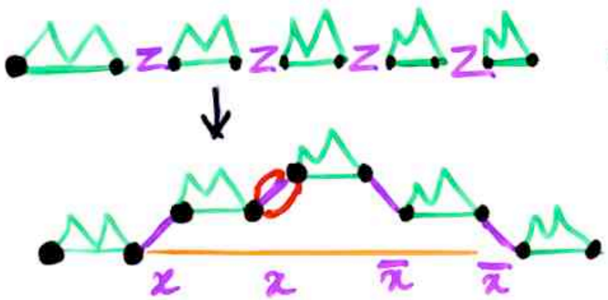
- $\theta : f \longrightarrow (\chi(f), \hat{f})$ from Catalan factorization

  $|f| = n$, $f \in \{x, \bar{x}\}^*$ $\quad 0 \leq \chi(f) \leq |\hat{f}|_z$ $\quad$ Catalan word $|\hat{f}| = n$

- $\rho : g \longrightarrow (s, w)$

  Catalan word $|g| = 2n$ $\quad 0 \leq s \leq n$ $\quad$ Dyck word of length $2n$

notions
à expliquer
dans le cours

ρ 

$= g$

$= w$

- if $|g|_z \neq 0$

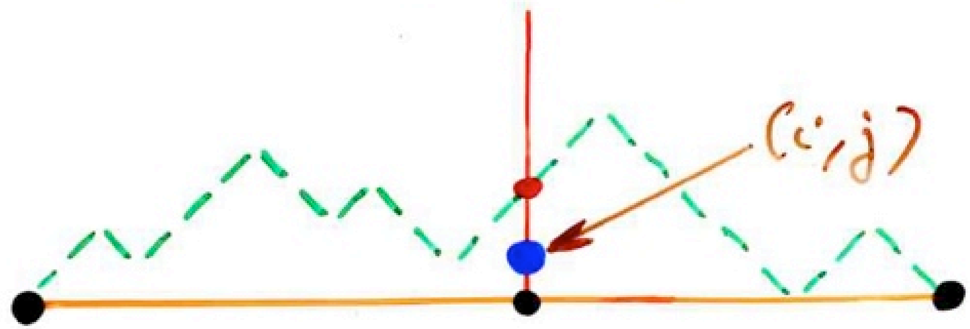
  $s =$ index of among the x's of w

  $1 \leq s \leq n$

- if $|g|_z = 0$, then $\rho(g) = (0, g)$

a) give a bijection between the words of $\{x, \bar{x}\}^{2n}$ and pointed Dyck paths

i.e. the pair $((i,j), \omega)$ where ω is a Dyck path of length $2n$ and (i,j) is a point of $\mathbb{N}\times\mathbb{N}$ "below" ω, as in fig:

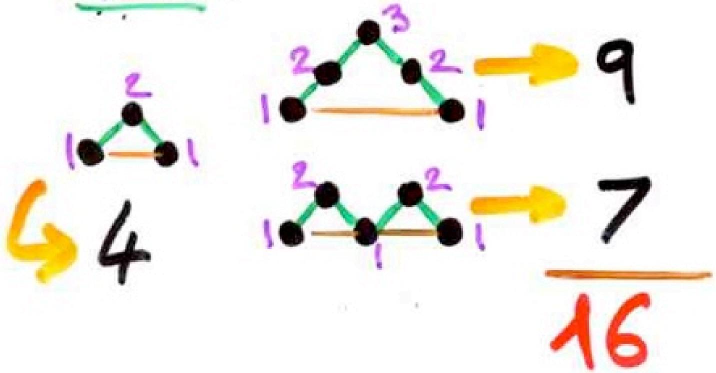


- Thus you have proved

$$\sum_{\substack{|\omega|=2n \\ \text{Dyck paths}}} \mathrm{Area}(\omega) = 4^n$$

where $\mathrm{Area}(\omega)$ = total nb of points of $\mathbb{N}\times\mathbb{N}$ "below" the path.

check:



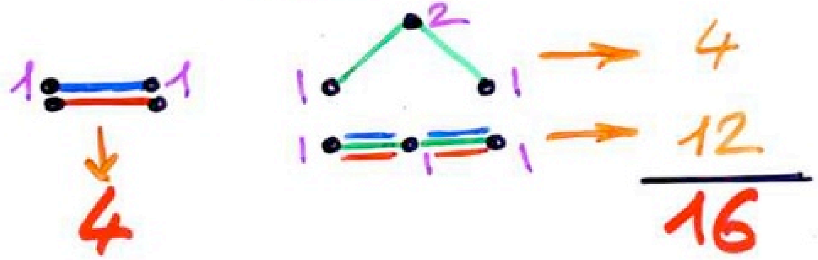
$\Rightarrow$ average area $\sim n^{3/2}$

– **Rem.** An analogous proof for 2-col. Motzkin paths shows that

$$\sum_{\omega} \text{Area}(\omega) = 4^n$$

2-col Motzkin path

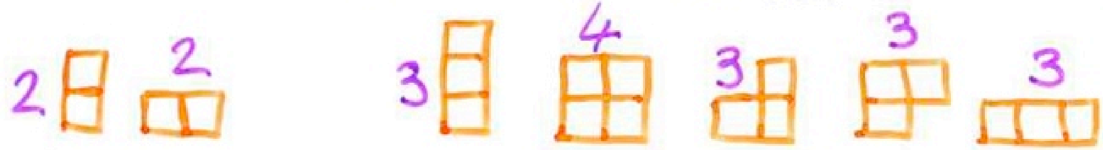
$|\omega| = n$



Cor. $$\sum_{P} \text{area}(P) = 4^{n-1}$$

stair case polygons q-Polya

perimeter $2n+2$



4 16

$\Downarrow$ average area $\sim n^{3/2}$

Schwarzler, in paper Fürlinger, Hofbauer q-Catalan JCTA, 40 (1985) 248–264

D. Rogers, L. Shapiro, W-J. Woan (1996)

b) Using the bijection between

Dyck path $\longleftrightarrow$ complete binary tree

$|\omega| = 2n$ $\qquad$ $|B| = 2n+1$

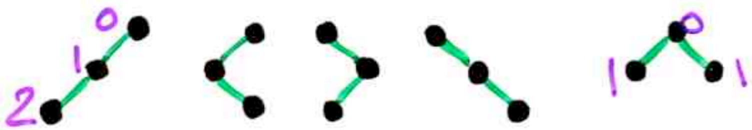
deduce the following fact, classical in computer science:

define the (internal) path length of a binary tree B as the sum of the height of all the vertices s of B.

Then deduce:

$$\text{average path length (for binary tree } |B| = n) = \frac{4^n}{C_n} - (3n+1)$$

check:



path length → 3 3 3 3 2 $\xrightarrow{\text{total}}$ 14

average $= \frac{14}{5}$

$$\frac{4^3}{C_3} - (3\times 3+1) = \frac{64}{5} - 10 = \frac{14}{5} \quad \text{(ouf!)}$$

problème:

démonstration que les racines des polynômes de couplages des graphes sont des nombres réels en utilisant la notion de chemins arborescents et d'empilements de pièces

Tree-like paths on a graph and matching polynomial.

G graph (finite, non-oriented)

<u>Def</u>- Matching polynomial of G.

$$M(G) = \sum_{\substack{\alpha \\ \text{matching}}} (-1)^{|\alpha|} x^{\mathrm{ip}(\alpha)}$$

(or $M(G;x)$)

<u>remark</u>- $\underset{\text{reciprocal}}{M^*(G)} = \sum_{\alpha} (-x^2)^{|\alpha|}$

Def- Tree-like path on $G = (V, E)$
vertices, edges

$\omega = (s_0, s_1, \dots, s_n)$ with $s_i \in V$,
is a tree-like path iff: for every $i = 1, \dots, n-1$,
define by recurrence $\text{Stone}_i(\omega) = \eta_i$,
self-avoiding path going from s_0
to s_i, such that $\eta_0 = (s_0)$
and for $i+1$:

$$\begin{cases} \bullet \text{ if } s_{i+1} \notin \eta_i, \text{ then } \eta_{i+1} = (\eta_i, s_{i+1}) \\ \bullet \text{ else } s_{i+1} = x_{j_i - 1}, \text{ with} \end{cases}$$

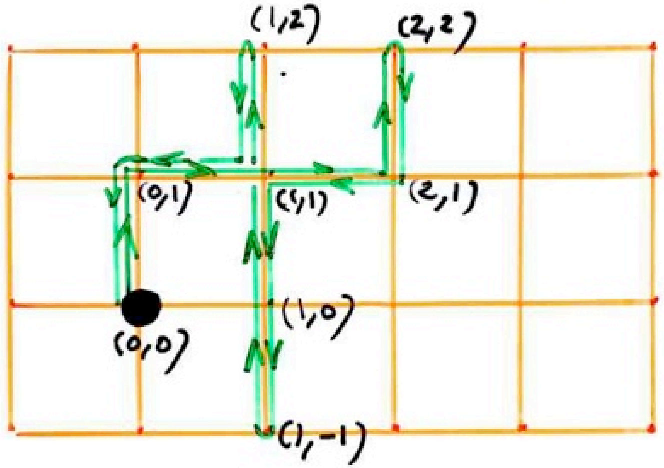
$$\eta_i = (s_0 = x_0, \dots, x_{j_i - 1}, x_{j_i} = s_i).$$

$$\eta_{i+1} = (s_0 = x_0, \dots, x_{j_i - 1} = s_{i+1})$$

examples: – projections of Dyck path (on $\mathbb{N}$)
– bilateral Dyck path (on $\mathbb{Z}$)

examples: Path on a tree.

example: $G = \mathbb{Z}^2$ square-lattice.



$$\omega = \{ (0,0)\ (0,1)\ (1,1)\ (2,1)\ (2,2)\ (2,1)\ (1,1)$$
$$(1,0)\ (1,-1)\ (1,0)\ (1,1)\ (1,2)\ (1,1)$$
$$(0,1)\ (0,0) \}$$

a) Find a bijection between the set $TLP_n(G, s_0)$ of tree-like paths on G, going from s_0 to s_0, of length $2n$, and the set $Pyr_n(G, s_0)$ of pyramids P of "dimers" on G such that

$$s_0 \in \pi(\max(P))$$

(projection of the maximal piece of P)

b) Deduce that the generating function for tree-like paths on G, is:

$$\sum_{n \geq 0} a_n t^{2n} = \frac{M^*(G - s_0; t)}{M^*(G; t)},$$

where $a_n = |TLP_n(G, s_0)|$.

c) For (G, s_0), find a tree (T, r_0) (r_0: root of the tree), such that the set $TLP_n(G, s_0)$ is in bijection with the set of paths ω on T, of length $2n$, starting and ending in r_0.

d) Deduce an expression for the generating function of tree-like paths on G in term of the adjacency matrix of T (considered as a graph)

e) Deduce that for any graph G, the zeros of the matching polynomial $M(G;x)$ are real numbers