Fluid and Diffusion Models of an M–designed Call Center with Retrials

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Outline

1. Model of an M–designed call center with retrials

2. Approximation via a deterministic fluid process

3. Refinement to a diffusion model

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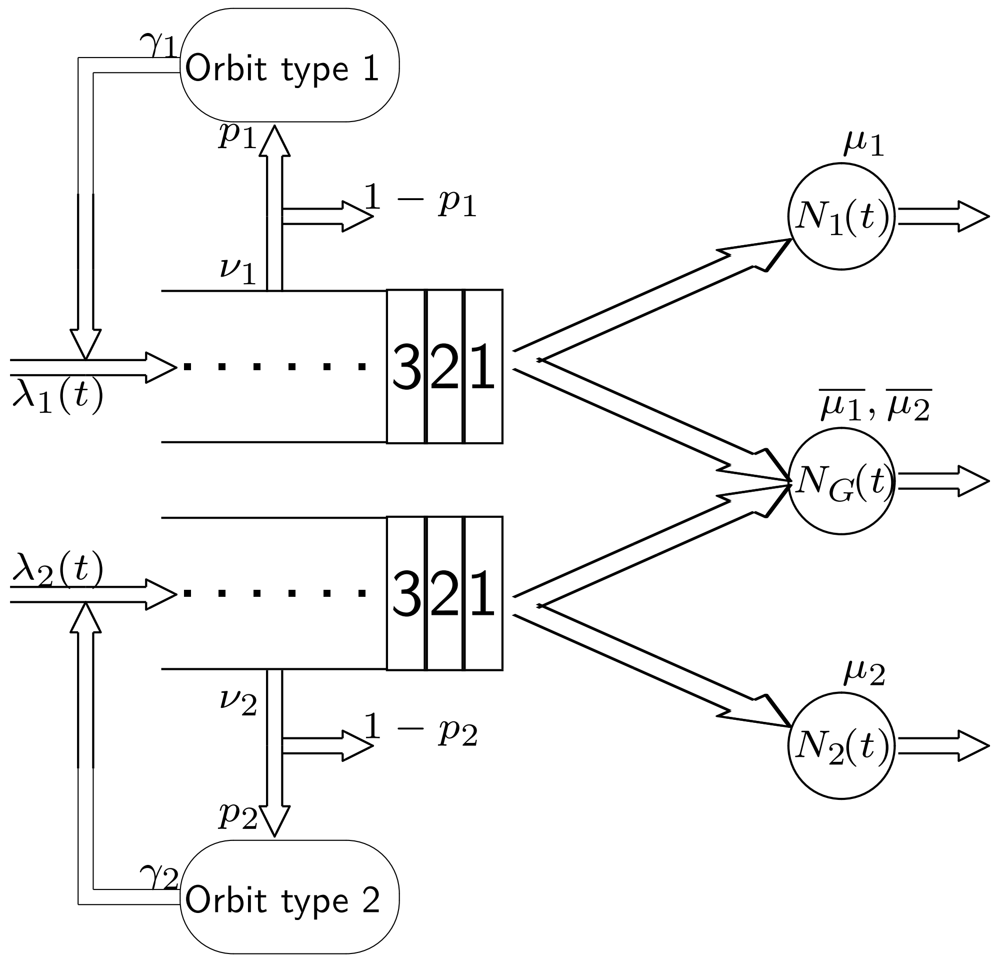
1. Model of an M–designed call center with retrials



Assumptions

- Customers arrive according to independent, non–homogeneous Poisson–processes with rates $\lambda_1(t)$ and $\lambda_2(t)$

- $N_1(t), N_2(t)$, and $N_G(t)$ are the time–dependent numbers of available agents

- Exponentially distributed service times with rates μ_1, μ_2, $\overline{\mu_1}$, and $\overline{\mu_2}$.

- Preemtive priority of type–1–customers

- Exponentially distributed times–to–abandon with rates ν_1 and ν_2

Assumptions
continued

- Impatient customers recall with probability p_1 and p_2 after exponentially distributed times–in–orbit with rates γ_1 and γ_2

- Infinite waiting room

2. Approximation via a deterministic fluid process

- $\boldsymbol{Q(t)} = (Q_1(t), Q_2(t), Q_{\mathcal{O}1}(t), Q_{\mathcal{O}2}(t))_{t \in \mathbb{R}_0^+}$ denotes the number of customers in the system and the orbits

- $\boldsymbol{Q^F(t)}$ is the fluid–process vector refering to the stochastic process vector

- The processes are characterized through a system of nonlinear differential equations

- **Justification:** Strong law of large number applied to a scaled version of $\boldsymbol{Q(t)}$ [1]

[1] see A. Mandelbaum, W. Massey, and M. Reiman, Strong Approximations for Markovian Service Networks, *QUESTA*, 30, 149-201, November 1998

Differential equations for the population in system

$$\frac{d}{dt}\boldsymbol{Q_1(t)} = \boldsymbol{\lambda_1(t)} - \boldsymbol{\nu_1}\left\{\boldsymbol{Q_1(t)} - \boldsymbol{N_1(t)} - \boldsymbol{N_G(t)}\right\}^+ - \boldsymbol{\mu_1}\min\left\{\boldsymbol{Q_1(t)}, \boldsymbol{N_1(t)}\right\}$$
$$- \overline{\boldsymbol{\mu_1}}\min\left\{\boldsymbol{N_G(t)}, \left\{\boldsymbol{Q_1(t)} - \boldsymbol{N_1(t)}\right\}^+\right\} + \boldsymbol{\gamma_1(t)Q_{\mathcal{O}1}(t)}$$

$$\frac{d}{dt}\boldsymbol{Q_2(t)} = \boldsymbol{\lambda_2(t)} - \boldsymbol{\nu_2}\left\{\boldsymbol{Q_2(t)} - \boldsymbol{N_2(t)} - \left\{\boldsymbol{N_G(t)} - \left\{\boldsymbol{Q_1(t)} - \boldsymbol{N_1(t)}\right\}^+\right\}^+\right\}^+$$
$$- \boldsymbol{\mu_2}\min\left\{\boldsymbol{Q_2(t)}, \boldsymbol{N_2(t)}\right\} + \boldsymbol{\gamma_2(t)Q_{\mathcal{O}2}(t)}$$
$$- \overline{\boldsymbol{\mu_2}}\min\left\{\left\{\boldsymbol{Q_2(t)} - \boldsymbol{N_2(t)}\right\}^+, \left\{\boldsymbol{N_G(t)} - \left\{\boldsymbol{Q_1(t)} - \boldsymbol{N_1(t)}\right\}^+\right\}^+\right\}$$

$$\frac{d}{dt}\boldsymbol{Q_{\mathcal{O}1}(t)} = \boldsymbol{p_1\nu_1}\left\{\boldsymbol{Q_1(t)} - \boldsymbol{N_1(t)} - \boldsymbol{N_G(t)}\right\}^+ - \boldsymbol{\gamma_1(t)Q_{\mathcal{O}1}(t)}$$

$$\frac{d}{dt}\boldsymbol{Q_{\mathcal{O}2}(t)} = \boldsymbol{p_2\nu_2}\left\{\boldsymbol{Q_2(t)} - \boldsymbol{N_2(t)} - \left\{\boldsymbol{N_G(t)} - \left\{\boldsymbol{Q_1(t)} - \boldsymbol{N_1(t)}\right\}^+\right\}^+\right\}^+$$
$$- \boldsymbol{\gamma_2(t)Q_{\mathcal{O}2}(t)},$$

for all $\boldsymbol{t} \in \mathbb{R}_{\boldsymbol{0}}^+$ where $\{X\}^+$ is the maximum of 0 and X.

3. Refinement to a diffusion model

Assumption:

$$\boldsymbol{Q}^n(\boldsymbol{t}) \stackrel{d}{=} n\boldsymbol{Q}^{\boldsymbol{F}}(\boldsymbol{t}) + \sqrt{n}\boldsymbol{Q}^{\boldsymbol{D}}(\boldsymbol{t}) + o\big(\sqrt{n}\big)$$

Justification: The functional central limit theorem states

$$\lim_{n\to\infty} \sqrt{n} \left[\frac{1}{n}\boldsymbol{Q}^n(\boldsymbol{t}) - \boldsymbol{Q}^{\boldsymbol{F}}(\boldsymbol{t})\right] \stackrel{d}{=} \boldsymbol{Q}^{\boldsymbol{D}}(\boldsymbol{t})$$

Consequence: $\boldsymbol{Q}^{\boldsymbol{D}}(\boldsymbol{t})$ is a solution to a system of functional equations and Gaussian processes if the set of critical times is negligible.

Functional equation in matrix form

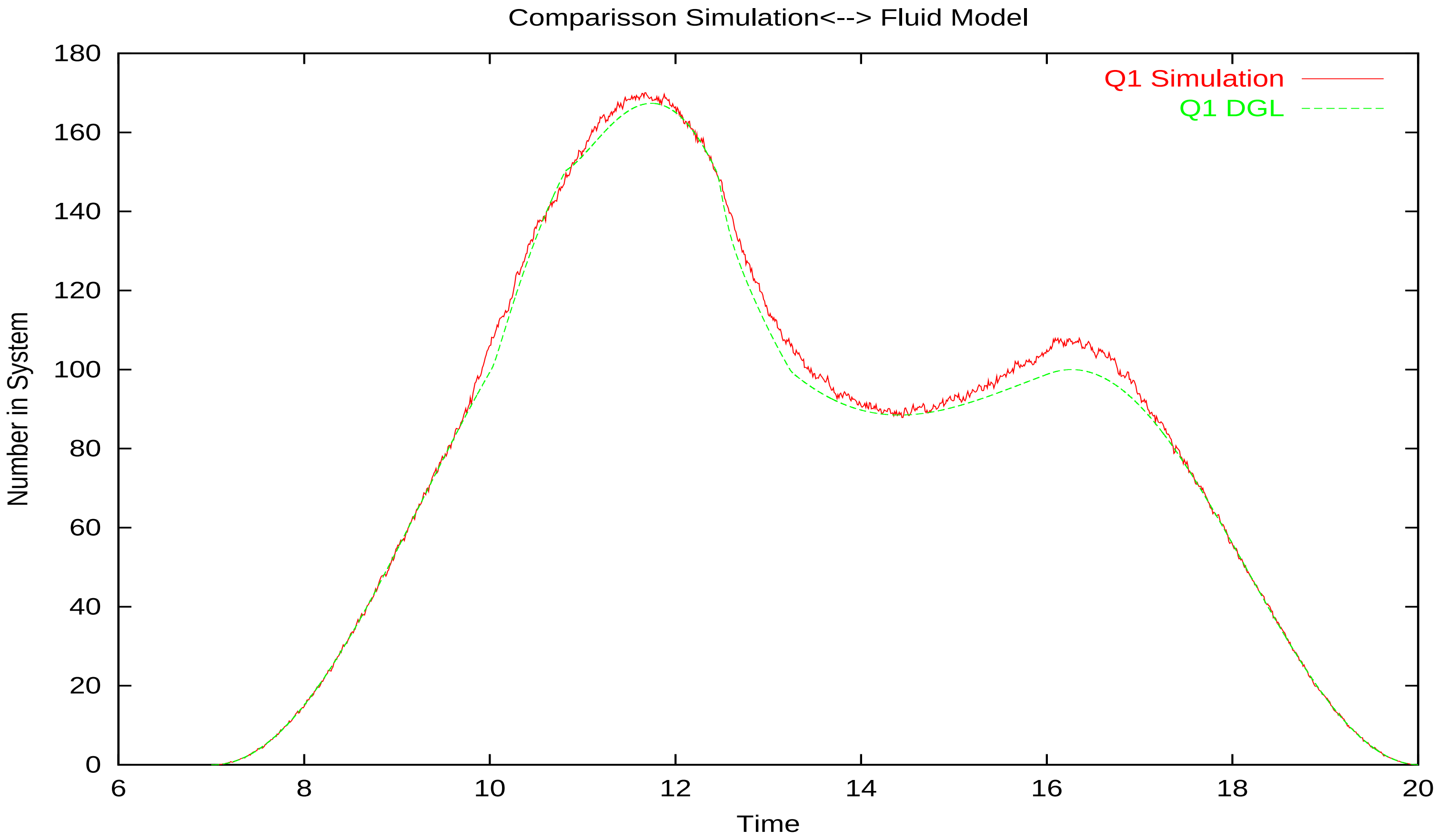
$$\boldsymbol{Q}^{\boldsymbol{D}}(\boldsymbol{t}) = \boldsymbol{Q}^{\boldsymbol{D}}(\boldsymbol{0}) + \int_0^t \boldsymbol{Q}^{\boldsymbol{D}}(\boldsymbol{s}) \cdot \boldsymbol{A}(\boldsymbol{s})\, ds + \boldsymbol{B}\left(\int_0^t \frac{d}{ds}\boldsymbol{Q}^{\boldsymbol{F}}(\boldsymbol{s})\, ds\right)$$

where $\boldsymbol{B}(\boldsymbol{t})$ is a row vector of standard Brownian motions and $\boldsymbol{A}(\boldsymbol{t})$ is the Jacobian of $\frac{d}{dt}\boldsymbol{Q}(\boldsymbol{t})$ differentiated at $\boldsymbol{Q}^{\boldsymbol{F}}(\boldsymbol{t})$.

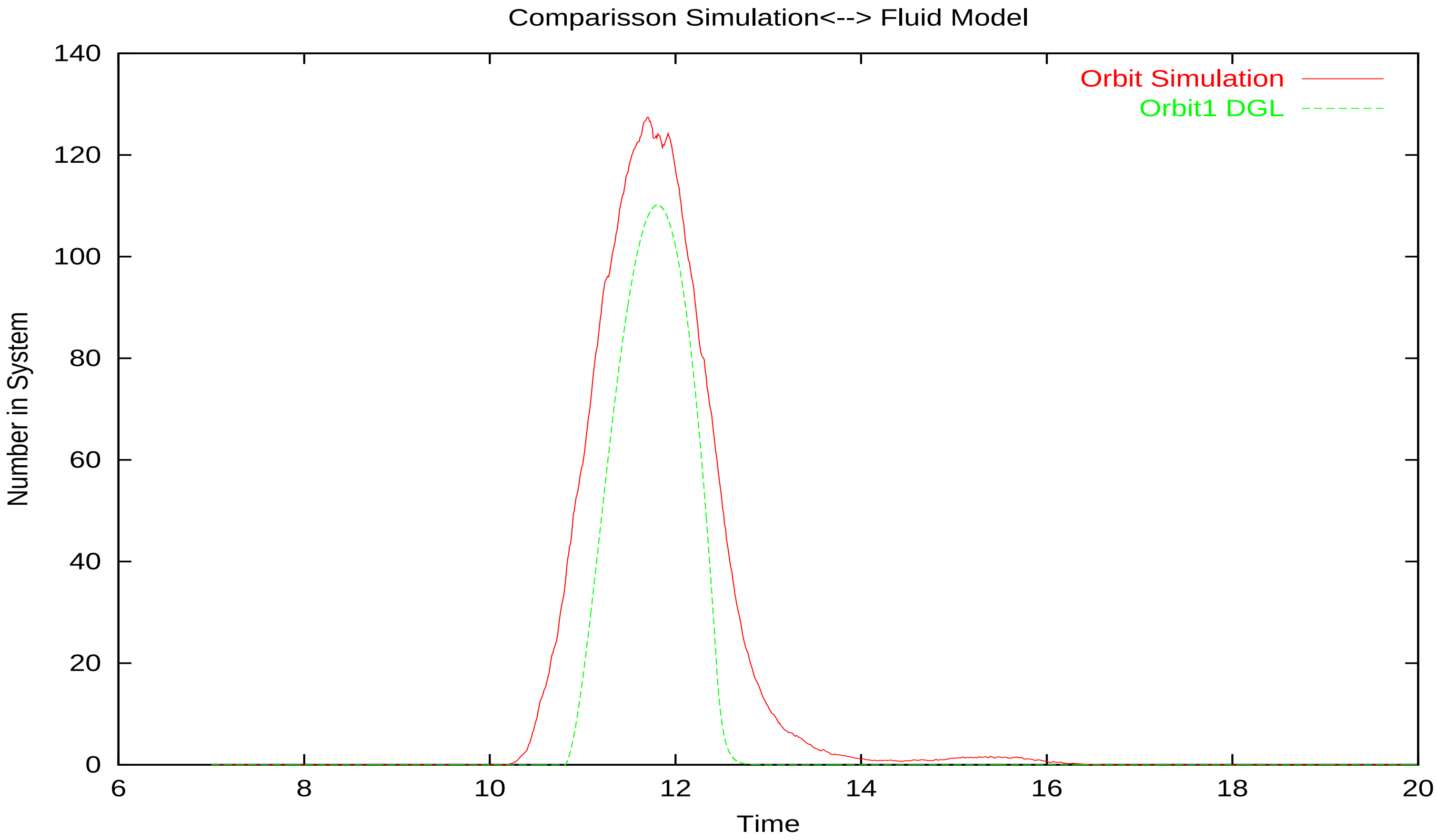
4. First results

- The fluid processes describe the mean processes accurately
- The variances of the simulation model are well approximated by the solutions to the differential equations for the variances and covariances
- Recallers have a significant effect on the number in system and the performance measures

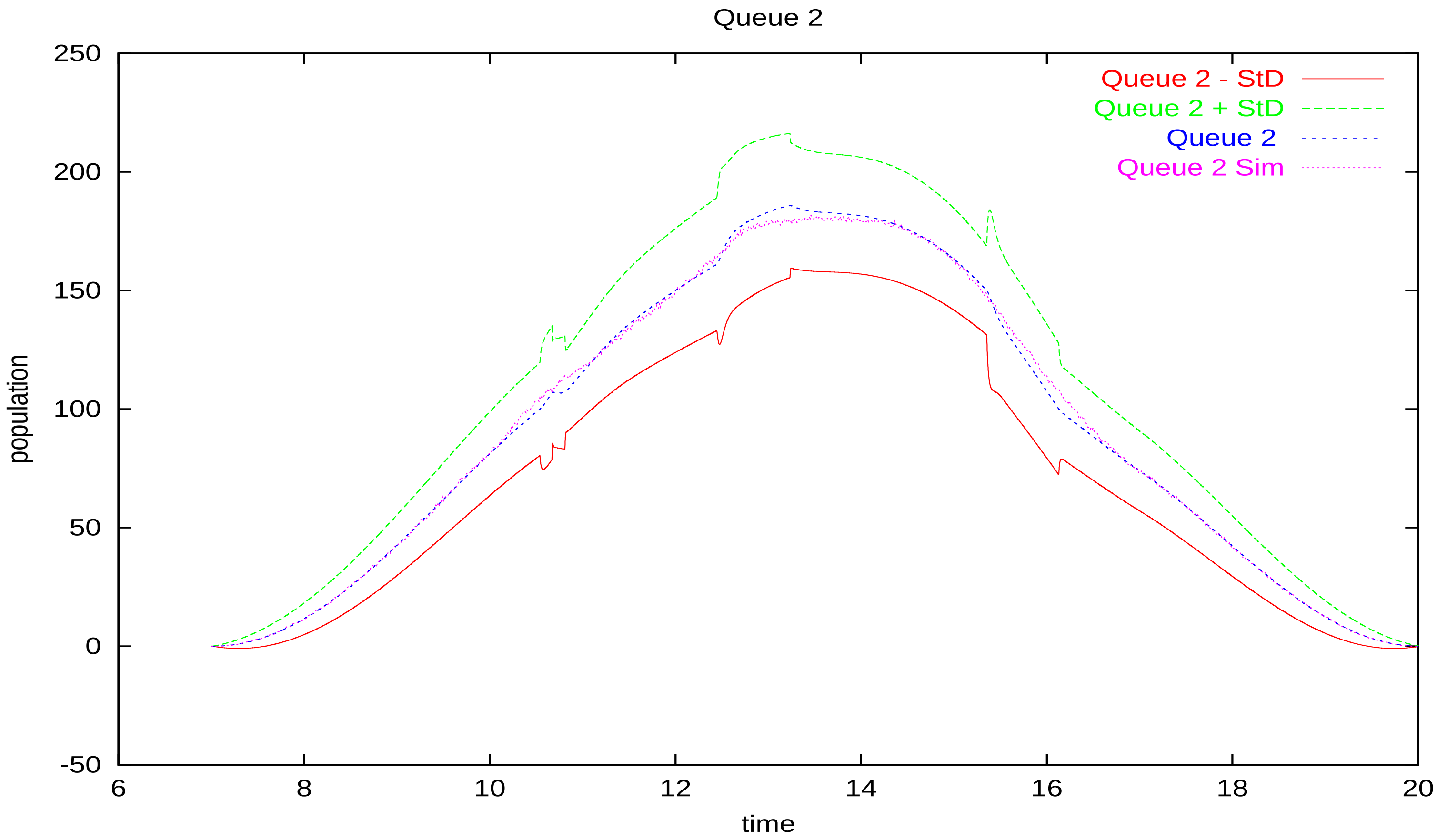
Customers in queue 1



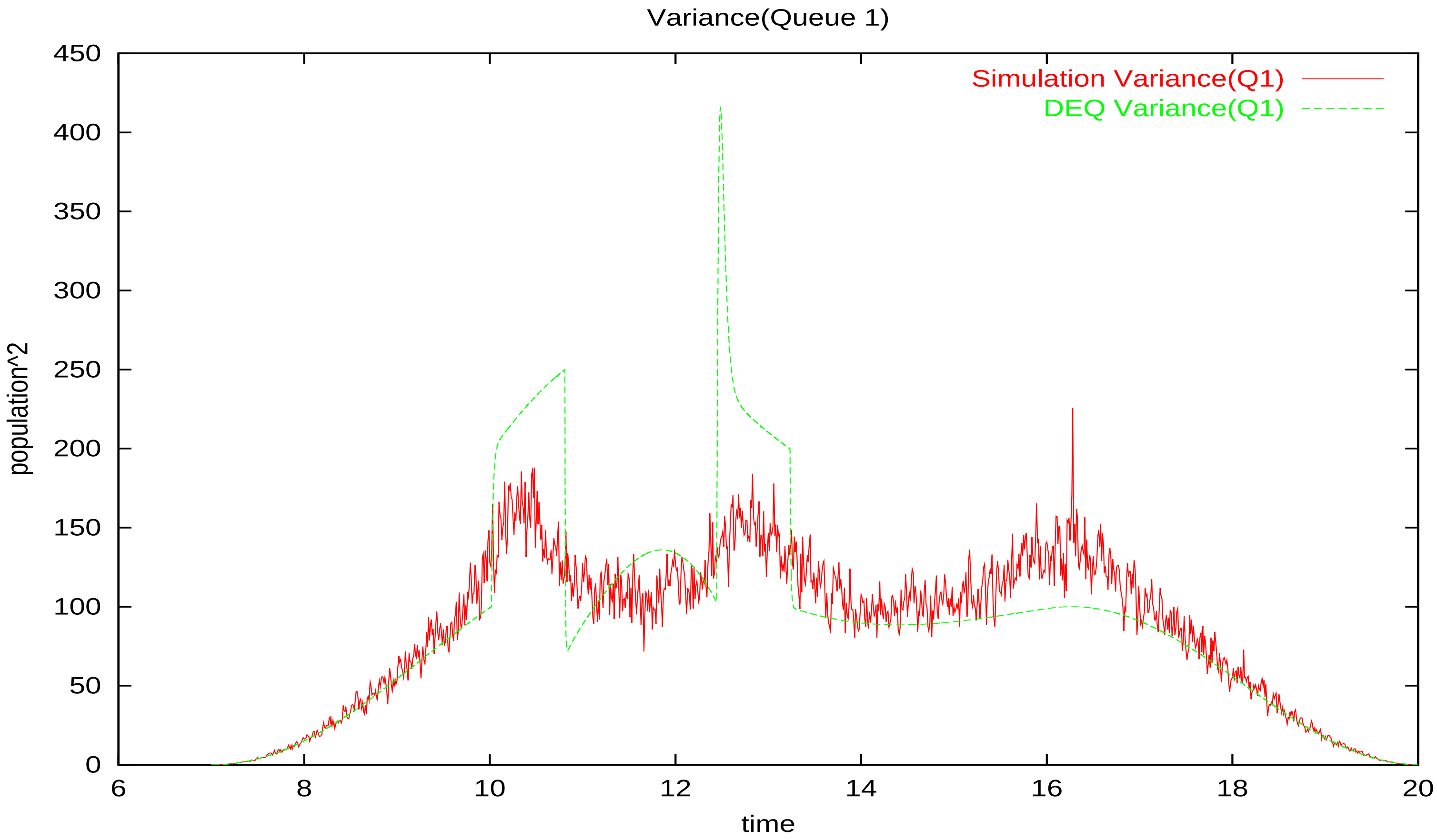
Customers in orbit 1



Customers in queue 2 with variances

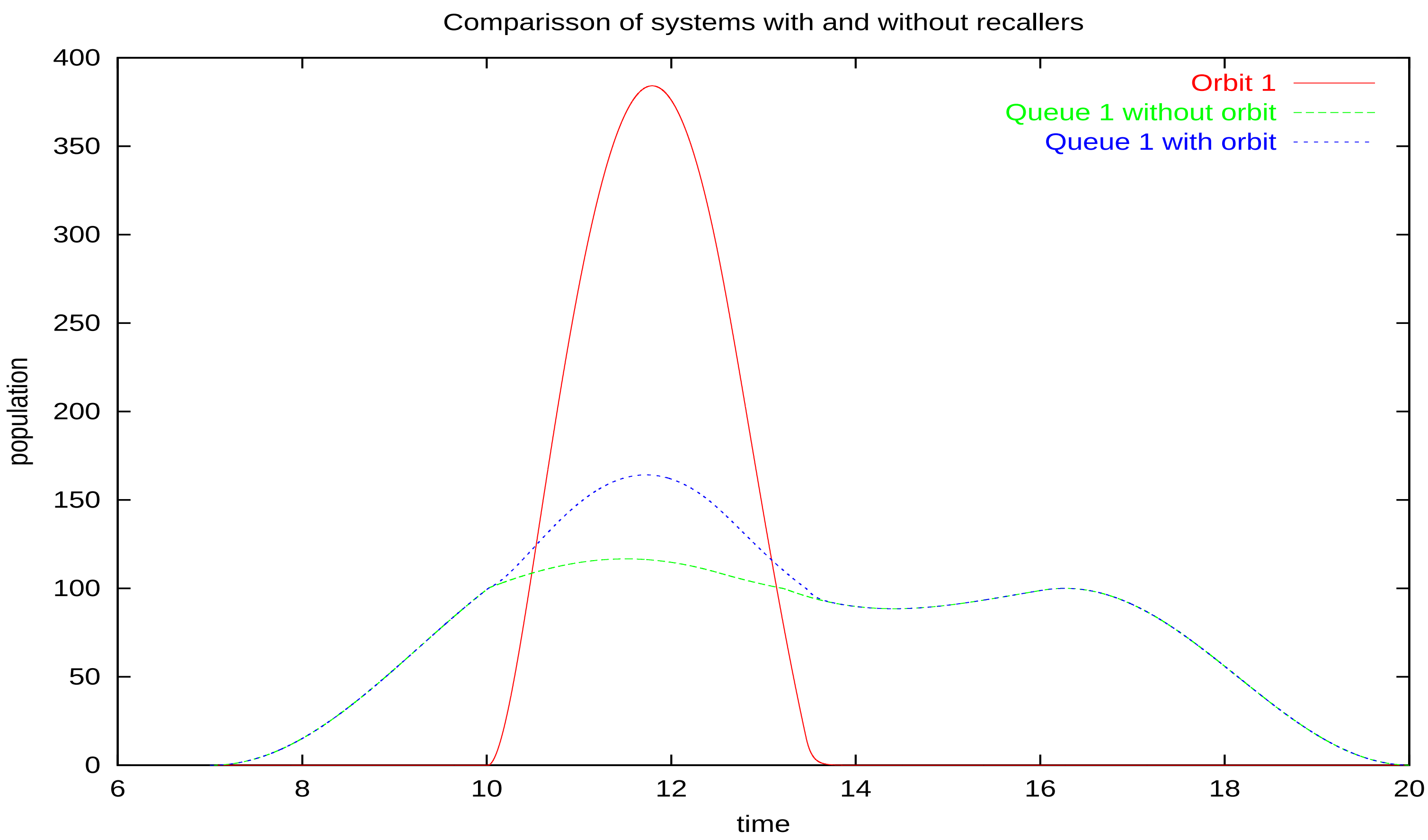


Variances



Variance(Queue 1)
Simulation Variance(Q1)
DEQ Variance(Q1)
population^2
time
450
400
350
300
250
200
150
100
50
0
6
8
10
12
14
16
18
20

Influence of recaller on the population in system



5. Prospects and questions

- More simulation studies
- Optimisation of server staffing
- What happens if the processes are not Gaussian?
- Which are useful performance measures for profit maximisation or cost minimisation?
- Should the variances be involved in optimisation?
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