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Constructing Stein manifolds after Eliashberg

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Abstract

We will discuss the existence theory of Stein structures, from Eliashberg's pioneering work to new theorems that complete the picture. Stein manifolds, or ane analytic varieties, have been studied by complex analysts for most of the past century. Their importance mandates the study of two basic existence questions: Which smooth manifolds admit Stein structures, and which open subsets of complex manifolds become Stein after isotopy (using the complex structure inherited from the ambient space)? Building on Eliashberg's fundamental work of the late 1980s, one can now completely answer these questions. In complex dimension $n \neq 2$, Eliashberg showed that an almost-complex manifold is Stein (after homotopy of the almost-complex structure) if and only if it has a handle decomposition with all indices $\leq n$. A similar statement applies to open subsets of complex manifolds. When $n = 2$, one obtains the corresponding characterizations by imposing additional delicate conditions. Alternatively, one can eliminate the extra conditions by passing to the topological category and invoking Freedman theory. As an application, one obtains Stein open subsets of $\mathbb{C}^2$ realizing uncountably many diffeomorphism types of exotic $\mathbb{R}^4$s. More generally, every open subset of $\mathbb{C}^2$ with a handle decomposition of all indices ≤ 2 is topologically isotopic to uncountably many diffeomorphism types of Stein open subsets.