Polyfolds

a new technology by Hofer-Wysocki-Zehnder

brought to you by Katrin Wehrheim (MIT)

This is an attempt at explaining the underlying philosophy and the two revolutionary foundation stones of a new technology & language that aims to overcome a fundamental problem in the differential geometric treatment of moduli spaces:

TRANSVERSALITY

"Wovon man nicht sprechen kann, darüber muß man schweigen"

"Whereof one cannot speak, thereof one must be silent"

(Ludwig Wittgenstein, Tractatus Logico-Philosophicus)

Disclaimer: I do not

a) mean to question the usefulness of all other "classical"-geometric or "virtual" approaches to transversality

b) even hint at the full strength and degree of generality of the polyfold technology

I do hope to give a taste of the flavour of a new technology, which I believe will make transversality proofs more user friendly – for those willing to learn the interface language. The vocabulary is small, but somewhat mindboggling ... and very fascinating!

$\boxed{\text{"classical" transversality \& gluing}}$

$$\begin{array}{c}\mathcal{Y}=\Omega^{0,1}(u^*TW)\\ \downarrow \uparrow s=\bar\partial \\ \mathcal{M}=\bar{s}^{-1}(0) \subset \mathcal{X}=\{u:\Sigma\to W\}\end{array}$$

→ describes moduli spaces as zero sets $\mathcal{M}=\bar{s}^{-1}(0)$

of Fredholm sections in Banach bundles

→ identifies ends of noncompact moduli spaces with fibre products of lower dimensional moduli spaces

TO DO: * (geometric) construction of perturbations $s \approx s' \pitchfork$ 0-section ; compatible with gluing

* "the usual" analysis (100s of pages if done ~~properly~~ lovingly; little remains strictly quotable when setup changes)

Defect: smooth structure on compactified moduli space (with boundary & corners) requires additional constructions

based on: IMPLICIT FUNCTION THEOREM : $\boxed{Ds|_{s^{-1}(0)} \text{ surjective (i.e. } s \pitchfork 0\text{-sect.)} \Rightarrow s^{-1}(0) \text{ manifold}}$

actual idea/wish: generalize ↓

TRANSVERSALITY: $Y \xrightarrow{} X$ ($\downarrow\uparrow s$) vector bundle over compact manifold X

⇒ $s \pitchfork$ 0-section for generic C^∞-section s

(and $s_1^{-1}(0) \sim s_2^{-1}(0)$ cobordant for $s_1, s_2 \pitchfork$ 0-sec.)

↳ to ∞-dim. function spaces

Polyfold Fredholm theory & operations

→ describes compactified moduli spaces as zero sets of "Fredholm" sections in "polyfold" bundles

→ identifies (codim. 1) boundaries of moduli spaces with fibre products of lower dimensional moduli spaces

AND encodes
- counts of 0-dim. moduli spaces
- relations from 1-dim. moduli spaces

in a general algebraic structure : "operations"

TO DO: understand compactified moduli space as a set, define appropriate ambient space & section, then quote

TRANSVERSALITY & IMPLICIT FUNCTION THEOREM

manifold version – for symmetries use orbifold version (without "M")

$\mathcal{X}$ M-polyfold, $\mathcal{Y} \to \mathcal{X}$ strong M-polybundle
with fibres for compact perturbations

$s : \mathcal{X} \to \mathcal{Y}$ sc^∞-Fredholm section ; $s^{-1}(0)$ compact
Fredholm after "filling"

$\Rightarrow \exists\, p : \mathcal{X} \to \mathcal{Y}$ sc^+-section (arbitrarily small & supported near $s^{-1}(0)$)
compact operator

such that $(s+p)^{-1}(0)$ is a smooth compact manifold with boundary with corners.

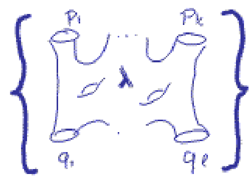
(And $(s+p_1)^{-1}(0) \sim (s+p_2)^{-1}(0)$ cobordant for different choices p_1, p_2)

Application to SFT

$$\mathcal{Y} \qquad \mathcal{Y}_u = \Omega^{0,1}(u^*TW) \quad L^p$$

$$\downarrow \ \uparrow\, s(u) = \bar\partial u$$

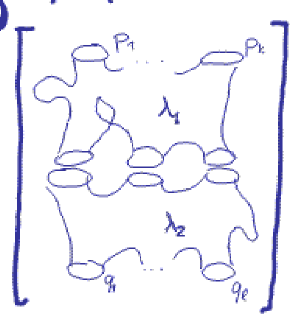
<u>"classical"</u>: $\widehat{\mathcal{M}}^p_q(\lambda) = s^{-1}(0) \subset \mathcal{B}^p_q(\lambda) = \left\{ \begin{array}{l} u : \Sigma \to W \quad W^{1,p} \\ \text{ends} \to p, q,\ [u] = \lambda \end{array} \right\}$

$\mathcal{M}^p_q(\lambda) = \widehat{\mathcal{M}}^p_q(\lambda) / \text{reparametrization} =$ 

$\overline{\mathcal{M}}^p_q(\lambda) = \{\text{holomorphic buildings}\}$

<u>"new age"</u>: $\overline{\mathcal{M}}^p_q(\lambda) = \bar{s}^{-1}(0)$

$\overline{\mathcal{B}}^p_q(\lambda) = \left\{ \begin{array}{l} (u_1, \dots u_k) \text{ not necc. holomorphic building} \quad W^{1,p} \\ \text{of } \underline{\text{any}} \text{ height } k;\ \text{ends } p, q;\ [u_1 \# \dots u_k] = \lambda \end{array} \right\} / \text{reparametrization}$



$\overline{\mathcal{Y}}_{(u_1, \dots, u_k)} = \Omega^{0,1}(u_1^*TW) \times \dots \times \Omega^{0,1}(u_k^*TW)$ $\quad \underline{W^{1,p}}$ strong

$\bar{s}(u_1, \dots, u_k) = (\bar\partial u_1, \dots, \bar\partial u_k)$

<u>issues & inspirations</u>:

② dimension of fibres jumps : "up by ∞" at the boundary

⇝ M-polyfolds & -bundles modelled on <u>"splicing cores"</u>

① action of reparametrization group is not smooth

E.g. $\Phi : \mathbb{R} \times \mathcal{C}^1(\mathbb{R}) \longrightarrow \mathcal{C}^1(\mathbb{R})$ $\qquad$ time shift

$\qquad (\tau, u) \longmapsto \tau * u\,(t) = u(\tau + t)$

6 $\quad D_{(\tau_0, u_0)}\Phi : (T, V) \longmapsto \tau_0 * V + T \cdot (\tau_0 * \dot{u}_0) \in \mathcal{C}^1(\mathbb{R})$ if $u_0 \in \mathcal{C}^2(\mathbb{R})$ $\ /\!/\ \mathcal{C}^k \ \mathcal{C}^{k+1}$

$\boxed{\text{scale calculus}}$

A **sc-structure** on Banach space E is a sequence $\mathbb{E}=(E_m)_{m\in\mathbb{N}_0}$ of subspaces $E_m\subset E$ with Banach norms $\|\cdot\|_{E_m}$ such that • $E_0=E$

$\forall m\geq 0$ • $E_{m+1}\hookrightarrow E_m$ compact, • $E_\infty:=\bigcap_{n\geq 0}E_n$ dense in E_m

Ex.: ~~$E=\mathcal{C}^1(\mathbb{R}),\ E_m=\mathcal{C}^{m+1}(\mathbb{R})$~~ $\rightsquigarrow E_\infty=\mathcal{C}^\infty(\mathbb{R})$ but $\mathcal{C}^{m+2}\hookrightarrow\mathcal{C}^{m+1}$ only bounded

$E=W^{1,2}(\mathbb{R})\ ,\ E_m=W^{m+1,2}_{\delta_m}(\mathbb{R})=e^{-\delta_m|t|}\cdot W^{m+1,2}(\mathbb{R})\qquad 0=\delta_0<\delta_1<\dots$

$\varphi:\mathbb{E}\to\mathbb{F}$ is **sc^0** if $\varphi|_{E_m}:E_m\to F_m$ is $\mathcal{C}^0$ $\forall m\geq 0$

time shift $\Phi:\mathbb{E}\to\mathbb{F}$ is sc^0, i.e. $\Phi:\underbrace{\mathbb{R}\times W^{m+1,2}_{\delta_m}}_{E_m}\to\underbrace{W^{m+1,2}_{\delta_m}}_{F_m}$

$\varphi:\mathbb{E}\to\mathbb{F}$ is **sc^1** if

- $\varphi:E_1\to F_0$ differentiable & induces $D\varphi(x_1):E_0\to F_0$ $\forall x_1\in E_1$
- $T\varphi:T\mathbb{E}\to T\mathbb{F}$, $(x,h)\mapsto(\varphi(x),D\varphi(x)h)$ is sc^0 $\left(\text{i.e. } E_{m+1}\times E_m\to F_m,\ (x,h)\mapsto D\varphi(x)h\ \ \mathcal{C}^0\ \forall m\right)$

φ is **sc^{k+1}** if $T\varphi$ is sc^k ; φ is **sc^∞** if it is sc^k $\forall k$

Φ is sc^∞ , in particular $\Phi:\mathbb{R}\times W^{m+1,2}_{\delta_m}\to W^{1,2}$ is $\mathcal{C}^m$

e.g. $D\Phi:\mathbb{R}\times W^{m+2,2}_{\delta_{m+1}}\times\mathbb{R}\times W^{m+1,2}_{\delta_m}\longrightarrow W^{m+1,2}_{\delta_m}$, $(\tau,u,T,V)\mapsto\tau*V+T\cdot(\tau*\dot{u})$ is $\mathcal{C}^0$ $\forall m$

Chain rule: $\varphi:\mathbb{E}\to\mathbb{F},\ \psi:\mathbb{F}\to\mathbb{G}$ sc^1 $\Rightarrow$ $\psi\circ\varphi$ sc^1, $T(\psi\circ\varphi)=T\psi\circ T\varphi$

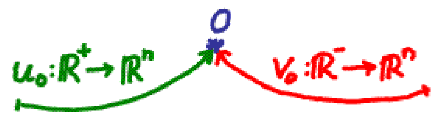
Proof crucially uses compactnes of $E_{m+1}\hookrightarrow E_m$

② $\rightsquigarrow$ Gluing & Antigluing $\rightsquigarrow$ Splicings & dimension jumps

<u>Goal</u>: charts for space $\mathcal{X}^{E}_{g}(N)$ of level 1,2,3,... buildings

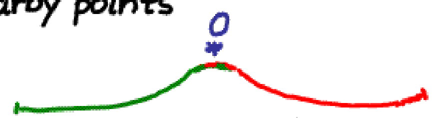
<u>Goal'</u>: —— " —— $\mathcal{X}$ of broken & unbroken paths (in Morse theory)
... locally near crit. point $0 \in \mathbb{R}^n$ (and no $/\mathbb{R}$ yet)

$(u_0, v_0) \in \mathcal{X}$



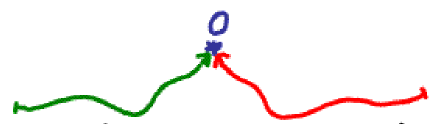
(length $R=\infty$ boundary point)

has nearby points



$w : [0,R] \to \mathbb{R}^n$

(length $1 \ll R < \infty$ interior point)



$(u : \mathbb{R}^+ \to \mathbb{R}^n,\ v : \mathbb{R}^- \to \mathbb{R}^n)$

(length $R=\infty$ boundary point)

parametrize a neighbourhood of (u_0, v_0) in $\mathcal{X}$:

$$[0,\tfrac{1}{2}) \times W^{1,2}(\mathbb{R}^+,\mathbb{R}^n) \times W^{1,2}(\mathbb{R}^-,\mathbb{R}^n) \xrightarrow[\text{!not injective!}]{\Phi} \mathcal{X}$$

$\Downarrow r \mapsto R(r) = e^{1/r}$

$(e^2, \infty]$ "gluing profile"

$$(r,h,k) \longmapsto \begin{cases} (u_0+h,\ v_0+k) & ; r=0 \quad (R=\infty) \\ \oplus_{R(r)}(u_0+h,\ v_0+k) & ; r>0 \quad (R=e^{1/r}) \end{cases}$$

pregluing

fiber wise projection to complement of kernel $\downarrow \pi$

φ M-polyfold chart

$$\bigcup_{r\in[0,\frac{1}{2})} \{r\} \times \ker \ominus_{R(r)}$$

splicing core

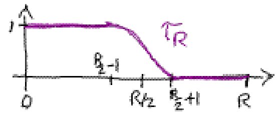
antigluing

Pregluing & Anti(pre)gluing define an isomorphism

$$\oplus_R \times \ominus_R : W^{1,2}(\mathbb{R}^+,\mathbb{R}^n) \times W^{1,2}(\mathbb{R}^-,\mathbb{R}^n) \xrightarrow{\sim} W^{1,2}([0,R],\mathbb{R}^n) \times W^{1,2}(\mathbb{R},\mathbb{R}^n)$$

hence $\ker \ominus_R$ is a complement of $\ker \oplus_R$.

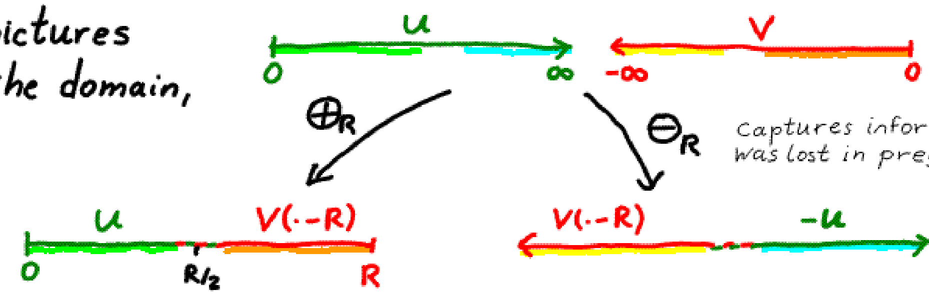
In formulas,



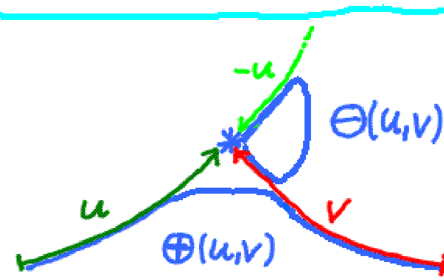
$$(\oplus_R \times \ominus_R)\begin{pmatrix} u \\ v \end{pmatrix} := \begin{pmatrix} \tau_R\, u + (1-\tau_R)\, v(\cdot - R) \\ -(1-\tau_R)\, u + \tau_R\, v(\cdot - R) \end{pmatrix}$$

is an isomorphism since the matrix $\begin{pmatrix} \tau & 1-\tau \\ -(1-\tau) & \tau \end{pmatrix}$ is

In pictures of the domain,



In picture of the image,



A splicing consists of

$V \subset \mathbb{R}^k \times [0,\infty)^{k'}$ open set (of gluing parameters)

$\mathbb{E}$ Banach space with scale structure

Gluing & Antigluing Example

$V = [0, \frac{1}{2})$

$E = W^{1,2}(\mathbb{R}^+, \mathbb{R}^n) \times W^{1,2}(\mathbb{R}^-, \mathbb{R}^n)$

$E_m = W^{m+1,2}_{\delta_m}(\ldots) \times W^{m+1,2}_{\delta_m}(\ldots)$

$\pi : V \times \mathbb{E} \to \mathbb{E}$ sc^∞ family of projections $(\pi_v^2 = \pi_v)$

$(v, e) \mapsto \pi_v e$

$$\pi_r(h,k) = \underset{R = e^{1/r}}{(\oplus_R \times \ominus_R)^{-1}}(\oplus_R(h,k), 0) = \left(\begin{cases} \frac{\tau_R^2}{\tau_R^2 + (1-\tau_R)^2} h + \frac{\tau_R(1-\tau_R)}{\tau_R^2+(1-\tau_R)^2} k(\cdot - R) & ; r>0 \\ h & ; r=0 \end{cases}, \begin{cases} \sim \\ \sim \end{cases} \right)$$

Its splicing core is $\bigcup_{v \in V} \{v\} \times \operatorname{im} \pi_v \subset V \times \mathbb{E}$

The Gluing & Antigluing splicing core is the "fibration"

$\bigcup_{r\in[0,1/2)} \{r\}\times \mathrm{im}\,\pi_r$ of subspaces $\mathrm{im}\,\pi_r = \ker \ominus_{R(r)} \subset W^{1,2}(\mathbb{R}^+)\times W^{1,2}(\mathbb{R}^-)$

$$\| \begin{cases} \overset{\oplus_R}{\cong} W^{1,2}([0,R]) & ; \begin{pmatrix} r>0 \\ R<\infty \end{pmatrix} \\ W^{1,2}(\mathbb{R}^+)\times W^{1,2}(\mathbb{R}^-) & ; \begin{pmatrix} r=0 \\ R=\infty \end{pmatrix} \end{cases}$$

over the set $[0,1/2)$ of gluing parameters.

Now, the map $\varphi := \Phi|_K = \bigcup_{r\in[0,1/2)} \ominus_{R(r)} : K \to \mathcal{X}$ is injective, and will serve as chart for the M-polyfold $\mathcal{X}$ – see below.

finite dimensional example of a splicing core

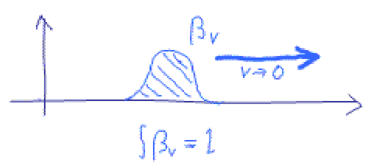
⚠ This example just illustrates the dimension jumps in splicing cores. In all applications, the fibres of splicing cores will be ∞-dimensional.

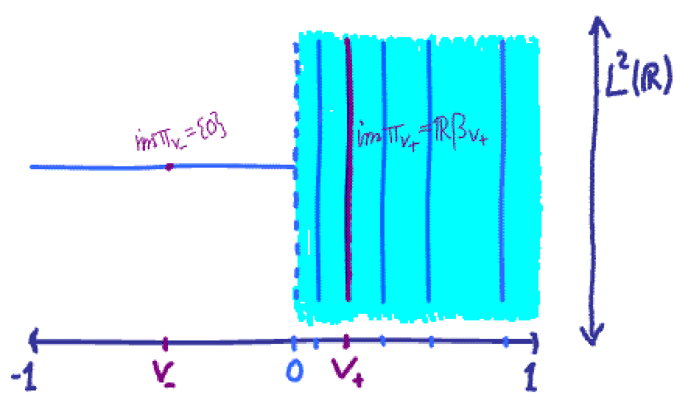
$V = (-1,1)$

$E = L^2(\mathbb{R})$

$\pi_v f = \beta_v \left(\int f\cdot\beta_v\right)$

$$\beta_v = \begin{cases} \mathrm{bump}(\cdot - e^{1/v}) & ; v>0 \\ 0 & ; v\le 0 \end{cases}$$





Check that π is sc^0: Fix $f\in L^2(\mathbb{R})$, then $\int_{\mathbb{R}} f\cdot\beta_v \xrightarrow[v\to 0]{} 0$,

and hence $\lim_{v\to 0} \pi_v f = \lim_{v\to 0} \beta_v\cdot\left(\int f\cdot\beta_v\right) = 0 = \pi_0 f$.

$\mathcal{X}$ 2^{nd} countable Hausdorff space

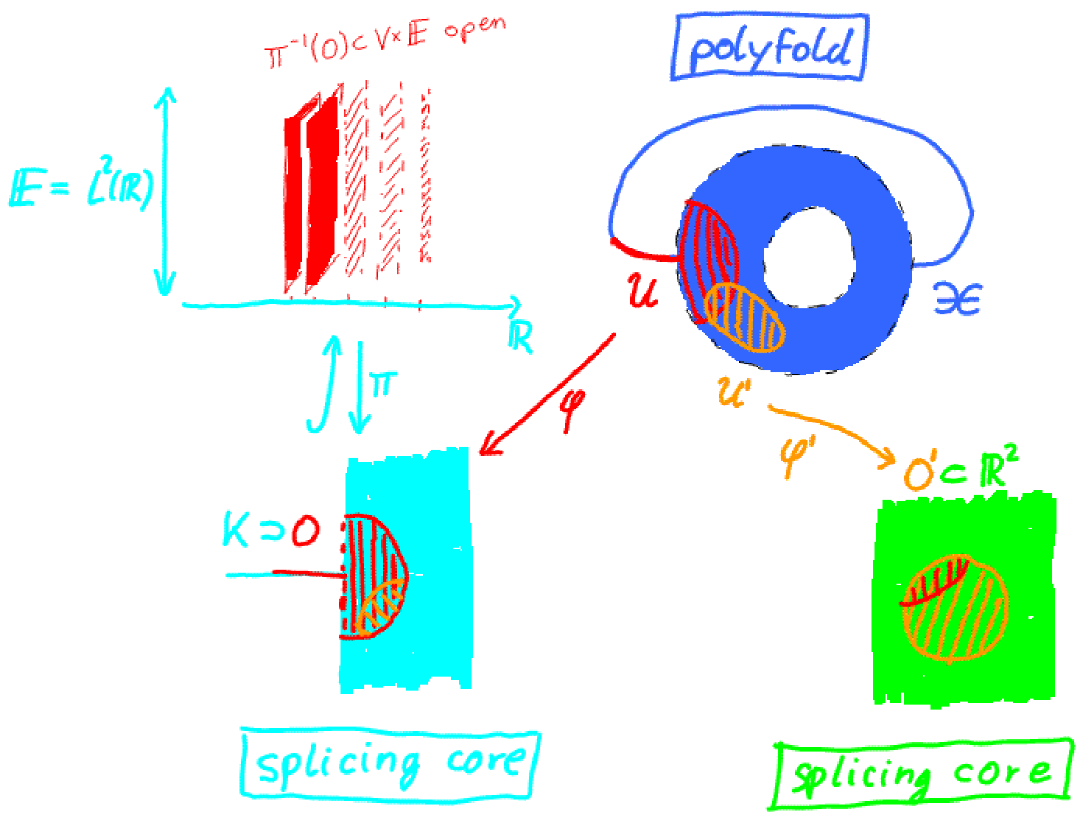
An M-polyfold chart is $\mathcal{X} \underset{open}{\supset} \mathcal{U} \underset{homeom.}{\xrightarrow{\varphi}} \mathcal{O} \underset{open}{\subset} K$ splicing core

Charts $\mathcal{U}_i \xrightarrow{\varphi_i} O_i \subset K_i \subset V_i \times \mathbb{E}_i$ are compatible if

$$\begin{array}{ccccccccc} \boxed{V_1 \times \mathbb{E}_1} & \xrightarrow{\pi} & K_1 & & \mathcal{X} & & & & \\ \cup & & \cup & & \cup & & & & \\ \pi^{-1}(\varphi_1(\mathcal{U}_1 \cap \mathcal{U}_2)) & \xrightarrow{\pi} & \varphi_1(\mathcal{U}_1 \cap \mathcal{U}_2) & \xrightarrow{\varphi_1^{-1}} & \mathcal{U}_1 \cap \mathcal{U}_2 & \xrightarrow{\varphi_2} & K_2 & \hookrightarrow & \boxed{V_2 \times \mathbb{E}_2} \end{array}$$

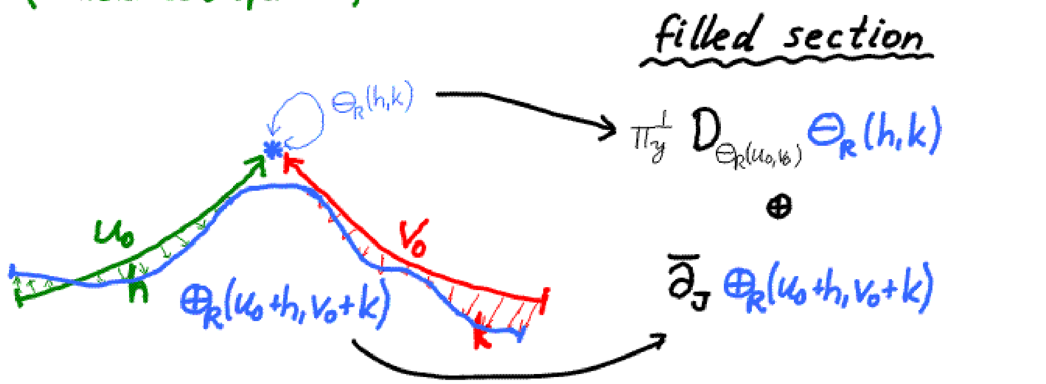
is sc^∞

An M-polyfold structure on $\mathcal{X}$ is a maximal collection of compatible M-polyfold charts covering $\mathcal{X}$.



An **M-polybundle** $\mathcal{Y} \xrightarrow{\pi} \mathcal{X}$ is a sc^∞ surjection π between M-polyfolds $\mathcal{X}, \mathcal{Y}$ "with some gluing parameters" and linear fibres.

A **Fredholm section** $s : \mathcal{X} \to \mathcal{Y}$ is a sc^∞ map, $\pi \circ s = Id$, that "can locally be filled up to a Fredholm map $V \times E_{\mathcal{X}} \to V \times E_{\mathcal{Y}_{fibre}}$"
(usually by the linearized operator)



The zero set of a transverse Fredholm section in a M-polybundle is a smooth manifold.