Invariants of exact Lagrangian cobordisms

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Introduction

Consider the standard contact $(\mathbf{R}^3, \xi = \ker \alpha)$, where $\alpha = dz - ydx$.

A Legendrian knot $L \subset (\mathbf{R}^3, \xi)$ is a knot which is everywhere tangent to ξ, i.e., satisfies $dz - ydx = 0$.

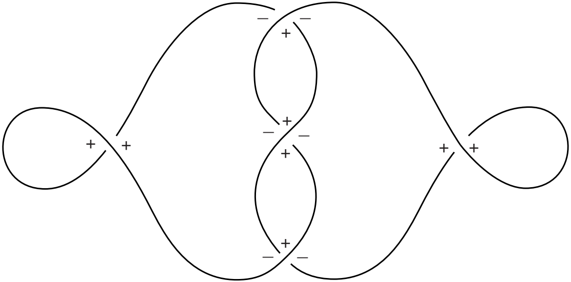

Figure: A right-handed trefoil in the xy-projection (Lagrangian projection).

Legendrian knot invariants

Given an oriented Legendrian knot L, there are two "classical invariants", the *Thurston-Bennequin number* and the *rotation number*. With respect to the xy-projection π, they are given as follows:

1. $tb(L)$ is the *writhe* of $\pi(L)$, i.e., the number of positive crossings minus the number of negative crossings.
2. $r(L)$ is the degree of the Gauss map (or winding number) of $\pi(L)$.

Example: For the right-handed trefoil example, $tb(L) = 1$ and $r(L) = 0$.

Lagrangian cobordisms

The *symplectization* of $(\mathbf{R}^3, \xi)$ is $\mathbf{R} \times \mathbf{R}^3$ with the symplectic form $\omega = d(e^t\alpha) = e^t(dt \wedge \alpha + d\alpha)$, where t is the first coordinate. This will be our ambient manifold for today.

Definition

Let L, L' be Legendrian knots. A Lagrangian cobordism Λ from L to L' *is an* embedded *Lagrangian surface* $($*i.e.*, $\omega|_\Lambda = 0)$ *which agrees with a cylinder over L for $t \gg 0$ and a cylinder over L' for $t \ll 0$.*

Remark: A cylinder $\mathbf{R} \times L$ over a Legendrian knot L is Lagrangian in the symplectization.

Exact Lagrangian cobordisms

Today our Lagrangians Λ will be *exact*: The symplectic form $\omega = d(e^t\alpha)$ is an exact symplectic form. By the Lagrangian condition, $\omega|_\Lambda = d(e^t\alpha)|_\Lambda = 0$. We say Λ is *exact* if $e^t\alpha|_\Lambda = dF$, where F is a function on Λ.

The exactness will be important when we discuss the TQFT properties of Lagrangian cobordisms.

Basic Observations

Theorem (Chantraine)

If Λ is a Lagrangian cobordism from L to L', then

$$tb(L) - tb(L') = -\chi(\Lambda).$$

If $L' = \emptyset$, then, combining with the slice Bennequin inequality (due to Kronheimer-Mrowka/Rudolph):

$$tb(L) \pm r(L) \leq 2g_4(L) - 1,$$

where $g_4(L)$ is the 4-ball genus, we have:

Corollary

If L bounds a Lagrangian surface, then $tb(L) = 2g_4(L) - 1$ and $r(L) = 0$.

Examples

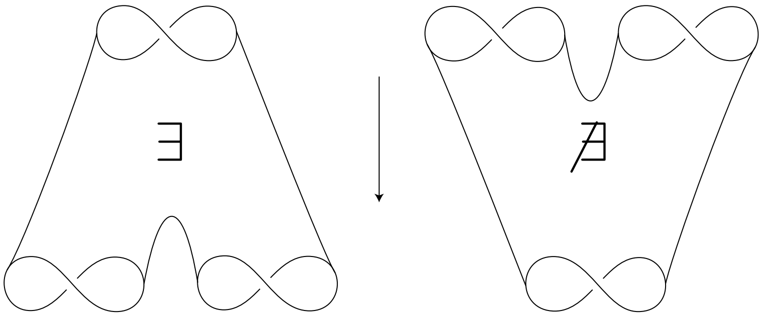

Figure: The left-hand cobordism exists, whereas the right-hand one does not.

Constructions

Theorem

There exists an exact Lagrangian cobordism for the following:

1. *Legendrian isotopy from L to L'.*
2. 0-*resolution at a* contractible *crossing of L in the xy-projection.*
3. *Capping off a $tb = -1$ unknot with a disk.*

Definition

A contractible *crossing of L is a crossing so that $z_1 - z_0$ can be shrunk to zero without affecting the other crossings. (Here z_1 is the z-coordinate on the upper strand and z_0 is the z-coordinate on the lower strand.)*

0-resolution

xy (Lagrangian) projection

xz (Front) projection

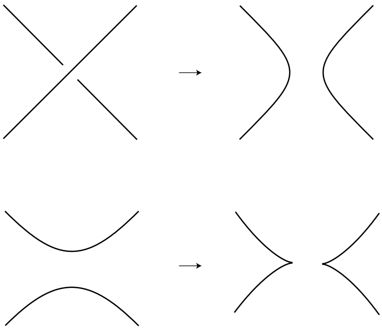

Figure: The 0-resolution.

An example

Example: (Right-handed trefoil) All three crossings in the middle are contractible. Hence it is possible to 0-resolve each of the b_1, b_2, b_3, in any order.

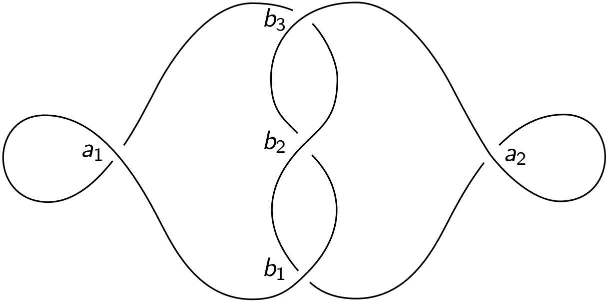

Figure: The right-handed trefoil in the *xy*-projection.

Decomposability

Definition

Λ *is* decomposable *if it is obtained by stacking exact Lagrangians corresponding to the following three operations given above:*

1. *Legendrian isotopy;*
2. 0*-resolution;*
3. *Capping off an unknot.*

Conjecture

Every exact Lagrangian cobordism of L *to* $\emptyset$ *is decomposable.*

Legendrian contact homology

The combinatorial version is due independently to Chekanov and Eliashberg.

We will work over $\mathbf{Z}/2$-coefficients.

$\mathcal{A}(L)$ = free algebra with unit (= tensor algebra) generated by the double points of the xy-projection

The boundary map $\partial : \mathcal{A}(L) \to \mathcal{A}(L)$ is given as follows:

$$\partial y = \sum_P w_P,$$

where the sum is over all immersed polygons P with one positive corner at y and zero or more negative corners (in the xy-projection). If the vertices of P are y, x_1, x_2, x_3 in clockwise order, then $w_P = x_1 x_2 x_3$.

Legendrian contact homology, continued

Fact: $\partial^2 = 0$.

We can therefore define the *Legendrian contact homology* $HC(L)$ as the homology of $(\mathcal{A}(L), \partial)$.

For each Legendrian Reidemeister move $L \rightsquigarrow L'$ (i.e., Legendrian isotopy), there exists a corresponding combinatorial chain map which induces an isomorphism $\Phi : HC(L) \to HC(L')$.

Recall that every Legendrian isotopy gives rise to an exact Lagrangian cobordism. The combinatorial chain maps of Chekanov arise from the corresponding exact Lagrangian cobordism. (Work of Ekholm-Kálmán, in progress.)

TQFT/SFT package

Theorem

- *An exact*[a] *Lagrangian cobordism* Λ *from* L *to* L' *induces a map*

$$\Phi_\Lambda : HC(L) \to HC(L'),$$

which is an invariant of Λ *up to exact Lagrangian isotopy.*

- *Given exact Lagrangian cobordisms* Λ_1 *(from* L_1 *to* L_2*) and* Λ_2 *(from* L_2 *to* L_3*),*

$$\Phi_{\Lambda_1 \circ \Lambda_2} = \Phi_{\Lambda_2} \circ \Phi_{\Lambda_1}.$$

[a] The exactness is crucial here, to prevent boundary bubbling.

0-resolution map

Theorem

Let $L \rightsquigarrow L'$ be a 0-resolution[a] at a contractible double point x, and let Λ be an exact Lagrangian cobordism representing this 0-resolution. Then the combinatorial chain map

$$\Phi_\Lambda : \mathcal{A}(L) \to \mathcal{A}(L')$$

is given as follows:

$$x \mapsto 1,$$
$$y \mapsto y + \sum_P w_P,$$

where $y \neq x$ and the sum is over all immersed polygons P with two positive corners, one at x and the other at y, and zero or more negative corners. If the vertices of P are y, x_1, x_2, x, x_3 in clockwise order, for example, then $w_P = x_1 x_2 x_3$.

[a] $\exists$ minor technical conditions

Proof of Theorem

First change coordinates:

$$d(e^t(dz - ydx)) = d(e^t) \wedge dz + dx \wedge d(e^t y),$$

and set $u_1 = e^t, v_1 = z, u_2 = x, v_2 = e^t y$. Then Λ is a Lagrangian in the standard $(\mathbf{R}^4, \sum_i du_i \wedge dv_i)$, with an infinity in the u_1-direction.

Exact Lagrangian means that there is a lift of the Lagrangian to a Legendrian submanifold in the standard contact $\mathbf{R}^5$ with coordinates (w, u_1, v_1, u_2, v_2) and contact form $dw - \sum_i v_i du_i$.

Use a slight extension of Ekholm's gradient flow tree technique for Legendrian submanifolds, which in turn generalizes Fukaya and Oh's work on gradient flow trees.

An example

Example: (Right-handed trefoil)

We resolve the contractible crossings b_1, b_2, b_3 in six different ways.

For example, if we resolve b_1, b_2, b_3 in that order, then we obtain the map

$$\varepsilon : \mathcal{A}(L) \to A(\emptyset) = \mathbf{Z}/2,$$

$$b_1 \mapsto 1, \ b_2 \mapsto 0, \ b_3 \mapsto 0, \ a_i \mapsto 0.$$

Augmentations

A chain map $\varepsilon : \mathcal{A}(L) \to \mathbf{Z}/2$ is called an *augmentation*. (Augmentations are usually required to *linearize* contact homology and compute its E_1-term with respect to the word length filtration.)

Fact: Algebraically there are 5 augmentations $\varepsilon : \mathcal{A}(L) \to \mathbf{Z}/2$, where L is the right-handed trefoil.

Augmentations, continued

Theorem

1. *All* 5 *augmentations are* geometric, *i.e., arise from an exact Lagrangian cobordism from* L *to* $\emptyset$. *They are obtained by resolving* b_1, b_2, b_3 *in different orders.*
2. *All* 5 *exact Lagrangian cobordisms are distinct via exact Lagrangian isotopy.*

A conjecture

Conjecture

Every augmentation of a Legendrian knot is geometric.

I'm deliberately being vague here — for example, do we allow immersed exact Lagrangian cobordisms whose double points have certain Maslov indices, so that the map $HC(L) \to \mathbf{Z}/2$ is still well-defined and independent of immersed exact Lagrangian isotopy?

Khovanov homology

To each decomposable Λ, there exists a sequence

$$L = L_1, L_2, \ldots, L_n = \sqcup \text{ unknots},$$

where each $L_i \rightsquigarrow L_{i+1}$ is either a Reidemeister move or a 0-resolution.

In Khovanov homology, there exist maps corresponding to:

1. The inverse of a 1-resolution.
2. The creation of an unknot.
3. Reidemeister moves.

Elements in Khovanov homology

We need to mirror so that $Kh(L)$ becomes $Kh(\overline{L})$ and 0-resolutions become 1-resolutions.

Theorem (Jacobsson)

Given a cobordism Λ of L to $\emptyset$, there is a corresponding element c_Λ in $Kh(\overline{L})$.

If Λ is decomposable, then c_Λ is obtained by composing:

$$Kh(\emptyset) \to Kh(\sqcup \text{ unknots}) \to \cdots \to Kh(\overline{L}),$$

where all the arrows after the first are maps for Reidemeister moves or inverses of 1-resolution maps.

Relationship to Ng's and Plamenevskaya's works

Example: (Right-handed trefoil) If L is the right-handed trefoil, then c_Λ for the examples of Λ above are at homological grading 0 and q-grading -1.

1. The class c_Λ coincides with the transverse knot invariant of Plamenevskaya in Khovanov homology.
2. The class c_Λ lies on the line given by Ng which indicates an upper bound on the Thurston-Bennequin invariant.

Khovanov homology of left-handed trefoil

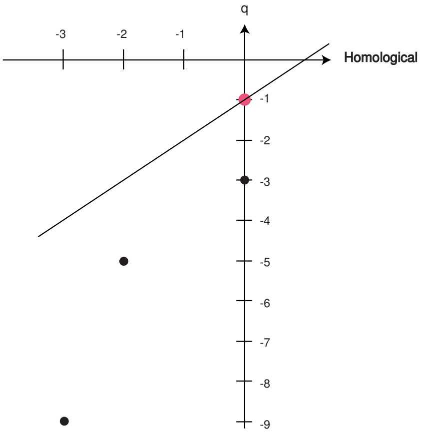

Figure: The Khovanov homology of the left-handed trefoil. Each dot represents a **Z**, and the red dot is the location of c_Λ.

Some questions

We can assign to each Legendrian knot L the collection

$$Lag(L) = \{\Lambda \mid \Lambda \text{ Lagrangian which bounds } L\},$$

and hence

$$C(L) = \{c_\Lambda \mid \Lambda \in Lag(L)\} \subset Kh(\overline{L})$$

Questions

1. *Is $C(L)$ a finite set? ($C(L) = \emptyset$ if $tb(L) \neq 2g_4(L) - 1$.)*
2. *Does $C(L)$ ever have more than one element?*

Happy Birthday, Yasha!