

MINIMUM VOLUME CUSPED HYPERBOLIC THREE-MANIFOLDS

DAVID GABAI, ROBERT MEYERHOFF, AND PETER MILLEY

1. INTRODUCTION

In this paper, we prove:

Theorem 1.1. *Let N be a one-cusped orientable hyperbolic 3-manifold with $\text{Vol}(N) \leq 2.848$. Then N can be obtained by Dehn filling all but one of the cusps of M , where M is one of the 21 cusped hyperbolic 3-manifolds listed in the table in figure 1.*

In [MM] the Dehn surgery spaces of the 21 manifolds listed in figure 1 are rigorously analyzed, producing a complete list of one-cusped manifolds with volume no greater than 2.848 which result from the Dehn fillings described above. We therefore obtain:

Corollary 1.2. *Let N be a 1-cusped orientable hyperbolic 3-manifold with $\text{Vol}(N) \leq 2.848$, then N is one of $m003$, $m004$, $m006$, $m007$, $m009$, $m010$, $m011$, $m015$, $m016$, or $m017$. (Notation as in the SnapPea census.)*

This corollary extends work of Cao and Meyerhoff who had earlier shown that $m003$ and $m004$ were the smallest volume cusped manifolds. Also, the above list agrees with the SnapPea census of one-cusped manifolds produced by Jeff Weeks ([W]), whose initial members are conjectured to be an accurate list of small-volume cusped manifolds.

Let N be a closed hyperbolic 3-manifold with simple closed geodesic γ and let N_γ denote the manifold $N \setminus \gamma$. Agol ([Ago]) discovered a formula relating $\text{Vol}(N)$ to $\text{Vol}(N_\gamma)$ and the tube radius of γ . Assuming certain results of Perelman, Agol and Dunfield (see [AST]) have further strengthened that result. A straightforward calculation (see [ACS]) using this stronger result, the $\log(3)/2$ theorem of [GMT], plus bounds on the density of hyperbolic tube packings by Przeworski, shows that a compact hyperbolic manifold with volume less than that of the Weeks manifold must be obtainable by Dehn filling on a cusped manifold with volume less than or equal to 2.848. The paper [MM] rigorously shows that the Weeks manifold is the unique compact hyperbolic 3-manifold of smallest volume obtained by filling any of the 10 manifolds listed in Corollary 1.2. We therefore obtain,

Corollary 1.3. *The Weeks manifold is the unique closed orientable hyperbolic 3-manifold of smallest volume.*

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m125	m129	m203	m202	m292	m295	m328
m329	m359	m366	m367	m391	m412	s596
s647	s774	s776	s780	s785	s898	s959

FIGURE 1. The cusped manifolds which generate all one-cusped hyperbolic 3-manifolds with volume ≤ 2.848 . The manifolds are denoted here as they appear in the SnapPea census.

The Weeks manifold is obtained is obtained by $(5, 1)$, $(5, 2)$ filling on the Whitehead link, or by $(2, 1)$ filling on the manifold m003 in the SnapPea census.

The proof of Theorem 1.1 is based on the Mom technology introduced in [GMM2]. Indeed, Figure 1 lists the collection of Mom-2 and Mom-3 manifolds enumerated in [GMM2], thus we have the following equivalent formulation:

Theorem 1.4. *Let N be a one-cusped orientable hyperbolic 3-manifold with $\text{Vol}(N) \leq 2.848$. Then N can be obtained by Dehn filling all but one of the cusps of M , where M is a hyperbolic Mom-2 or Mom-3 manifold.*

Recall that a Mom- n manifold is a 3-manifold M obtained by starting with $T \times [0, 1]$ where T is the 2-torus and attaching n 1-handles and n valence-3 2-handles to the $T^2 \times 1$ side. Furthermore, ∂M is a union of tori. Given N as in the theorem, the goal is to find a hyperbolic Mom-3 embedded in N , or in the terminology of [GMM2] show that N possesses an *internal Mom- n structure* for some $n \leq 3$. I.e., M satisfies the condition that the interior of M has a complete hyperbolic structure of finite volume and each component of ∂M bounds (to the outside) either a solid torus or a cusp.

In practice, we think of $T \times 0$ as the torus cutting off a maximal cusp neighborhood U . In the universal covering U lifts to a collection of horoballs $\{B_i\}$. To first approximation, when lifted to \mathbb{H}^3 , the cores of the 1-handles of the Mom- n structure will be geodesic arcs connecting two B_i 's. (Being maximal, some B_i 's will be tangent to each other and these points of tangency will also be viewed as 1-handles.) The cores of the 2-handles, when lifted to \mathbb{H}^3 will correspond to totally geodesic hexagons whose sides alternately lie on the 1-handles and boundaries of B_i 's.

Using the 2.848 volume bound we will show that N possesses a *geometric Mom- n structure*, where $n \leq 3$. This means that we will find n $\pi_1(N)$ -orbits of geodesic arcs and n $\pi_1(N)$ -orbits of geodesic hexagons with boundaries on the geodesic arcs and B_i 's as in the previous paragraph. With some luck, when thickened up, these geodesics and hexagons will descend to an internal Mom- n structure on N . In reality, when projected to N , these hexagons may self-intersect in undesirable ways and/or the resulting handle structures may be unsuitable for various technical reasons. Much of this paper is devoted to the process of promoting a geometric Mom- n structure $n \leq 3$ which is *torus-friendly* to a hyperbolic internal Mom- k structure, $k \leq n$. (See definition 2.2.)

This paper is organized as follows. In Section 2 we give a detailed definition of geometric Mom- n structure. In Section 3 we present several useful geometric lemmas that will be used extensively in the rest of the paper. Then in Sections 4 and 5 we show that if N satisfies the hypotheses of Theorem 1.1, then N must contain a geometric Mom- n structure which is torus-friendly. This part of the proof,

while theoretically simple, is computationally complicated and was completed with the use of computer assistance; the use of rigorous floating-point computations is discussed in Section 5.

Sections 6, 7, and 8 are concerned with the process of promoting the geometric Mom- n structure produced in Section 4 and 5 to an internal Mom- n structure of the type described in [GMM2]. There is a natural geometric object associated to a geometric Mom- n structure, consisting of a thickened copy of the cusp torus ∂U (which corresponds to our $T^2 \times [0, 1]$) together with a “one-handle” for every orthogonal geodesic arc in the Mom- n structure and a “2-handle” for every hexagon. However there are three key conditions that this geometric object must meet before it fits the definition of an internal Mom- n structure. Each of Sections 6, 7, and 8 are devoted to one of these three conditions, and to showing that either the condition holds or else we can replace our geometric Mom- n with a “simpler” structure, for some appropriate definition of “simpler”. Section 6 is concerned with whether or not the geometric object associated to the Mom- n is embedded in N , in particular whether the various handles have undesirable intersections or self-intersections. Section 7 is concerned with whether the components of the complement have the correct topology, and Section 8 is concerned with whether or not our Mom- n structure has “simply-connected lakes” in the language of Matveev. Having completed these three sections, we find that the geometric Mom- n structure produced in Sections 4 and 5 will have evolved into an internal Mom- k structure for some $k \leq n$.

At that point, the proof of Theorem 1.1 reduces to an application of Theorems 4.1 and 5.1 of [GMM2]. Together those two theorems imply that if N has an internal Mom- n structure with $n \leq 3$ then N contains an embedded submanifold M which is a hyperbolic manifold with boundary whose interior is homeomorphic to one of the manifolds in figure 1 and such that $N - M$ is a disjoint union of solid tori and cusps. This is the desired result. Section 9 summarizes this argument formally.

It should be noted that while the list in Figure 1 is precisely the list of manifolds produced by Theorem 5.1 of [GMM2], it is somewhat redundant for the purposes of Theorem 1.1 of this paper. The manifold *s776* is a three-cusped manifold from which many of the two-cusped manifolds on the list—experimentally, everything up to and including *m391*—can be recovered by Dehn filling. Hence the 21 manifolds in Figure 1 could be reduced to a list of 9 manifolds with no effect on Theorem 1.1. However, we use the longer list here to be consistent with [GMM2].

Finally in Section 9 we will discuss some of the ideas used in [MM] to rigorously analyze various Dehn fillings of the manifolds of Figure 1.

2. DEFINITION OF A GEOMETRIC MOM- n

For the rest of this paper, N will refer to an orientable one-cusped hyperbolic 3-manifold. Suppose N is such a manifold; then N possesses a *maximal cusp neighborhood* which is a closed set whose interior is homeomorphic to $T^2 \times (0, \infty)$, with the property that each torus $T^2 \times \{x\}$ has constant sectional curvature in N . The term “maximal” here means that this cusp neighborhood is not a proper subset of any other closed subset of N with this property.

T bounds a horoball when lifted to \tilde{T} in the universal covering \mathbb{H}^3 of N . In practice we think of T as a maximal cusp bounding a horoball. In \mathbb{H}^3 , the cores of

the various 1-handles will be orthogonal geodesic arcs (or points) connecting $\pi_1(N)$ -translates of this horoball. The cores of the 2-handles will be totally geodesic discs with boundary alternately on horoballs and these geodesic arcs.

We prove Theorem 1.1 using *geometric Mom-n structures*, defined below.

In the universal cover \mathbb{H}^3 of N , the maximal cusp neighborhood lifts to a collection of horoballs $\{B_i\}$; any two such horoballs have disjoint interiors but maximality implies that some pairs will be tangent at their boundaries. Choose one such horoball and denote it B_∞ . In the upper half-space model $\{(x, y, z) | z > 0\}$ of \mathbb{H}^3 , we may assume after conjugation by some element of $Isom^+(\mathbb{H}^3)$ that B_∞ will be precisely the set $\{(x, y, z) | z \geq 1\}$; then every other B_i will appear as a sphere with center (x_i, y_i, z_i) and radius z_i for some $0 < z_i \leq \frac{1}{2}$. Let $H \subset \pi_1(N)$ be the subgroup which fixes B_∞ , so that B_∞/H is homeomorphic to the cusp neighborhood. Let d_E denote the distance function in the subspace (Euclidean) metric along the boundary of B_∞ .

The *center* of a horoball $B_j \neq B_\infty$ is the limiting point of the horoball on the sphere at infinity. For example, if B_j appears in the upper half-space model as a sphere centered at (x_j, y_j, z_j) with radius z_j , then the center of the horoball is the point $(x_j, y_j, 0)$. We will sometimes refer to the complex number $x_j + iy_j$ as the center of B_j in this case, and define the center of B_∞ to be ∞ ; then the center of a horoball is always an element of $\hat{\mathbb{C}} = \mathbb{C} \cup \infty$.

In addition to this we will define the *orthocenter* of a horoball $B_j \neq B_\infty$ to be the point on ∂B_∞ which is closest to B_j . Uniqueness is guaranteed by the fact that the boundary of any horoball has positive sectional curvature in \mathbb{H}^3 , being a Euclidean surface in a negatively curved space. In the upper half-space model, the orthocenter of B_j is just the point on the surface $z = 1$ directly above the center of B_j .

Given two horoballs A and B , neither equal to B_∞ , we will say that A and B are in the same *orthoclass* if either A and B lie in the same H -orbit or there exists some $g \in \pi_1(N)$ such that $g(A) = B_\infty$ and $g(B_\infty) = B$. In the latter case we say that A and B lie in *conjugate* H -orbits. We denote the orthoclasses by $\mathcal{O}(1)$, $\mathcal{O}(2)$, and so forth. For any $B \in \mathcal{O}(n)$ we call $d(B, B_\infty)$ the *orthodistance* and denote it $o(n)$; this is clearly well-defined. Order the orthoclasses $\mathcal{O}(1)$, $\mathcal{O}(2)$, \dots in such a way that the corresponding orthodistances are non-decreasing: $0 = o(1) \leq o(2) \leq \dots$. We will refer to this as the *orthodistance spectrum*. In addition we also define $e_n = \exp(o(n)/2)$, and refer to the sequence $1 = e_1 \leq e_2 \leq \dots$ as the *Euclidean spectrum*. Note that if $A \in \mathcal{O}(n)$, then in the upper half-space model the point on ∂A which is closest to B_∞ must appear to be at a height of $\exp(-d(A, B_\infty))$, which equals e_n^{-2} since $d(A, B_\infty) = o(n)$. The choice of the word ‘‘Euclidean’’ actually comes from Lemma 3.4.

Closely related to the orthoclasses are another set of equivalence classes which we will call the *orthopair classes*. These are just the equivalence classes of the action of $\pi_1(N)$ on the set of unordered pairs of horoballs $\{A, B\}$. It follows immediately from the definition that A and B are in the same orthoclass if and only if $\{A, B_\infty\}$ and $\{B, B_\infty\}$ lie in the same orthopair class. Hence we will occasionally abuse notation and denote the orthopair classes by $\mathcal{O}(1)$, $\mathcal{O}(2)$, \dots as well. The definition of orthodistance in this context is clear.

Definition 2.1. A (p, q, r) -triple (or equivalently a triple of type (p, q, r)) is a triple of horoballs $\{B_1, B_2, B_3\}$ with the property that $\{B_1, B_2\} \in \mathcal{O}(p)$, $\{B_2, B_3\} \in \mathcal{O}(q)$, and $\{B_3, B_1\} \in \mathcal{O}(r)$, possibly after re-ordering.

Now we come to the key definition of this paper.

Definition 2.2. A geometric Mom- n structure is a collection of n triples of type $(p_1, q_1, r_1), \dots, (p_n, q_n, r_n)$, no two of which are equivalent under the action of $\pi_1(N)$, and such that the indices p_i, q_i , and r_i all come from the same n -element subset of \mathbb{Z}_+ .

We will occasionally drop the word “structure” when our meaning is otherwise clear.

A geometric Mom- n will be said to be torus-friendly if $n = 2$ or if $n = 3$ and the Mom-3 does not possess exactly two triples of type (p, q, r) for any set of distinct positive indicies p, q , and r . (The geometrical implications of this term will be explained in Section 7).

So, for example, a $(1, 1, 3)$ -triple and a $(1, 3, 3)$ -triple would constitute a geometric Mom-2, while a $(1, 1, 2)$ -triple and a $(1, 1, 3)$ -triple would not. A $(1, 1, 2)$ -triple, a $(1, 1, 3)$ -triple, and a $(1, 2, 3)$ -triple, however, would constitute a geometric Mom-3 which furthermore is torus-friendly. A $(1, 1, 2)$ -triple and two $(1, 2, 3)$ -triples which are not equivalent under the action of $\pi_1(M)$ form a geometric Mom-3 which is not torus-friendly. Although this definition can clearly be generalized, in this paper we will only be discussing geometric Mom- n 's where $n = 2$ or 3 , and where the indices all come from the set $\{1, 2, 3, 4\}$.

The connection between geometric Mom- n 's and the internal Mom- n structures of [GMM2] is clear. The term “geometric” is meant to highlight the fact that this definition does not include any of the topological assumptions that are part of the definition of an internal Mom- n structure, such as embeddedness. Nevertheless, the correspondence between geometric and internal Mom- n structures is the key to proving Theorem 1.1.

3. GEOMETRICAL LEMMAS

Throughout this chapter we will be using the upper half-space model of \mathbb{H}^3 . We will take certain facts, listed below, as given; a reader who is interested may refer to [Fen] for more information.

First, the orientation-preserving isometries of \mathbb{H}^3 can be identified with the matrix group $\mathrm{PSL}(2, \mathbb{C})$ in a natural way. Each element of $\mathrm{PSL}(2, \mathbb{C})$ acts on the sphere at infinity $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ by the corresponding Mobius transformation, i.e.

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} : z \mapsto \frac{az + b}{cz + d}$$

if $z \in \mathbb{C}$, and $\infty \mapsto \frac{a}{c}$. The action of $\mathrm{PSL}(2, \mathbb{C})$ on \mathbb{H}^3 itself can be expressed similarly, using quaternions. In the upper half-space model, the point (x, y, t) in \mathbb{H}^3 with $t > 0$ can be associated to the quaternion $x + yi + tj$; then the action of $\mathrm{PSL}(2, \mathbb{C})$ can be expressed as

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} : x + yi + tj \mapsto (a(x + yi + tj) + b)(c(x + yi + tj) + d)^{-1}$$

The resulting quaternion will always be equal to $u + vi + sj$ for some real (u, v, s) with $s > 0$.

If $g \in \mathrm{PSL}(2, \mathbb{C})$ is not the identity we can determine from the trace of g whether or not g is hyperbolic, parabolic, or elliptic as an isometry (note that trace is only defined up to sign in $\mathrm{PSL}(2, \mathbb{C})$). For example, g is an elliptic isometry (that is, g is a rotation about a line in \mathbb{H}^3) if and only if $\mathrm{tr}(g) = \pm 2 \cos(\theta/2)$, where $\theta \in [-\pi, \pi]$ is the angle of rotation of g . Note that in contrast to the hyperbolic and parabolic cases, an elliptic isometry cannot be an element of $\pi_1(N)$, where N is a 1-cusped hyperbolic 3-manifold.

One final fact about $\mathrm{PSL}(2, \mathbb{C})$ that we will use: if $g = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ then the image of B_∞ under g will be a horoball which appears as a Euclidean ball of diameter $|c|^{-2}$. More generally, the subset $B(t) = \{(x, y, z) | z \geq t\}$ of \mathbb{H}^3 will be mapped to a horoball which appears as a Euclidean ball of diameter $t^{-1}|c|^{-2}$. This can be demonstrated as follows. Since $g(\infty) = \frac{a}{c}$, the image of $B(t)$ will be a horoball with center $\frac{a}{c} \in \mathbb{C}$. Therefore the orthocenter of $g(B(t))$ will correspond to the quaternion $\frac{a}{c} + \delta j$ where δ is the diameter we seek. This is just the point where $g(B(t))$ intersects the line in \mathbb{H}^3 from ∞ to $\frac{a}{c}$. Therefore the pre-image of this point under g is the point where $B(t)$ intersects the line from $g^{-1}(\infty) = -\frac{d}{c}$ to $g^{-1}(\frac{a}{c}) = \infty$. In other words,

$$\frac{a}{c} + \delta j = g \left(-\frac{d}{c} + tj \right)$$

Direct calculation with quaternions then yields the desired result.

We now begin enumerating the geometrical lemmas that we will use in the rest of the paper. The following lemma first appears (using different language) in [Ada].

Lemma 3.1. *Every orthoclass consists of two H -orbits.*

Proof: It is fairly clear from the definition that each orthoclass contains no more than two H -orbits. If an orthoclass contains exactly one H -orbit, then we must have $g^{-1}(B_\infty) = hg(B_\infty)$ for some $h \in H$, $g \notin H$. In other words, $ghg \in H$, and therefore $(gh)^2 \in H$. But as an isometry gh must be either hyperbolic, parabolic, or the identity. In the first case $(gh)^2$ would also be hyperbolic (and hence not in H), and in the last two we would have $gh \in H$ and hence $g \in H$. Either case is a contradiction. \square

The following lemma appears in [CM]; a related lemma for compact manifolds appears in [GMM]. It is reproduced here for the sake of completeness.

Lemma 3.2. *If A and B are both in $\mathcal{O}(n)$, and if $g \in \pi_1(N)$ is an isometry such that $g(B) = B_\infty$, then $g(A) \notin \mathcal{O}(n)$.*

Proof: There are two cases to consider: either A and B lie in different H -orbits or else they lie in the same H -orbit.

In the first case, by the definition of an orthoclass $g(B_\infty) = h(A)$ for some $h \in H$. By replacing g with $h^{-1}g$, therefore, we may assume that $g(B_\infty) = A$. Suppose that $g(A)$ is in $\mathcal{O}(n)$. Then either $g(A) = h_1(B)$ for some $h_1 \in H$, in which case we have $g^2(B_\infty) = h_1g^{-1}(B_\infty)$, or else $g(A) = h_1(A)$, in which case we have $g^2(B_\infty) = h_1g(B_\infty)$.

Either way we have $g^2 = h_1g^{\pm 1}h_2$ for some $h_2 \in H$. Let

$$g = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \quad h_i = \begin{bmatrix} 1 & k_i \\ 0 & 1 \end{bmatrix}, \quad i \in \{1, 2\}$$

Expanding both sides of the equation $g^2 = h_1 g^{\pm 1} h_2$ and taking the $(2, 1)$ -entry of the resulting matrix on each side, we get $c(a+d) = \pm c$. Therefore the square of the trace of g is 1, which implies that $g \in \pi_1(N)$ is elliptic of order 3, a contradiction.

The other case is when A and B lie in the same H -orbit, i.e. $A = h(B)$ for some $h \in H$. Suppose that $g(A) \in \mathcal{O}(n)$ and that $g(A)$ lies in the same H -orbit as A and B , i.e. $g(A) = k(B)$ for some $k \in H$. Then $g(A)$ and $g(B_\infty)$ both lie in $\mathcal{O}(n)$ but must lie in different H -orbits (see Lemma 3.1). Furthermore $gk^{-1}(g(A)) = g(B) = B_\infty$, and $gk^{-1}(B_\infty) = g(B_\infty)$. Therefore replacing B with $g(A)$, A with $g(B_\infty)$, and g with gk^{-1} reduces the problem to the previous case. On the other hand, suppose $g(A) \in \mathcal{O}(n)$ but $g(A)$ does not lie in the same H -orbit as A and B . Then $g(A) = kg(B_\infty)$ for some $k \in H$. Since $A = h(B) = hg^{-1}(B_\infty)$, we have $ghg^{-1} = kgl$ for some $l \in H$. This last equation, after some manipulation (note that h and l commute), implies that $k^{-1}(gl)h = (gl)^2$, which leads to a contradiction just as in the first case. This completes the proof of the lemma. \square

Another way of phrasing the above result is to say that the unordered pair $\{A, B\}$ does not lie in the same orthopair class as the pairs $\{A, B_\infty\}$ and $\{B, B_\infty\}$. Hence we immediately get the following:

Corollary 3.3. *There are no (n, n, n) -triples.*

\square

The following lemma will be quite useful when studying horoball diagrams on the surface of B_∞ :

Lemma 3.4. *Let A, B be two horoballs not equal to B_∞ and let p, q be their orthocenters. Suppose $A \in \mathcal{O}(m)$ and $B \in \mathcal{O}(n)$, and suppose that $d(A, B) = O(r)$. Then the distance between p and q along the surface of B_∞ is given by $d_E(p, q) = e_r / (e_m e_n)$.*

Proof: Using the upper half-space model, we may assume that $p = (0, 0, 1)$ and $q = (x, 0, 1)$ where $x = d_E(p, q)$. Then A is a ball of height e_m^{-2} tangent to S_∞^2 at 0 and B is a ball of height e_n^{-2} tangent to S_∞^2 at $(x, 0)$. Consider the following isometry given as an element of $\text{PSL}(2, \mathbb{C})$:

$$\sigma = \begin{bmatrix} 1 & 0 \\ -x^{-1} & 1 \end{bmatrix}$$

Since σ is a parabolic isometry which fixes 0, σ preserves A . And σ sends x to ∞ ; therefore σ sends B to a horoball centered at infinity. The height of $\sigma(B)$ will be the same as the height of the image of the point $(x, 0, e_n^{-2})$, which is readily determined to be $x^2 e_n^{-2}$. Therefore the distance between $A = \sigma(A)$ and $\sigma(B)$ is the logarithm of the ratio of $x^2 e_n^{-2}$ to the diameter of A , which implies that

$$e_r^2 = \frac{x^2 e_n^{-2}}{e_m^{-2}}$$

which proves the lemma. \square

The next two lemmas concern pairs of lines joining the centers of horoballs in \mathbb{H}^3 ; they will be used extensively in discussing when geometric Mom- n 's are embedded in Section 6.

Lemma 3.5. *Suppose $A, B, C,$ and D are disjoint horoballs in \mathbb{H}^3 with no horoball contained in the interior of another (or equivalently, with no two having the same center on the sphere at infinity). Let λ_1 be the line joining the centers of A and $C,$ λ_2 be the line joining the centers of B and $D,$ and let x be the distance between λ_1 and $\lambda_2.$ Then*

$$e^{\frac{d(A,B)+d(C,D)}{2}} + e^{\frac{d(A,D)+d(B,C)}{2}} = e^{\frac{d(A,C)+d(B,D)}{2}} \cosh x$$

Corollary 3.6. *If $\{A, B\} \in \mathcal{O}(h), \{B, C\} \in \mathcal{O}(j), \{C, D\} \in \mathcal{O}(k), \{D, A\} \in \mathcal{O}(l), \{A, C\} \in \mathcal{O}(m),$ and $\{B, D\} \in \mathcal{O}(n),$ then*

$$e_h e_k + e_j e_l = e_m e_n \cosh x$$

Proof: Arrange the horoballs in the upper half-space model in such a way that A is the horoball at infinity with height 1 and C is centered at 0. Let $z, w \in \mathbb{C}$ be the centers of B and D respectively. To compute the distance between the two lines λ_1 and $\lambda_2,$ we turn to [Fen] which says that

$$\cosh(x + iy) = -\frac{1}{2} \operatorname{tr}(\Lambda_1 \Lambda_2)$$

where

$$\Lambda_1 = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix} \in \operatorname{PSL}(2, \mathbb{C})$$

is an elliptic element of order 2 fixing the line from 0 to ∞ while

$$\Lambda_2 = \frac{i}{w-z} \begin{bmatrix} w+z & -2wz \\ 2 & -w-z \end{bmatrix} \in \operatorname{PSL}(2, \mathbb{C})$$

is an elliptic element of order 2 fixing the line from z to $w,$ and where y is the relative angle between the two lines along the shortest arc between them, which is only defined modulo $\pi.$ Then by direct calculation,

$$\begin{aligned} \cosh(x + iy) &= -\frac{i}{2(w-z)} \operatorname{tr} \left(\begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix} \begin{bmatrix} w+z & -2wz \\ 2 & -w-z \end{bmatrix} \right) \\ &= -\frac{i}{2(w-z)} (i(w+z) - i(-w-z)) \\ &= \frac{w+z}{w-z} \end{aligned}$$

To determine $\cosh x$ note that as y varies, $\cosh(x + iy) = \cosh x \cos y + i \sinh x \sin y$ varies along an ellipse in the complex plane. It is straightforward to determine that the foci of this ellipse are at -1 and $1,$ and that therefore for any $y,$

$$|\cosh(x + iy) - 1| + |\cosh(x + iy) + 1| = 2 \cosh x$$

Therefore,

$$\begin{aligned} \cosh x &= \frac{1}{2} \left(\left| \frac{w+z}{w-z} - 1 \right| + \left| \frac{w+z}{w-z} + 1 \right| \right) \\ &= \frac{|w| + |z|}{|w-z|} \end{aligned}$$

So $|w| + |z| = |w-z| \cosh x.$ But by Lemma 3.4, $|w|^2 = e^{d(C,D)-d(A,C)-d(A,D)},$ $|z|^2 = e^{d(C,B)-d(A,C)-d(A,B)},$ and $|w-z|^2 = e^{d(B,D)-d(A,B)-d(A,D)}.$ The result follows. \square

Lemma 3.7. *Let N be a cusped hyperbolic manifold with cusp neighborhood T , and suppose A, B, C , and D are all horoballs in \mathbb{H}^3 which are lifts of T . Define λ_1, λ_2 , and x as in the previous lemma. If there exists an element $g \in \pi_1(N)$ such that $g(A) = B$ and $g(C) = D$, and if $d(A, C) = d(B, D)$ is less than or equal to $2 \log(1.5152)$ then $x \geq 0.15$.*

Proof: Suppose $x < 0.15$; we will establish a contradiction. For sake of notation, suppose that $\{A, C\}$ and $\{B, D\}$ are both elements of $\mathcal{O}(k)$. Note that this implies that $e_k \leq 1.5152$. Arrange the four horoballs in the upper half-space model so that A is centered at infinity with height e_k and C is centered at 0 with height e_k^{-1} . Then suppose that

$$g = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{PSL}(2, \mathbb{C})$$

We wish to compute x in terms of a, b, c , and d in the same fashion as in the previous lemma. Since g sends 0 to $\frac{b}{d}$ and ∞ to $\frac{a}{c}$, by the same arguments as before we have that for some y ,

$$\begin{aligned} \cosh(x + iy) &= \frac{\frac{a}{c} + \frac{b}{d}}{\frac{a}{c} - \frac{b}{d}} \\ &= \frac{ad + bc}{ad - bc} \\ &= ad + bc \\ &= 2ad - 1 \end{aligned}$$

where the last two steps used $ad - bc = 1$. Hence

$$ad = \frac{1}{2} (\cosh(x + iy) + 1)$$

Since $x < 0.15$, this implies that ad lies strictly inside an ellipse in the complex plane with foci at 0 and 1, whose boundary intersects the real axis at the points $\frac{1}{2}(\cosh 0.15 + 1)$ and $\frac{1}{2}(1 - \cosh 0.15)$.

Now recall that if $c \neq 0$ then the element g sends a horoball of height t centered at infinity to a horoball of height $t^{-1}|c|^{-2}$ (whereas if $c = 0$ then g fixes the point at infinity). So since $g(A) = B$ and since B and A have disjoint interiors, we must have $e_k^{-1}|c|^{-2} \leq e_k$, or in other words $|c| \geq e_k^{-1} \geq 1.5152^{-1}$. But C and $g(A) = B$ also have disjoint interiors, so let

$$h = \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix} \in \mathrm{PSL}(2, \mathbb{C})$$

Then h swaps A and C ; therefore $hg(A)$ and $h(C) = A$ have disjoint interiors. Since

$$hg = \begin{bmatrix} ic & id \\ ia & ib \end{bmatrix}$$

this implies that $e_k^{-1}|ia|^{-2} \leq e_k$, or in other words $|a| \geq e_k^{-1} \geq 1.5152^{-1}$. Similarly, A and $D = g(C) = gh(A)$ have disjoint interiors, which implies that $|d| \geq 1.5152^{-1}$, and C and D have disjoint interiors, so $h(C) = A$ and $h(D) = hgh(A)$ have disjoint interiors, which implies that $|b| \geq 1.5152^{-1}$. Combining these facts we have $|ad| \geq 1.5152^{-2}$ and $|ad - 1| = |bc| \geq 1.5152^{-2}$. Hence we can conclude that ad lies somewhere in the shaded region indicated in Figure 2.

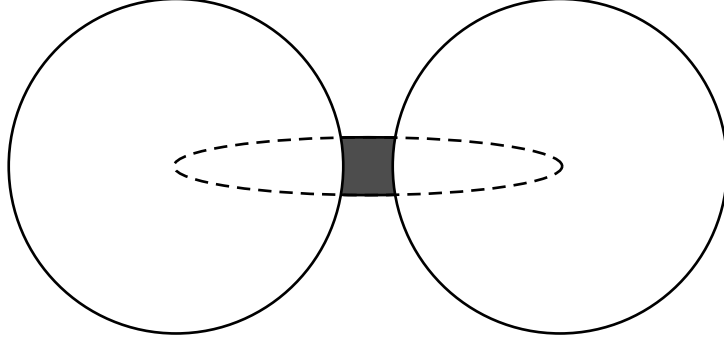


FIGURE 2. The possible range of ad in the complex plane. The shaded region is defined by the equation $|z-0|+|z-1| \leq \cosh 0.15$ (i.e. the solid ellipse bounded by the curve $\frac{1}{2}(\cosh(0.15+it)+1)$ where t is real) and the equations $|z| \geq 1.5152^{-2}$ and $|z-1| \geq 1.5152^{-2}$.

It's worthwhile at this point to sketch out the motivation for the argument that follows. Roughly speaking, we've shown that ad is approximately equal to $1/2$. Given the stated lower bounds on $|a|$ and $|d|$, this implies that $|a|$ and $|d|$ are each approximately equal to $\sqrt{1/2}$. If it were the case that $ad = 1/2$ and $|a| = |d| = \sqrt{1/2}$, then $a+d$ would have to be a real number between $-\sqrt{2}$ and $\sqrt{2}$ which would imply that $g \in \pi_1(N)$ is elliptic, a contradiction. Since we only have approximate equality in the previous statement, we wish to show that g is "approximately elliptic". More rigorously, we wish to show that $a+d$ lies close enough to the real interval from $-\sqrt{2}$ to $\sqrt{2}$ to ensure that $g^n(A)$ or $g^n(C)$ intersects A for some n (specifically $n = 2, 3,$ or 4), a contradiction which will complete the proof of the lemma.

To make this argument work we need to break the problem into two cases depending on the value of ad . The first case will be when $\Re(ad) \geq 1/2$ and ad lies in the region in the above figure; the second case is when $\Re(ad) \leq 1/2$. We proceed with the proof in the first case.

So suppose $\Re(ad) \geq 1/2$ and ad lies in the region in the above figure; we wish to determine a domain for the trace $\tau = a+d$ of g . Suppose that $a = |a|e^{i\theta}$ and $d = |d|e^{i\phi}$; then

$$\tau = e^{i\left(\frac{\theta+\phi}{2}\right)} \left((|a|+|d|) \cos \frac{\theta-\phi}{2} + i(|a|-|d|) \sin \frac{\theta-\phi}{2} \right)$$

Hence τ lies in an ellipse, centered at the origin, whose major axis has length $2(|a|+|d|)$ and whose minor axis has length $2||a|-|d||$, and whose major axis is tilted away from the real axis by half the argument of ad .

Note that we have upper bounds on all three of these quantities. First, the argument of ad is bounded since ad is contained in a bounded simply-connected region which does not contain 0. Second, since $|ad|$ is bounded above and $|a|$ and $|d|$ are both bounded below, the point $(|a|, |d|)$ is contained in a region in the first quadrant of the real plane which is bounded by the lines $x = 1.5152^{-1}$ and

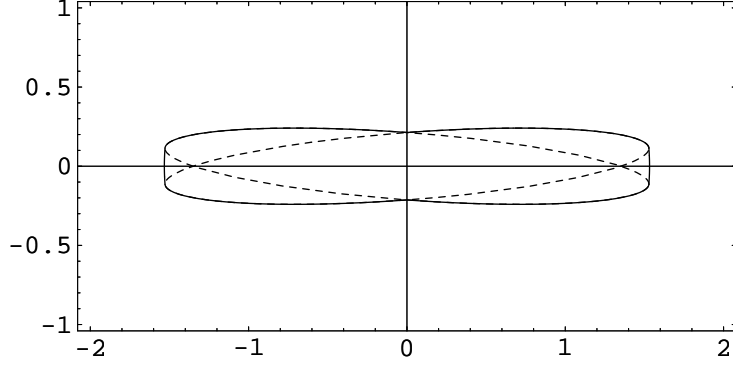


FIGURE 3. The region of possible values for $a + d$ (solid contour), generated by sweeping out ellipses with constant major axis and minor axis length, parameterized by the angle of inclination with the x -axis. The dashed contours represent the beginning and ending ellipses.

$y = 1.5152^{-1}$ and the hyperbola $xy = D$ where D is the maximum possible value of $|ad|$. This implies that $|a| + |d|$ and $||a| - |d||$ are both bounded as well. Specifically, we can state the following:

$$\begin{aligned} \frac{\theta + \phi}{2} &\leq 0.07473 \\ |a| + |d| &\leq 1.5323 \\ ||a| - |d|| &\leq 0.2124 \end{aligned}$$

These facts together imply that τ must be contained in a bounded region near the origin, as sketched out in figure 3.

Now consider g^2 , g^3 , and g^4 ; these elements do not fix ∞ since g does not. Similarly they do not fix 0. By direct calculation,

$$\begin{aligned} g^2 &= \begin{bmatrix} * & b\tau \\ c\tau & * \end{bmatrix} \\ g^3 &= \begin{bmatrix} * & b(\tau^2 - 1) \\ c(\tau^2 - 1) & * \end{bmatrix} \\ g^4 &= \begin{bmatrix} * & b(\tau^3 - 2\tau) \\ c(\tau^3 - 2\tau) & * \end{bmatrix} \end{aligned}$$

(Here $*$ is used to denote entries whose value is unimportant.) Since A and $g^n(A)$ (respectively C and $g^n(C)$) have disjoint interiors, by the same arguments as before all three of the quantities $|c\tau|$, $|c(\tau^2 - 1)|$, and $|c(\tau^3 - 2\tau)|$ (respectively $|b\tau|$, $|b(\tau^2 - 1)|$, and $|b(\tau^3 - 2\tau)|$) must be no less than 1.5152^{-1} . Since $|bc| = |ad - 1|$ and since ad lies in a bounded simply connected region which does not contain 1, $\sqrt{|bc|}$ is bounded above and hence $|\tau|$, $|\tau^2 - 1|$, and $|\tau^3 - 2\tau|$ are bounded below. Specifically

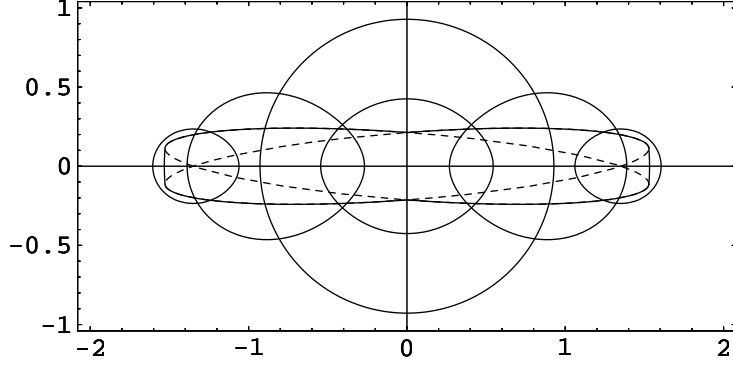


FIGURE 4. The region from figure 3, contained inside the union of the regions $|\tau| < 0.9281$, $|\tau^2 - 1| < 0.9281$, and $|\tau^3 - 2\tau| < 0.9281$.

we have

$$\begin{aligned} |\tau| &\geq 0.9281 \\ |\tau^2 - 1| &\geq 0.9281 \\ |\tau^3 - 2\tau| &\geq 0.9281 \end{aligned}$$

But as demonstrated in figure 4, the intersection of the three regions described by the above inequalities is disjoint from the previously determined range of possible values of τ , a contradiction. This completes the proof in the first case.

The proof in the second case is nearly identical, only the specific numbers differ: replace 0.07473 with 0.08599, 1.5323 with 1.4262, 0.2124 with 0.1062, and 0.9281 with 0.8698. This completes the proof of the lemma. \square

The remaining lemmas in this chapter will be quite arbitrary and geometrical, but they will be quite useful in the next chapter when we begin to compute the volume of manifolds which do not possess geometric Mom- n 's. We start with a lemma and definition which first appeared in [CM]:

Lemma 3.8. *Let B be a horoball centered at infinity in the upper half-space model of \mathbb{H}^3 , whose boundary has Euclidean height $1/b$. Let A be a half-space, i.e. one of the two connected components of the complement of a plane; assume that A does not contain the point at infinity, and that the plane which defines A appears in the upper half-space model as a Euclidean hemisphere with radius $1/a$. Assume that $a < b$, so that $A \cap B$ is non-empty. Then the volume of $A \cap B$ is*

$$\pi \left(\frac{b^2}{2a^2} - \frac{1}{2} + \log \frac{a}{b} \right)$$

Definition 3.9. *Denote the above quantity by $\text{lessvol}(a, b)$.*

Proof of Lemma 3.8: Without loss of generality, assume that the plane defining A appears as a Euclidean hemisphere centered at the origin. Then convert to cylindrical coordinates; the desired volume is given by the following integral:

$$\int_{\frac{1}{b}}^{\frac{1}{a}} \int_0^{2\pi} \int_0^{\sqrt{\frac{1}{a^2} - z^2}} \frac{r}{z^3} dr d\theta dz$$

The result follows immediately. \square

The following lemma is a straightforward exercise in trigonometry; the proof is left to the reader:

Lemma 3.10. *Let A and B be two circular disks in the plane with radius a and b respectively, such that the centers of A and B are c units apart. Suppose that $|a - b| \leq c \leq a + b$, i.e. suppose that the two disks overlap but neither disk is entirely contained in the interior of the other. Then*

$$\text{Area}(A \cap B) = a^2 f\left(\frac{a^2 - b^2 + c^2}{2ac}\right) + b^2 f\left(\frac{-a^2 + b^2 + c^2}{2bc}\right)$$

where $f(x) = \cos^{-1}(x) - x\sqrt{1 - x^2}$, if $a \neq b$. If $a = b$, then

$$\text{Area}(A \cap B) = 2a^2 f\left(\frac{c}{2a}\right)$$

Definition 3.11. *Define the function $\text{overlapArea}(a, b, c)$ to equal the right-hand side of the first equation above if $a \neq b$, and the right-hand side of the second equation above if $a = b$.*

Finally, we provide a companion to the above lemma. The function $f(x)$ described above, while relatively simple, turns out to be unsuitable for rigorous floating-pointing computation. As will be explained in more detail in Section 5, a polynomial approximation to $f(x)$ will be extremely useful. Hence we provide the following:

Lemma 3.12. *Let A and B be two circular disks in the plane with radius a and b respectively, such that the centers of A and B are c units apart. Suppose that $|a - b| \leq c \leq a + b$, i.e. suppose that the two disks overlap but neither disk is entirely contained in the interior of the other. Then*

$$\text{Area}(A \cap B) \leq a^2 g\left(\frac{a^2 - b^2 + c^2}{2ac}\right) + b^2 g\left(\frac{-a^2 + b^2 + c^2}{2bc}\right)$$

where $g(x) = (\frac{5}{3} - \frac{\pi}{2})x^5 + \frac{1}{3}x^3 - 2x + \frac{\pi}{2}$, if $a \neq b$. If $a = b$, then

$$\text{Area}(A \cap B) \leq 2a^2 g\left(\frac{c}{2a}\right)$$

Definition 3.13. *Define the function $\text{overlapApprox}(a, b, c)$ to equal the right-hand side of the first inequality above if $a \neq b$, and the right-hand side of the second inequality above if $a = b$.*

Proof: Comparing the above lemma to Lemma 3.10, clearly it would suffice to show that $g(x) \geq f(x)$ for all $x \in [-1, 1]$ where $f(x) = \cos^{-1}(x) - x\sqrt{1-x^2}$. Unfortunately, this cannot be the case as both $f(x) - \frac{\pi}{2}$ and $g(x) - \frac{\pi}{2}$ are odd functions. However, it is true that $g(x) \geq f(x)$ for all $x \in [0, 1]$; we will use this fact in what follows. Let $h(x) = g(x) - f(x)$. We note the following facts about $h(x)$, which are readily verified: $h(x)$ is an odd function, $h(\pm 1) = h(0) = 0$, $h(x) > 0$ if $0 < x < 1$, and $h(x)$ has a single local maximum at $x_{max} = 0.928\dots$. By Lemma 3.10 it suffices to show that

$$a^2 h\left(\frac{a^2 - b^2 + c^2}{2ac}\right) + b^2 h\left(\frac{-a^2 + b^2 + c^2}{2bc}\right) \geq 0$$

if $a \neq b$.

Let P be the center of the first circle, Q the center of the second, and let R be one of the points where the circles intersect. Let θ be the angle of the triangle PQR at P , and let ϕ be the angle at Q . Applying the law of cosines for PQR to the above inequality, we get

$$a^2 h(\cos \theta) + b^2 h(\cos \phi) \geq 0$$

Note that if $\theta = 0$ or π then $\phi = 0$ or π and vice-versa. Since $h(1) = h(-1) = 0$ the lemma is true in either case. So suppose that $\theta, \phi \in (0, \pi)$. Note that since θ and ϕ are angles of a (possibly flat) triangle this also implies that $\theta + \phi \in (0, \pi)$. Then we can apply the law of sines to the above inequality to get

$$\left(\frac{c \sin \phi}{\sin(\theta + \phi)}\right)^2 h(\cos \theta) + \left(\frac{c \sin \theta}{\sin(\theta + \phi)}\right)^2 h(\cos \phi) \geq 0$$

Letting $x = \cos \theta$ and $y = \cos \phi$, it suffices to show that

$$(1 - y^2)h(x) + (1 - x^2)h(y) \geq 0$$

where $x < 1$, $y < 1$, and $x > -y$ (since $\theta < \pi - \phi$). Let $H(x, y)$ equal the left-hand side of the above inequality. Note that $H(x, 1) = H(1, y) = 0$ since $h(1) = 0$, and $H(x, -x) = 0$ for $x \in [-1, 1]$ since $h(x)$ is odd. Hence the lemma will be proved if we can show that $H(x, y)$ has a non-negative value at any local minimum in the interior of the triangle bounded by the lines $x = 1$, $y = 1$, and $x + y = 0$. Since $h(x) \geq 0$ if $0 \leq x \leq 1$, $H(x, y) \geq 0$ whenever $0 \leq x \leq 1$ and $0 \leq y \leq 1$. Also, $H(x, y) = H(y, x)$. Therefore it suffices to examine local minima in the interior of the triangle bounded by the lines $x = 1$, $y = 0$, and $x + y = 0$. We now show that there are no such local minima in that region.

Computing the gradient, we see that if (x_0, y_0) is a local extreme point of $H(x, y)$ then

$$\begin{aligned} (1 - y_0^2)h'(x_0) &= 2x_0h(y_0) \\ (1 - x_0^2)h'(y_0) &= 2y_0h(x_0) \end{aligned}$$

If (x_0, y_0) is in the interior of the triangle described above then $0 < x_0 < 1$ and $-1 < y_0 < 0$. Then from the first equation, since $h(y_0) < 0$ we must have $h'(x_0) < 0$, and from the second equation since $h(x_0) > 0$ we must have $h'(y_0) < 0$. Therefore $x_{max} < x_0 < 1$, and $-1 < y_0 < -x_{max}$. Now suppose further that (x_0, y_0) is a local minimum. Then we must have $\partial^2 H / \partial x^2 > 0$ at that point, which implies that

$$h''(x_0) > \frac{2h(y_0)}{1 - y_0^2}$$

But it can be readily computed that the maximum value of $h''(x)$ on the interval $[x_{max}, 1]$ is less than -1.6 , while the minimum value of $2h(y_0)/(1 - y_0^2)$ on the interval $[-1, -x_{max}]$ is greater than -0.6 , contradicting the above inequality. This completes the proof of the lemma. \square

4. BOUNDING THE PARAMETER SPACE

In Section 5 we will show that a one-cusped hyperbolic 3-manifold with volume less than or equal to 2.848 has a geometric Mom-2 or Mom-3 structure. This proof involves constructing volume estimates in terms of the first three non-trivial elements of the Euclidean spectrum e_2 , e_3 , and e_4 , and then evaluating those formulas with rigorous computer assistance. This requires us to restrict our attention from the space of all possible values of (e_2, e_3, e_4) to a compact subset of that space. Doing so is the purpose of this section.

Specifically, we wish to prove the following:

Proposition 4.1. *Let N be a one-cusped hyperbolic 3-manifold with $\text{Vol}(N) \leq 2.848$. Then $e_2 \leq 1.4751$; furthermore, one of the following statements must be true:*

- $e_3 \leq 1.5152$, or
- N contains a geometric Mom-2 structure consisting only of $(1, 1, 2)$ -triples and $(1, 2, 2)$ -triples.

Note that this effectively provides upper bounds on e_2 and e_3 for manifolds with $\text{Vol}(N) \leq 2.848$; lower bounds are trivially provided by $e_k \geq e_1 = 1$. This proposition does not provide an upper bound for e_4 , but we can do without as described in the next section.

The proof of Proposition 4.1 depends on several complicated estimates of area and volume and spans multiple cases. Similar arguments, presented in less detail for brevity, will be used for the proofs in Section 5, and hence this section should perhaps be considered a “warm-up” for those results.

We begin with the following simple result:

Lemma 4.2. *The volume of a one-cusped hyperbolic three-manifold N is at least*

$$\frac{e_2^4 \sqrt{3}}{2} - \pi (e_2^2 - 1 - 2 \log e_2)$$

Corollary 4.3. *If $e_2 > 1.4751$ then $\text{Vol}(N) > 2.848$.*

Proof: Let $\{p_i\} \subset \partial B_\infty$ be the set of orthocenters of horoballs belonging to $\mathcal{O}(1)$. According to lemmas 3.2 and 3.4, $d_E(p_i, p_j) \geq e_2$ for all $i \neq j$. Therefore ∂B_∞ can be packed by circles of radius $e_2/2$ centered at each point p_i . There are two H -orbits of such points by lemma 3.1, and therefore the area of $\partial B_\infty/H$ is at least $\pi(e_2^2)/2$. Since our packing is by circles of uniform radius, we can immediately improve this bound by a factor of $\sqrt{12}/\pi$ (the density of the hexagonal circle packing) to get

$$\text{Area}(\partial B_\infty/H) \geq e_2^2 \sqrt{3}$$

However we want to estimate volume, not area. The volume of B_∞/H is exactly $\text{Area}(\partial B_\infty/H)/2$, and this is a lower bound on the volume of N , but this lower bound only accounts for the volume inside the cusp neighborhood. We would like our estimate to count some of the volume outside the cusp neighborhood as well.

We do this by “inflating” the cusp neighborhood in a uniform fashion. In the upper half-space model, this corresponds to replacing B_∞ with a new horoball C which is centered at infinity but whose Euclidean height is some positive number which may be less than one. In particular we choose C to have height $1/e_2$. Then

$$\begin{aligned} \text{Vol}(C/H) &= \frac{e_2^2}{2} \text{Area}(\partial B_\infty/H) \\ &= \frac{e_2^4}{2} \sqrt{3} \end{aligned}$$

but of course C/H is no longer embedded in N ; it is only immersed. To get a valid lower bound for $\text{Vol}(N)$ we must “cut off” C/H by subtracting the volumes of those regions where C/H extends beyond the Ford domain corresponding to B_∞/H . The maximum height of a Ford face corresponding to $A \in \mathcal{O}(2)$ is just $1/e_2$; therefore C/H only extends past the Ford faces corresponding to $\mathcal{O}(1)$ -horoballs and not $\mathcal{O}(2)$ -horoballs (or $\mathcal{O}(n)$ -horoballs for $n > 2$). Hence at most we must subtract twice the volume of the region where the horoball C intersects the half-space corresponding to some $B \in \mathcal{O}(1)$. That volume is given by the lessvol function (see Definition 3.9); specifically,

$$\begin{aligned} \text{Vol}(N) &\geq \text{Vol}(C/H) - 2 \text{lessvol}(1, e_2) \\ &= \frac{e_2^4 \sqrt{3}}{2} - \pi (e_2^2 - 1 - 2 \log e_2) \end{aligned}$$

This is the desired result.

The corollary follows since the above function is easily verified to be increasing in e_2 . \square

In practice, the area of the cusp torus $\partial B_\infty/H$ will usually be far greater than our crude estimate of $e_2^2 \sqrt{3}$, because the density of the packing of ∂B_∞ by circles around the orthocenters of the $\mathcal{O}(1)$ -horoballs is typically nowhere near optimal. One strategy to try and capture the extra area between the circles is simply to use larger circles: specifically, use circles of radius $e_3/2$ instead of $e_2/2$ to estimate the area. This has the advantage of incorporating e_3 into the area estimate, which will lead to the upper bound on e_3 that we seek. The disadvantage, of course, is that by increasing the radius we can no longer guarantee that each circle will be embedded in $\partial B_\infty/H$: the larger circles may overlap.

However, such overlaps imply the existence of triples which may be part of a geometric Mom- n structure. If two circles of radius $e_3/2$ overlap then the distance between their centers is less than e_3 , which implies by Lemma 3.4 that the corresponding horoballs, together with B_∞ , must form a $(1, 1, 2)$ -triple. If there are no such triples, then our new expanded circles will not overlap. This leads to the following:

Lemma 4.4. *Let N be a one-cusped hyperbolic three-manifold whose horoball diagram contains no triples of type $(1, 1, 2)$. Then*

$$(1) \quad \text{Vol}(N) \geq \frac{e_2^2 e_3^2 \sqrt{3}}{2} - \pi (e_2^2 - 1 - 2 \log e_2)$$

Corollary 4.5. *Under the assumptions of the lemma, if $e_3 > 1.8135$ then $\text{Vol}(N) > 2.848$.*

Proof: Let $\{p_i\} \subset \partial B_\infty$ be defined as in the proof of Theorem 4.2. Since the horoball diagram for N contains no $(1, 1, 2)$ handles, $d_E(p_i, p_j) \geq e_3$ for all $i \neq j$, and ∂B_∞ can be packed by circles of radius $e_3/2$ centered at each point p_i . The rest of the proof proceeds just as in the proof of Theorem 4.2.

To prove the corollary, note that the right-hand side of equation (1) is both increasing in e_2 for fixed e_3 and vice-versa. When $e_3 = 1.8135$ and $e_2 = 1$, $\text{Vol}(N) > 2.848$ and the result follows. \square

The upper bound on e_3 obtained from Corollary 4.5 is unfortunately too large to be useful. The next step is to improve upon it by accounting for the effect of the horoballs in $\mathcal{O}(2)$ upon $\text{Area}(\partial B_\infty)$. If e_2 is small, then we expect to be able to construct circles around the orthocenters of these horoballs which will be almost as large as the circles around the orthocenters of the horoballs in $\mathcal{O}(1)$, increasing our estimate. While if e_2 is large, then e_3 must be large as well, increasing the area of the circles constructed in the proof of Lemma 4.4.

Thus we wish to estimate $\text{Area}(\partial B_\infty)$ using two sets of circles: one set corresponding to the elements of $\mathcal{O}(1)$ as before, and another set corresponding to the elements of $\mathcal{O}(2)$. But we need to decide how large the circles in this new second set will be. Also, whatever new circles we construct may intersect the previously constructed circles corresponding to the $\mathcal{O}(1)$ horoballs. In particular, if N possesses a $(1, 2, 2)$ -triple then $\mathcal{O}(2)$ -horoballs and $\mathcal{O}(1)$ -horoballs will be close enough for these circles to possibly intersect. This can be accounted for using the function $\text{overlapArea}(a, b, c)$ from Lemma 3.10, leading to the following:

Lemma 4.6. *Let N be a one-cusped hyperbolic 3-manifold containing no triples of type $(1, 1, 2)$ and at most one triple of type $(1, 2, 2)$. Furthermore, assume that $e_2 \leq 1.4751$ and $e_3 \leq 1.8135$. Then*

$$\begin{aligned} \text{Area}(\partial B_\infty/H) &\geq 2\pi \left(\frac{e_3}{2}\right)^2 + 2\pi \left(\frac{e_3}{e_2} - \frac{e_3}{2}\right)^2 \\ &\quad - 2 \text{overlapArea} \left(\frac{e_3}{e_2} - \frac{e_3}{2}, \frac{e_3}{2}, 1\right) \\ &\quad - \text{overlapArea} \left(\frac{e_3}{e_2} - \frac{e_3}{2}, \frac{e_3}{e_2} - \frac{e_3}{2}, \frac{1}{e_2^2}\right) \end{aligned}$$

and furthermore

$$(2) \quad \text{Vol}(N) \geq \text{Area}(\partial B_\infty/H) \frac{e_3^2}{2} - \pi \left(e_3^2 - 1 - 2 \log e_3 + \frac{e_3^2}{e_2^2} - 1 - 2 \log \frac{e_3}{e_2} \right)$$

Corollary 4.7. *Under the assumptions of Lemma 4.6 if $e_3 > 1.4751$ then $\text{Vol}(N) > 2.848$.*

Proof: Let $\{p_i\}$ be the set of orthocenters of $\mathcal{O}(1)$ horoballs as before, and let $\{q_i\}$ be the set of orthocenters of $\mathcal{O}(2)$ horoballs. Assume that we have already constructed circles of radius $e_3/2$ around each p_i as in Lemma 4.4. As before, since there is not a $(1, 1, 2)$ -triple these circles will not overlap.

Now in addition to this, construct circles of radius $e_3/e_2 - e_3/2$ around each of the points q_i . These circles may conceivably overlap either the previous circles or each other. Suppose that the circle around q_i overlaps the circle around p_j for some i and j , and let B_i and B_j be the corresponding horoballs. Then clearly

$d_E(q_i, p_j) < e_3/e_2$, which by Lemma 3.4 implies that $d(B_i, B_j) < o(3)$. Therefore (B_i, B_j, B_∞) must be a triple of type $(1, 2, 2)$ (or a triple of type $(1, 1, 2)$, but we're explicitly excluding that case right now).

Or suppose that the circles around q_i and q_j overlap for some i and j , and let B_i and B_j be the corresponding horoballs. If $d(B_i, B_j) \geq o(3)$, then by Lemma 3.4 we must have $d_E(q_i, q_j) \geq e_3/e_2^2$. But it is easy to show that $2(e_3/e_2 - e_3/2) \leq e_3/e_2^2$ for all positive e_2 and e_3 ; therefore if the circles are indeed overlapping then $d(B_i, B_j) < o(3)$. Therefore (B_i, B_j, B_∞) must again be a triple of type $(1, 2, 2)$ (recall that a triple of type $(2, 2, 2)$ is impossible by Lemma 3.2).

In summary, any overlaps between the new circles and the old ones, or between the new circles and each other, arise due to the presence of a triple of type $(1, 2, 2)$. And we have supposed that there is no more than one such triple up to the action of G .

After taking the quotient by the action of H we are left with two new circles of radius $e_3/e_2 - e_3/2$ and up to three new cases where one circle overlaps another. First, a $(1, 2, 2)$ -triple implies that $d_E(q_i, q_j) = 1/e_2^2$ for some i and j ; therefore in $\partial B_\infty/H$ we may see either the two new circles overlapping each other or else we may see one of the new circles overlapping itself. Secondly, a $(1, 2, 2)$ -triple implies that $d_E(q_i, p_j) = 1$ for some i and j ; in $\partial B_\infty/H$ we may see up to two instances of a new circle being overlapped by an old one.

Therefore the area of $\partial B_\infty/H$ is at least the area of the two new circles plus the two old circles minus the three possible overlaps. Using Lemma 3.10, this proves the first half of the lemma.

There is a technical issue that must be addressed: Lemma 3.10 is not valid if the circles in question do not overlap, or if one circle is contained in the interior of the other. Since we're using Lemma 3.10 twice we need to ensure that both of the following sets of inequalities hold:

$$\begin{aligned} \left| \frac{e_3}{e_2} - e_3 \right| &\leq 1 && \leq \frac{e_3}{e_2} \\ 0 &\leq e_2^{-2} && \leq \frac{2e_3}{e_2} - e_3 \end{aligned}$$

All of the above inequalities can be verified by elementary means when $1 \leq e_2 \leq 1.4751$ and $e_2 \leq e_3 \leq 1.8135$. Hence our use of Lemma 3.10 is valid. Note this also confirms that the new circles actually contribute to the area of $\partial B_\infty/H$.

To find a lower bound of the volume of N , we inflate the cusp neighborhood as in the proof of Theorem 4.2. This time we obtain a horoball C centered at infinity with Euclidean height $1/e_3$. Then $\text{Vol}(C/H) = \frac{1}{2} \text{Area}(\partial B_\infty/H) e_3^2$. And while C/H extends beyond the Ford domain corresponding to B_∞/H , at worst it only extends past the Ford faces corresponding to $\mathcal{O}(1)$ -horoballs and $\mathcal{O}(2)$ -horoballs. Hence,

$$\text{Vol}(N) \geq \text{Area}(\partial B_\infty/H) \frac{e_3^2}{2} - 2 \text{lessvol}(1, e_3) - 2 \text{lessvol}(e_2, e_3)$$

This proves the second half of the lemma. To prove the corollary, note that the resulting volume bound is increasing in e_3 for fixed e_2 and decreasing in e_2 for fixed e_3 . Hence the minimum value of the volume bound over the domain $1 \leq e_2 \leq$

1.4751, $1.4751 \leq e_3 \leq 1.8135$ occurs when $e_2 = e_3 = 1.4751$, and at the point $\text{Vol}(N) > 2.848$. \square

The next step is to perform the same analysis in the case where the horoball diagram for N contains exactly one triple of type $(1, 1, 2)$ and no triples of type $(1, 2, 2)$. (Note that if N had at least one of each type of triple then N would have a geometric Mom-2.) This analysis is similar to that of the previous case, and therefore in what follows some details are omitted.

Lemma 4.8. *Let N be a one-cusped hyperbolic 3-manifold whose horoball diagram contains one triple of type $(1, 1, 2)$. Then*

$$\text{Area}(\partial B_\infty/H) \geq 2\pi \left(\frac{e_3}{2}\right)^2 - \text{overlapArea}\left(\frac{e_3}{2}, \frac{e_3}{2}, e_2\right)$$

and furthermore

$$(3) \quad \text{Vol}(N) \geq \text{Area}(\partial B_\infty/H) \frac{e_2^2}{2} - \pi(e_2^2 - 1 - 2 \log e_2)$$

Corollary 4.9. *Under the same assumptions as in Lemma 4.8, if $e_3 > 2.1491$ then $\text{Vol}(N) > 2.848$.*

Proof: As in the proof of Lemma 4.4, we wish to tile $\partial B_\infty/H$ with disks of radius $e_3/2$ centered at the centers of the $\mathcal{O}(1)$ -horoballs. But since in this case there is by assumption a single $(1, 1, 2)$ -triple, such disks will overlap exactly once. This, together with Lemma 3.10, proves the first part of the lemma. (It is a trivial matter to confirm that 3.10 applies; the relevant inequality is $0 \leq e_2 \leq e_3$.) To prove the second part, we inflate the cusp to obtain a horoball C centered at infinity and with Euclidean height $1/e_2$, and then proceed just as in Theorem 4.2.

The resulting bound on $\text{Vol}(N)$ is increasing in e_3 for fixed e_2 and vice-versa, and when $e_2 = 1$ and $e_3 = 2.1491$ we get $\text{Vol}(N) > 2.848$; this proves the corollary. \square

We now wish to improve the bound on e_3 by mimicking the argument used in the previous case. That is, we wish to construct additional circles in the horoball diagram of radius $e_3/e_2 - e_3/2$ corresponding to the $\mathcal{O}(2)$ horoballs as before. As in the previous case, such circles are small enough that the only overlaps between them and the circles of radius $e_3/2$ will arise as a result of the presence of a $(1, 1, 2)$ -triple (of which we assume there is at most one) or a $(1, 2, 2)$ -triple (of which we will assume there are none at all).

Here we run into a problem, however, when we try to confirm that Lemma 3.10 applies, or in other words when we try to confirm that the new circles are both overlapped by the old ones and not completely contained within the old ones. According to Lemma 3.4, in the presence of a $(1, 1, 2)$ triple the orthocenter of a $\mathcal{O}(2)$ horoball will be at a distance of $1/e_2$ from the center of some $\mathcal{O}(1)$ horoball. Therefore Lemma 3.10 applies if and only if the following inequalities hold:

$$\left| \frac{e_3}{e_2} - e_3 \right| \leq \frac{1}{e_2} \leq \frac{e_3}{e_2}$$

The right-hand inequality is trivially true since $e_3 \geq 1$ but the left-hand inequality simplifies to $e_3 \leq (e_2 - 1)^{-1}$. This last inequality is *not* always true in the region $1 \leq e_2 \leq 1.4751$, $e_2 \leq e_3 \leq 2.1491$. So we have to be a little more clever. Note that

the inequality $e_3 \leq (e_2 - 1)^{-1}$ fails when e_3 and e_2 are both large, but the worst case in the previous lemma occurred when e_2 was small. Hence a more sophisticated version of Corollary 4.9 is required, one which lets us restrict our attention to a smaller region which does not intersect the curve $e_3 = (e_2 - 1)^{-1}$:

Corollary 4.10. *Under the assumptions of Lemma 4.8, if $e_3 > 2.1491 - (e_2 - 1)$ then $\text{Vol}(N) > 2.848$.*

Proof: This follows from two observations. First, that the volume estimate in Lemma 4.8 is increasing in e_3 for fixed e_2 . Second, if $e_3 = 2.1491 - (e_2 - 1)$ then $\text{Vol}(N) > 2.848$, as can be readily verified by direct computation. \square

Note that the curve $e_3 = (e_2 - 1)^{-1}$ lies above the line $e_3 = 2.1491 - (e_2 - 1)$. Thus with the preceding corollary in hand, we can now prove the following:

Lemma 4.11. *Let N be a one-cusped hyperbolic 3-manifold containing no triples of type $(1, 2, 2)$ and one triple of type $(1, 1, 2)$. Furthermore, assume that $e_2 \leq 1.4751$ and $e_3 \leq 2.1491 - (e_2 - 1)$. Then*

$$\begin{aligned} \text{Area}(\partial B_\infty/H) &\geq 2\pi \left(\frac{e_3}{2}\right)^2 + 2\pi \left(\frac{e_3}{e_2} - \frac{e_3}{2}\right)^2 - \text{overlapArea}\left(\frac{e_3}{2}, \frac{e_3}{2}, e_2\right) \\ &\quad - 2\text{overlapArea}\left(\frac{e_3}{e_2} - \frac{e_3}{2}, \frac{e_3}{2}, \frac{1}{e_2}\right) \end{aligned}$$

and furthermore

$$\text{Vol}(N) \geq \text{Area}(\partial B_\infty/H) \frac{e_3^2}{2} - \pi \left(e_3^2 - 1 - 2 \log e_3 + \frac{e_3^2}{e_2^2} - 1 - 2 \log \frac{e_3}{e_2} \right)$$

Corollary 4.12. *Under the assumptions of Lemma 4.11, if $e_3 > 1.5152$ then $\text{Vol}(N) > 2.848$.*

Proof: As promised this proof will be somewhat light on details due to the extreme similarity to the previous case. Let $\{p_i\}$ be the orthocenters of the $\mathcal{O}(1)$ horoballs and $\{q_i\}$ the orthocenters of the $\mathcal{O}(2)$ horoballs as before. Construct circles of radius $e_3/2$ around the p_i 's. As in Lemma 4.8 there is one overlap between those two disks, caused by the $(1, 1, 2)$ -triple. In addition, we also have circles of radius $e_3/e_2 - e_3/2$ centered at the q_i 's. These circles do not intersect each other since there are by assumption no $(1, 2, 2)$ -triples in N ; however since there is one $(1, 1, 2)$ -triple there will be two overlaps between these circles and the circles of radius $e_3/2$. This together with Lemma 3.10 and our observations after Corollary 4.10 prove the first part of the lemma. To prove the second part, we inflate the cusp to obtain a horoball C centered at infinity with Euclidean height $1/e_3$, which at worst extends past the Ford faces corresponding to the $\mathcal{O}(1)$ -horoballs and $\mathcal{O}(2)$ -horoballs, then apply the lessvol function as before. The resulting bound on $\text{Vol}(N)$ is increasing in e_3 for fixed e_2 , and decreasing in e_2 for fixed e_3 , and when $e_2 = 1.4751$ and $e_3 = 1.5152$ we get $\text{Vol}(N) > 2.848$. This proves the corollary. \square

We now finally have enough tools to prove the main result of this section:

Proof of Proposition 4.1: Suppose that N is such that $\text{Vol}(N) \leq 2.848$. Then $e_2 \leq 1.4751$ by Corollary 4.3.

Suppose that N does *not* contain a geometric Mom-2 of the type described in the proposition; we wish to show that this implies $e_3 \leq 1.5152$. Our assumption implies that N cannot contain:

- Two or more (1, 1, 2)-triples
- Two or more (1, 2, 2)-triples
- Both a (1, 1, 2)-triple and a (1, 2, 2)-triple.

So if N contains no (1, 1, 2)-triples, then it contains at most one (1, 2, 2)-triple and Corollary 4.7 applies. If N contains exactly one (1, 1, 2)-triple, then it must contain no (1, 2, 2)-triples and Corollary 4.12 applies, completing the proof. \square

5. FINDING A GEOMETRIC MOM- n , $n = 2$ OR 3

The goal of this section is to strengthen the results of the previous section to obtain the following:

Proposition 5.1. *Let N be a one-cusped hyperbolic 3-manifold with $\text{Vol}(N) \leq 2.848$. Then N possesses a geometric Mom- n structure for $n = 2$ or 3 which is torus-friendly.*

The technique used to prove the above theorem is identical in principle to the techniques used to prove Proposition 4.1. Namely, we assume that N does not possess such a geometric Mom- n structure and then construct a lower bound on the area of the cusp torus. Specifically we will construct circles around the orthocenters of horoballs in $\mathcal{O}(1)$, $\mathcal{O}(2)$, and $\mathcal{O}(3)$. The circles we will construct will be large enough that they will overlap one another, but we can use the lack of a Mom-2 or Mom-3 structure to carefully limit the number of such overlaps that will occur. Once we have a bound on the area of the cusp torus we can use that (along with a careful estimate of the volume of the manifold outside the cusp neighborhood) to find a bound on the volume of N . The difficulties, as we will see, are in the number of cases to be considered and the complex nature of the volume bounds that result. But for now, we begin with the following:

Definition 5.2. *Let $e_{max} = \min(e_4, 1.5152)$, and let*

$$A_0 = \sum_{i=1}^3 2\pi \left(e_{max} \left(e_i^{-1} - \frac{1}{2} \right) \right)^2$$

Recall that by the results of the previous section we may assume that $e_3 \leq 1.5152 < 2$; therefore $e_{max}(e_i^{-1} - 1/2)$ is always positive for $i \leq 3$. Furthermore, if p_i and p_j are the orthocenters of horoballs $B_i \in \mathcal{O}(i)$ and $B_j \in \mathcal{O}(j)$, where $i, j \in \{1, 2, 3\}$, and if $d(B_i, B_j) \geq o(4)$, then $d_E(p_i, p_j) \geq e_4/e_i e_j$ by Lemma 3.4. Then we have the following:

$$\begin{aligned} \frac{e_4}{e_i e_j} - e_{max} \left(e_i^{-1} - \frac{1}{2} \right) - e_{max} \left(e_j^{-1} - \frac{1}{2} \right) &\geq \frac{e_4}{e_i e_j} (1 - e_i)(1 - e_j) \\ &\geq 0 \end{aligned}$$

where the last line follows since $e_i \geq 1$ for all i .

Therefore if we place a circle of radius $e_{max}(e_i^{-1} - 1/2)$ around the orthocenter of both of the horoballs in $\mathcal{O}(i)$, for $i \in \{1, 2, 3\}$, and if the manifold N does not possess *any* horoball triples involving $\mathcal{O}(1)$, $\mathcal{O}(2)$, and $\mathcal{O}(3)$, then those six circles will have disjoint interiors and therefore A_0 will be a lower bound for the area of

the cusp torus $\partial B_\infty/H$. Of course it is highly unlikely that N will possess no such triples, and thus we must consider the possibility that some of those six circles will overlap. This leads to our next definition:

Definition 5.3. *Let*

$$l_{i,j,k} = \text{overlapApprox} \left(e_{\max}(e_i^{-1} - \frac{1}{2}), e_{\max}(e_j^{-1} - \frac{1}{2}), c \right)$$

where $c = \min(e_k/(e_i e_j), e_{\max}(e_i^{-1} + e_j^{-1} - 1))$.

Recall that $\text{overlapApprox}(a, b, c)$ was defined in Definition 3.13, and by Lemma 3.12 it is always greater than or equal to the area of the intersection of a circle of radius a and a circle of radius b whose centers are c units apart, provided that $|a - b| \leq c \leq a + b$.

Hence if p_i, p_j are respectively the orthocenters of horoballs $B_i \in \mathcal{O}(i)$, $B_j \in \mathcal{O}(j)$, and if $d(B_i, B_j) = o(k)$, where i, j , and k are all in $\{1, 2, 3\}$, then $l_{i,j,k}$ will be greater than or equal to the amount of overlap between a circle of radius $e_{\max}(e_i^{-1} - 1/2)$ around p_i and a circle of radius $e_{\max}(e_j^{-1} - 1/2)$ around p_j . We need to check that the condition $|a - b| \leq c$ holds. Assuming that $e_i \leq e_j$, we have

$$\begin{aligned} c - |a - b| &\geq \frac{e_k}{e_i e_j} - e_{\max} |e_i^{-1} - e_j^{-1}| \\ &= \frac{1}{e_i e_j} (e_k - e_{\max}(e_j - e_i)) \\ &\geq \frac{1}{e_i e_j} (1 - 1.5152(1.5152 - 1)) \\ &\geq 0 \end{aligned}$$

where the second-to-last line used the fact that $1 \leq e_i \leq e_j \leq 1.5152$. Hence we do in fact have $|a - b| \leq c$, i.e. it is not the case that one circle lies entirely inside the other.

Note that it is entirely possible that $e_k/(e_i e_j)$ will be greater than $e_{\max}(e_i^{-1} + e_j^{-1} - 1)$, i.e. that the two circles don't overlap at all. In this case $l_{i,j,k}$ will simply equal 0 since $\text{overlapApprox}(a, b, a + b) = 0$ for all non-negative a and b .

Now suppose that the manifold N possesses a (i, j, k) -triple where i, j , and k are all elements of $\{1, 2, 3\}$. Then there will exist orthocenters p_i and p_j corresponding to horoballs $B_i \in \mathcal{O}(i)$ and $B_j \in \mathcal{O}(j)$ such that $d(B_i, B_j) = o(k)$, and similarly for each cyclic permutation of i, j , and k . In other words, an (i, j, k) -triple with $i, j, k \in \{1, 2, 3\}$ can cause up to three different overlaps in $\partial B_\infty/H$. But there will be no overlaps between the circles that we have constructed that do not come from such a triple. In other words we have established the following:

Lemma 5.4. *Suppose that N possesses horoball triples of the form $(i_1, j_1, k_1), \dots, (i_s, j_s, k_s)$, where $i_r, j_r, k_r \in \{1, 2, 3\}$ for each $r = 1, \dots, s$. Furthermore, suppose every such triple in N appears in the above list exactly once. Then*

$$\text{Area}(\partial B_\infty/H) \geq A_0 - \sum_{r=1}^s l_{i_r, j_r, k_r} + l_{j_r, k_r, i_r} + l_{k_r, i_r, j_r}$$

Once we have a bound on the area of $\partial B_\infty/H$, we can compute the volume of B_∞/H and thereby construct a bound on the volume of N exactly as in the

previous section. Recall that in the previous section we “inflated” the cusp neighborhood to get a larger neighborhood C/H which was only immersed in N , and then subtracted the volume of the regions where C/H extended beyond the Ford domain corresponding to B_∞/H . This allowed us to improve our volume bound by accounting for some of the volume of $N - B_\infty/H$. We will do the same here, “inflating” our cusp neighborhood by replacing B_∞ with a new horoball C centered at infinity but with Euclidean height $1/e_{max}$. Then we have the following:

Lemma 5.5. *Under the same assumptions as in Lemma 5.4,*

$$\begin{aligned} \text{Vol}(N) \geq & \frac{e_{max}^2}{2} \left(A_0 - \sum_{r=1}^s l_{i_r, j_r, k_r} + l_{j_r, k_r, i_r} + l_{k_r, i_r, j_r} \right) \\ & - \pi \left(-3 + e_{max}^2 (1 + e_2^{-2} + e_3^{-2}) + \log \left(\frac{e_2^2 e_3^2}{e_{max}^6} \right) \right) \end{aligned}$$

Unfortunately one very quickly discerns that the volume bounds obtained by the above lemma simply aren’t large enough for our purposes, particularly when e_2 , e_3 , and e_4 are small. For example, if $e_2 = e_3 = e_4 = 1$ one can quickly determine that $A_0 = 3\pi/2$ and that all the $l_{i,j,k}$ ’s are 0, resulting in a volume bound of $3\pi/4$, which is less than 2.848. Thus we need to find more area in the case where e_4 is small. Since a small value of e_4 implies that the $\mathcal{O}(4)$ -horoballs will be close to B_∞ , it is natural to try to construct circles around the orthocenters of the $\mathcal{O}(4)$ -horoballs to increase our estimate of the area of B_∞/H .

So suppose $e_4 < 1.5152$ (so $e_{max} = e_4$). We construct a circle of radius $1/(e_4 e_2) - e_4/e_2 + e_4/2$ around the orthocenter of each of the two horoballs in $\mathcal{O}(4)$. The reasoning behind the choice of this particular radius is as follows. If q_4 is the orthocenter of such a horoball, and if p_i is the orthocenter of $B_i \in \mathcal{O}(i)$ for $i \in \{2, 3, 4\}$ (assume $p_i \neq q_4$), we want the circles constructed around q_4 and p_i to have disjoint interiors. Since $d_E(p_i, q_4) \geq 1/(e_i e_4)$ by Lemma 3.4, this means that when $i = 2$ or 3 we require

$$\frac{1}{e_i e_4} - \left(\frac{1}{e_4 e_2} - \frac{e_4}{e_2} + \frac{e_4}{2} \right) - \left(\frac{e_4}{e_i} - \frac{e_4}{2} \right) \geq 0$$

and when $i = 4$ we require

$$\frac{1}{e_4^2} - 2 \left(\frac{1}{e_4 e_2} - \frac{e_4}{e_2} + \frac{e_4}{2} \right) \geq 0$$

The first inequality is equivalent to $(e_2^{-1} - e_i^{-1})(e_4 - e_4^{-1}) \geq 0$, which is clearly true for $i = 2$ or 3 . The second inequality simplifies to

$$\frac{e_4 - 1}{e_4^2 e_2} (2e_4^2 + 2e_4 - e_2(e_4^2 + e_4 + 1)) \geq 0$$

But $1 \leq e_2 \leq e_4$, and the polynomial $2x^2 + 2x - x(x^2 + x + 1)$ is always positive for $1 \leq x \leq 1.5152$, and hence each factor on the left-hand side of the above inequality is always non-negative, proving the inequality.

Hence our new circles will not intersect each other, nor will they intersect the previous circles that were created around the orthocenters of the $\mathcal{O}(2)$ or $\mathcal{O}(3)$ -horoballs. Now suppose q_4 is the orthocenter of $B_4 \in \mathcal{O}(4)$ as before, and suppose p_1 is the orthocenter of a horoball $B_1 \in \mathcal{O}(1)$. Suppose further that $d(B_1, B_4) \geq o(2)$, and that hence $d_E(p_1, q_4) \geq e_2/e_4$ by Lemma 3.4. We wish to know if the circles

we've constructed around p_1 and q_4 will overlap; that is, we wish to verify the inequality

$$\begin{aligned} \frac{e_2}{e_4} - \left(\frac{1}{e_4 e_2} - \frac{e_4}{e_2} + \frac{e_4}{2} \right) - \frac{e_4}{2} &\geq 0 \\ \Leftrightarrow \frac{e_2 - 1}{e_4 e_2} (e_2 + 1 - e_4^2) &\geq 0 \end{aligned}$$

This inequality is *not* always true in the domain $1 \leq e_2 \leq e_4 \leq 1.5152$, particularly when e_2 is small and e_4 is large. However, the inequality *does* hold when we need it to hold, namely when e_2 and e_4 are both small. Hence in what follows we will make the additional assumption that $e_2 + 1 \geq e_4^2$, and take it for granted that if this assumption fails we don't need the extra area anyway.

With the additional assumption, we now have circles constructed around the orthocenters of the $\mathcal{O}(4)$ -horoballs which don't intersect each other, which don't intersect the circles around the centers of the $\mathcal{O}(2)$ -horoballs and $\mathcal{O}(3)$ horoballs, and which may intersect the circles constructed around a $\mathcal{O}(1)$ horoball but only if that horoball is less than $o(2)$ away from the $\mathcal{O}(4)$ -horoballs. In other words, the new circles do not intersect themselves or any of the previous circles unless N contains one or more $(1, 1, 4)$ -triples. If N contains exactly one such triple, there will be two overlaps to account for, while if N contains two or more such triples, we have a geometric Mom-2 structure. Thus we can conclude the following:

Lemma 5.6. *If $e_4 \leq 1.5152$ and $e_2 + 1 \geq e_4^2$, then either N contains a geometric Mom-2 structure consisting of two $(1, 1, 4)$ -triples or else the area estimate of Lemma 5.4 can be increased by*

$$2\pi \left(\frac{1}{e_4 e_2} - \frac{e_4}{e_2} + \frac{e_4}{2} \right)^2 - 2 \text{overlapApprox}(a, b, c)$$

where $a = e_4/2$, $b = 1/(e_4 e_2) - e_4/e_2 + e_4/2$, and $c = 1/e_4$.

Note that a and b are the radii of the circles constructed around the orthocenters of the $\mathcal{O}(1)$ -horoballs and $\mathcal{O}(4)$ -horoballs respectively, while c is the minimum possible distance between those orthocenters in the presence of a $(1, 1, 4)$ -triple. Also we are implicitly assuming that the conditions of Lemma 3.12 are met, namely that $|a - b| \leq c \leq a + b$. Fortunately the left-hand inequality simplifies to $(e_2 + 1 - e_4^2)/(e_2 e_4) \geq 0$, which is true by assumption, while the right-hand inequality simplifies to $(e_4^2 - 1)(e_2 - 1)/(e_2 e_4) \geq 0$.

We can also update our volume bounds:

Lemma 5.7. *If $e_4 \leq 1.5152$ and $e_2 + 1 \geq e_4^2$, then either N contains a geometric Mom-2 structure consisting of two $(1, 1, 4)$ -triples or else the volume estimate of Lemma 5.5 can be increased by*

$$\frac{e_4^2}{2} \left(2\pi \left(\frac{1}{e_4 e_2} - \frac{e_4}{e_2} + \frac{e_4}{2} \right)^2 - 2 \text{overlapApprox}(a, b, c) \right)$$

where $a = e_4/2$, $b = 1/(e_4 e_2) - e_4/e_2 + e_4/2$, and $c = 1/e_4$.

We now have in principle a procedure for proving Proposition 5.1. First, enumerate every possible combination of triples which does *not* include a geometric Mom-2 or Mom-3 structure which is torus-friendly. Second, for each combination construct the volume bounds from Lemma 5.5 and Lemma 5.7 above, which will

be a function of e_2 , e_3 , and e_4 . Third, show that each such volume bound never attains a value below 2.848.

The first step is the simplest, and in fact we can make it even simpler: we do not need to enumerate every possible combination of triples that do not include a Mom- n , merely the maximal ones, since adding additional triples only decreases the area bounds in Lemma 5.4 and Lemma 5.6. There are a total of eighteen maximal combinations of triples that must be considered:

- (1, 1, 2), (1, 1, 3)
- (1, 1, 2), (1, 3, 3)
- (1, 1, 2), (2, 2, 3)
- (1, 1, 2), (2, 3, 3)
- (1, 2, 2), (1, 1, 3)
- (1, 2, 2), (1, 3, 3)
- (1, 2, 2), (2, 2, 3)
- (1, 2, 2), (2, 3, 3)
- (1, 1, 3), (2, 2, 3)
- (1, 1, 3), (2, 3, 3)
- (1, 3, 3), (2, 2, 3)
- (1, 3, 3), (2, 3, 3)
- (1, 1, 2), (1, 2, 3), (1, 2, 3)
- (1, 2, 2), (1, 2, 3), (1, 2, 3)
- (1, 1, 3), (1, 2, 3), (1, 2, 3)
- (1, 3, 3), (1, 2, 3), (1, 2, 3)
- (2, 2, 3), (1, 2, 3), (1, 2, 3)
- (2, 3, 3), (1, 2, 3), (1, 2, 3)

The second step of the procedure is also simple; it can in fact be automated by a few lines of Mathematica code ([Mil]).

The third step, however, is daunting due to the complicated nature of the volume bounds that result from Lemmas 5.5 and 5.7. The expressions resulting from these two lemmas defy analysis by hand.

To handle these complicated expressions, we resort to computer assistance. Specifically, we use rigorous floating-point arithmetic of the type used in [GMT]. This is not standard interval arithmetic, so we take a moment to review the techniques involved here.

Definition 5.8. *An affine 1-jet $F = (f_0; f_1, f_2, f_3; f_\epsilon)$ consists of the set of all functions $f : [-1, 1]^3 \rightarrow \mathbb{R}$ such that $|f(x_1, x_2, x_3) - (f_0 + \sum f_i x_i)| \leq f_\epsilon$ for all $(x_1, x_2, x_3) \in [-1, 1]^3$. (Note that we require $f_\epsilon \geq 0$.)*

Note that in [GMT] complex 1-jets were used; however we only require real numbers here. [GMT] showed how, given two affine 1-jets F and G , a computer which meets IEEE standards for floating-point arithmetic can compute an affine 1-jet $H = (h_0; h_1, h_2, h_3; h_\epsilon)$ which “equals $F + G$ ”, in the sense that $f + g \in H$ for each $f \in F$ and $g \in G$. Specifically, let $h_i = f_i + g_i$ for $i = 0, 1, 2$, and 3, and let

$$h_\epsilon = (1 + \epsilon_a)(\epsilon_t + \epsilon_f)$$

where

$$\begin{aligned}\epsilon_t &= f_\epsilon + g_\epsilon \\ \epsilon_f &= \frac{EPS}{2} ((|f_0 + g_0| + |f_1 + g_1|) + (|f_2 + g_2| + |f_3 + g_3|)) \\ \epsilon_a &= 3EPS\end{aligned}$$

Here ϵ_t accounts for the ‘‘Taylor error’’, i.e. the maximum possible sup-norm distance between $f + g$ and the linear function represented by h as f and g vary over F and G respectively. Since we’re only computing a linear function of f and g anyway, ϵ_t only needs to account for the error carried over from the operands F and G ; for more complicated operations this term will be more significant. The quantity ϵ_f accounts for the floating-point error that may accrue from the calculation of h_0 through h_3 . Here EPS is a (small) computer-dependant constant that measures the granularity of the set of real numbers that the computer is capable of representing. Roughly, $1 + EPS$ will be the smallest real number strictly greater than 1 which has a floating-point representation on the computer in question. Finally ϵ_a accounts for the floating-point error that may accrue from calculating $\epsilon_t + \epsilon_f$. For any operation involving affine 1-jets, ϵ_a will always be of the form $nEPS$ where n is an integer roughly proportional to the base-2 logarithm of the number of arithmetic operations necessary to compute $\epsilon_t + \epsilon_f$. Constructed in this way, the error term h_ϵ will be large enough to account for the original error terms f_ϵ and g_ϵ and for the floating-point error that might accrue from calculating the terms of H . Similar constructions exist for ‘‘ $-F$ ’’, ‘‘ FG ’’, and ‘‘ F/G ’’ provided the range of g does not contain 0 for any $g \in G$.

For more specific details on this process, we refer the reader to Sections 5 and 6 of [GMT] which describes the theory behind these formulas and provides numerous examples.

Now suppose that we have a rational polynomial $p(x_1, x_2, x_3)$ and we wish to compute the range of possible values of p over a box $I_1 \times I_2 \times I_3$ in \mathbb{R}^3 . It is not possible to compute the exact range of possible values by computer due to floating-point error. However affine 1-jets do make it possible to rigorously determine an interval which must contain the range of possible values, as follows. Note that for clarity’s sake we start with a simplified version of the procedure and fill in certain troublesome details later.

Define X_1 to be the affine 1-jet $(a_1; b_1, 0, 0; 0)$ where $x \mapsto a_1 + b_1x$ is the unique nondecreasing linear bijection from $[-1, 1]$ to the interval I_1 . Define $X_2 = (a_2; 0, b_2, 0; 0)$ and $X_3 = (a_3; 0, 0, b_3; 0)$ similarly. Then compute the affine 1-jet $p(X_1, X_2, X_3)$ using the constructions in [GMT] in place of the usual arithmetic operations, and let $P = (p_0; p_1, p_2, p_3; p_\epsilon)$ denote the result. If $(x_1, x_2, x_3) \in I_1 \times I_2 \times I_3$, then trivially

$$(x_1, x_2, x_3) = (f_1(u, v, w), f_2(u, v, w), f_3(u, v, w))$$

where $(u, v, w) \in [-1, 1]^3$ and $f_i \in X_i$ for $i = 1, 2$, and 3 . (Specifically, we may choose $f_i(x) = a_i + b_i x$.) Therefore $p(x_1, x_2, x_3)$ must lie in the range of $p(f_1, f_2, f_3)$, which is an element of P . And therefore

$$p(x_1, x_2, x_3) \in [p_0 - |p_1| - |p_2| - |p_3| - p_\epsilon, p_0 + |p_1| + |p_2| + |p_3| + p_\epsilon]$$

In practice, there are several complications. First, the above construction implicitly assumes that the coefficients a_i and b_i have exact binary representations.

If they do not, then the affine 1-jets X_i must be modified to have a non-zero ϵ -term, representing the sup-norm distance between $a_i + b_i x$ and the actual unique nondecreasing linear bijection from $[-1, 1]$ to I_i . In practice it is usually simpler to replace I_i with a slightly larger interval whose endpoints do in fact have exact binary representations. Second, computing whether or not $p(x_1, x_2, x_3)$ lies in the interval described is itself a floating-point operation, and may introduce error. This can be dealt with using similar techniques to those used in calculating “F+G” in the first place. One final complication is that this whole technique will only be effective if p is very simple or if the intervals I_i are very small. If p is complicated (and the functions we’re interested in are very complicated) then it is usually necessary to subdivide $I_1 \times I_2 \times I_3$ into much smaller sub-boxes to achieve any kind of accuracy.

Nevertheless, we wish to apply the above techniques to compute a range of possible values for the volume bounds produced by Lemmas 5.5 and 5.7; if the computed ranges never include values less than or equal to 2.848 then we’ll be done. Unfortunately those volume bounds are not expressed as rational polynomials; we also need to be able to rigorously compute both natural logarithms and the minimum function, two operations that were never implemented in [GMT].

Fortunately while the natural logarithm is not a rational polynomial all of its derivatives are. This makes it possible to rigorously compute logarithms by using a Taylor approximation to $\log x$ at $x = 1$, and using Taylor’s theorem to compute an exact upper bound on the difference between $\log x$ and the polynomial approximation. Since Taylor’s theorem expresses this difference in terms of the derivatives of $\log x$ it is possible to incorporate it into the “Taylor error term” ϵ_t of the resulting affine 1-jet and still determine correct upper bounds for the corresponding ϵ_f and ϵ_a terms. Repeated use of the relation $\log(ax) = \log a + \log x$ (where $a \neq 1$ is just an arbitrary positive constant; $9/8$ was used in our implementation) allows the program to restrict its use of the Taylor approximation to regions close to $x = 1$, where the approximation is the most accurate.

It is tempting at this point to try and implement other non-polynomial functions in this way. For example, consider the function $\text{overlapArea}(a, b, c)$ from Definition 3.11: it is composed of polynomials, square roots, and the function $\text{acos}(x)$. Square roots were successfully implemented using affine 1-jets in [GMT], and the derivatives of $\text{acos}(x)$ are square roots of rational polynomials, so in theory it is possible to implement the function overlapArea with affine 1-jets. In practice, unfortunately, this works poorly. The function $f(x) = \text{acos}(x) - x\sqrt{1-x^2}$ which is used in the calculation of overlapArea is composed of two functions which have vertical tangent lines at the points $x = \pm 1$. When computing a function using affine 1-jets, the size of the resulting error term will always be proportional to the derivative of the function being computed; as a consequence, computing $f(x)$ for affine 1-jets which include $x = \pm 1$ in their range in practice causes the error term to grow so large as to make the entire 1-jet useless. This is the entire reason for the existence of Lemma 3.12. While $\text{overlapApprox}(a, b, c)$ is only an approximation to $\text{overlapArea}(a, b, c)$, as a polynomial function it is far more useful in this kind of computation.

Continuing, there remains one function to be implemented: a function $\text{Max0}(F)$ such that if $f \in F$ and $H = \text{Max0}(F)$ then $\max(f, 0) \in H$. (Then the identity $\min(f, g) = f - \max(f - g, 0)$ can be used to compute minimums.) If the computer can rigorously determine that 0 does not lie in the range of f for any $f \in F$ then

$Max0(F)$ equals either F itself if $f_0 > 0$ or else $Max0(F) = (0; 0, 0, 0; 0)$ if $f_0 < 0$. If the computer cannot exclude the possibility that 0 lies in the range of some $f \in F$ then $Max0(F)$ is defined to be equal to the 1-jet $(s; 0, 0, 0; s)$ where

$$s = \frac{1}{2}(1 + 3EPS)(f_0 + (|f_1| + |f_2|) + (|f_3| + f_\epsilon))$$

(The factor of $(1+3EPS)$ is there to account for floating-point error that may accrue during the calculation of the rest the expression.) Note that while this definition is technically correct in that $\max(f, 0)$ will lie in this affine 1-jet for all $f \in F$, from a practical standpoint it is a terrible definition as s is almost guaranteed to be much, much larger than the original error term f_ϵ . Fortunately (and unlike the case with $\arcsin(x)$ and square roots, above) in practice the cases where the $Max0$ function had to be called on affine 1-jets that may have contained 0 in their ranges were rare enough not to cause significant problems.

With these tools in hand we now consider the following:

Theorem 5.9. *Suppose N is a one-cusped hyperbolic 3-manifold with $\text{Vol}(N) \leq 2.848$. Then $e_2 \leq 1.4751 < 1.5152$ and furthermore N contains a geometric Mom- n structure which either*

- *is a geometric Mom-2 structure incorporating only the orthoclasses $\mathcal{O}(1)$ and $\mathcal{O}(2)$ and triples constructed from those three orthoclasses, or*
- *incorporates only the orthoclasses $\mathcal{O}(1)$, $\mathcal{O}(2)$, and $\mathcal{O}(3)$, and triples constructed from those three orthoclasses, and is torus-friendly, and in addition we have $e_3 \leq 1.5152$, or*
- *incorporates the orthoclasses $\mathcal{O}(1)$ and $\mathcal{O}(4)$ and a pair of $(1, 1, 4)$ -triples, and in addition we have $e_4 \leq 1.5152$.*

The proof of Theorem 5.9, using rigorous computer assistance, proceeds as follows. If $e_2 > 1.4751$ then $\text{Vol}(N) > 2.848$ by the results of the previous chapter. So, suppose otherwise. Furthermore, suppose that N does not contain a geometric Mom- n of one of the three types described. In particular, by Proposition 4.1 we may assume that $e_3 \leq 1.5152$, and we may assume that N contains at most one $(1, 1, 4)$ -triple unless $e_4 > 1.5152$.

Then let $(i_1, j_1, k_1), \dots, (i_s, j_s, k_s)$ be the complete list of triples in N satisfying the condition that $i_r, j_r, k_r \in \{1, 2, 3\}$ for all r . This list of triples must be a subcollection of one of the eighteen collections listed above. For each of those eighteen collections of triples, we can construct a lower bound on the volume of N via Lemma 5.5 which we will call $f_1(e_2, e_3, e_4)$. In addition, Lemma 5.7 provides a second, stricter lower bound in the case where $e_4 \leq 1.5152$ and $e_2 + 1 \geq e_4^2$; call this function $f_2(e_2, e_3, e_4)$. These bounds depend solely on the parameters e_2, e_3 , and e_4 and can be computed using only the four basic arithmetic operations, logarithms, the minimum function, and some constants. Furthermore we may assume that (e_2, e_3, e_4) lies in a compact subset of \mathbb{R}^3 . Technically we have not established an upper bound on e_4 , but if $e_4 > 1.5154$ then we may simply replace e_4 with 1.5152 in f_1 and still obtain a valid lower bound on the volume of N . We cannot do the same with f_2 , but in practice f_2 is only needed to improve the volume bound for small values of e_4 . Subdividing this compact region into a sufficiently small number of pieces and using the rigorous floating-point arithmetic techniques described above, we can rigorously demonstrate that $\max(f_1, f_2)$ never attains a value less than or equal to 2.848.

This approach has been successfully implemented ([Mil]) and used to prove Proposition 5.1. The resulting program requires approximately 80 minutes to establish that $\max(f_1, f_2) > 2.848$ on the parameter space $1 \leq e_2 \leq 1.4751$, $e_2 \leq e_3 \leq e_4 \leq 1.5152$ in each of the 18 cases. To do so the program subdivides each dimension of the parameter space into 2^8 subintervals. The program also establishes that $f_1 > 2.848$ whenever $1 \leq e_2 \leq 1.4751$, $e_2 \leq e_3 \leq 1.5152$, and $e_4 = 1.5152$, requiring approximately 10 seconds to do so; the number of dimensions in the parameter space has a tremendous affect on the running time. The programs themselves consist of just under 4800 lines of C++ code (close to half of which was generated automatically by a short Mathematica program), running in Redhat Linux on a four-processor PC. \square

6. EMBEDDING A GEOMETRIC MOM- n

Theorem 5.9 establishes conditions under which we can assume that a one-cusped hyperbolic 3-manifold N possesses a geometric Mom-2 or Mom-3 structure. Now what we wish to do is thicken the cellular complex Δ associated to that structure to obtain a full topological internal Mom- n structure as defined in [GMM2]. Examining that definition, we see that there are three obstacles to this. The first is that there is no guarantee that Δ is embedded in N ; this is the obstacle we will address in this section. The second obstacle is that the complement of Δ in N may have components whose boundaries are not tori, which violates the definition of an internal Mom- n structure; this obstacle will be addressed in the next section. The final obstacle, namely determining whether or not Δ is “full”, will be tackled in Section 8. We proceed with the question of embeddedness now; specifically we will to prove the following:

Theorem 6.1. *In Theorem 5.9, we may assume that the geometric Mom- n structure obtained has the property that the cellular complex $\Delta = T \cup \{\lambda_i\} \cup \{\sigma_j\}$ is embedded, where $\{\lambda_i\}$ is the set of 1-cells which are the projections in N of the shortest arcs joining the horoballs which form the orthopair classes of the Mom- n , and where $\{\sigma_j\}$ is the set of 2-cells which are the projections in N of the totally geodesic surfaces spanning the triples of horoballs in the geometric Mom- n structure and the arcs between them.*

For sake of notation, define the $\mathcal{O}(n)$ -edge to be the image in N of the shortest arc between any two horoballs which constitute an element of $\mathcal{O}(n)$. All of the lemmas in this chapter will assume that N is a one-cusped hyperbolic 3-manifold.

Lemma 6.2. *If the $\mathcal{O}(n)$ -edge intersects the cusp torus for any n , then $e_n > 1.5152$.*

Proof: Any edge which intersects the cusp torus lifts to an arc contained in a line which intersects the boundary of B_∞ , neither of whose endpoints are at infinity. Such a line clearly must be a half-circle of diameter ≥ 2 . But such a line must also join the centers of two horoballs and hence by Lemma 3.4 we must have $e_k/(e_m e_n) \geq 2$ for some $k \in \{1, 2, 3\}$ and for some m and n . This implies that $e_k \geq 2 > 1.5152$. \square

Lemma 6.3. *If any two-cell in Δ corresponding to an (n, m, k) -triple intersects the cusp torus then one of e_n , e_m , or e_k is greater than 1.5152.*

Proof: A two-cell which intersects the cusp torus lifts to a totally geodesic two-cell contained in an ideal hyperbolic triangle which intersects B_∞ , whose ideal vertices

are the centers of three horoballs A, B, C , with the property that $d(A, B) = o(n)$, $d(B, C) = o(m)$, and $d(C, A) = o(k)$. There are two possibilities. If one of the three edges of this ideal triangle also intersects B_∞ , then the previous lemma applies and we're done. So suppose none of the three edges intersect B_∞ , but instead the triangle intersects B_∞ at some point in its interior.

Let p be the highest point of the ideal triangle as viewed in the upper half-space model (here "highest" refers to the Euclidean or visual height), and let q be the highest point of the entire hyperbolic plane containing this triangle. Clearly if $p \neq q$ then p must be some point on an edge of the ideal triangle as close to q as possible, contradicting our supposition. So $p = q$ and q is contained in the ideal triangle. Now view the entire picture from the point at infinity: from above, the ideal triangle projects onto a Euclidean triangle in B_∞ joining the orthocenters of A, B , and C , and q projects to the circumcenter of this triangle. Since the projection of q lies in the interior of the triangle, and since the circumradius of the triangle is ≥ 1 , one of the sides of the triangle must have length greater than or equal to $\sqrt{2 - 2 \cos \frac{2\pi}{3}} = \sqrt{3}$. Wolog, assume that the side "from B to C" has this property. Therefore by Lemma 3.4 as before we have $e_k/(e_b e_c) \geq \sqrt{3}$ for some $k \in \{1, 2, 3\}$ and some b and c , and hence $e_k \geq \sqrt{3} > 1.5152$. \square

The next step is to establish whether or not any of the $\mathcal{O}(n)$ -edges contained in Δ can intersect one another. Some of the following lemmas are stronger than necessary to prove that this is not the case, but the stronger results will be used later in the argument. We begin with some definitions to simplify notation later:

Definition 6.4. Let $\delta_1 = 0.15$, $\delta_2 = \cosh^{-1}(1.5152^{-1} + 1.5152^{-2}) = 0.4337\dots$, and $\delta_3 = \cosh^{-1}(2/1.5152) = 0.7800\dots$

Note that the value of δ_1 comes from Lemma 3.7. The values of δ_2 and δ_3 are motivated by the following application of Corollary 3.6:

Lemma 6.5. Suppose A, B, C, D , and the various distances between them are defined as in Corollary 3.6. In particular suppose that $d(A, C) = o(m)$, $d(B, D) = o(n)$, and suppose that x is the shortest distance between the line joining the centers of A and C and the line joining the centers of B and D . Suppose further that $\max(e_m, e_n) \leq 1.5152$. If $x < \delta_2$ then all four of the distances $o(h) = d(A, B)$, $o(j) = d(B, C)$, $o(k) = d(C, D)$, and $o(l) = d(D, A)$ are strictly less than $\min(o(m), o(n))$. If $x < \delta_3$ then at least three of those four distances are strictly less than $\min(o(m), o(n))$.

Proof: Without loss of generality suppose that $o(m) \leq o(n)$, and that therefore $e_m \leq e_n \leq 1.5152$.

Suppose that $x < \delta_2$ and that $e_h \geq e_m$. Then we have the following:

$$\begin{aligned} e_m + 1 &\leq e_h e_k + e_j e_l \\ &= e_m e_n \cosh x \\ &< (1.5152^{-1} + 1.5152^{-2}) e_m e_n \end{aligned}$$

where the middle step used the result of Corollary 3.6. Dividing both sides by e_m and using the fact that $e_m \leq e_n \leq 1.5152$ we get

$$1 + \frac{1}{1.5152} \leq 1 + \frac{1}{e_m} < (1.5152^{-1} + 1.5152^{-2}) e_n \leq 1 + \frac{1}{1.5152}$$

which is a contradiction. Hence $e_h < e_m$, and similarly for e_j , e_k , and e_l .

Now suppose that $x < \delta_3$. If $e_h \geq e_m$ and $e_k \geq e_m$, then we have

$$\begin{aligned} e_m^2 + 1 &\leq e_h e_k + e_j e_l \\ &= e_m e_n \cosh x \\ &< \frac{2}{1.5152} e_m e_n \end{aligned}$$

Again, divide both sides by e_m . Then using $e_n \leq 1.5152$ and the AM-GM inequality we get

$$2 \leq e_m + \frac{1}{e_m} < \frac{2}{1.5152} e_n \leq 2$$

which is a contradiction. On the other hand, if $e_h \geq e_m$ and $e_j \geq e_m$, then we have

$$\begin{aligned} e_m + e_m &\leq e_h e_k + e_j e_l \\ &= e_m e_n \cosh x \\ &< \frac{2}{1.5152} e_m e_n \end{aligned}$$

Dividing both sides by e_m gives $2 < \frac{2}{1.5152} e_n \leq 2$, a contradiction. Hence by symmetry no two of e_h , e_j , e_k , and e_l can be greater than or equal to e_m ; therefore at least three of them are strictly less than e_m . This proves the lemma. \square

Lemma 6.6. *The $\mathcal{O}(1)$ -edge does not intersect or pass within less than δ_3 of any $\mathcal{O}(n)$ -edge where $e_n \leq 1.5152$.*

Proof: Apply Lemma 6.5 with $m = 1$ and $x < \delta_3$ to conclude that either three of $\{o(h), o(j), o(k), o(l)\}$ are less than $o(1) = 0$, which is ridiculous, or else $e_n > 1.5152$, a contradiction. \square

Lemma 6.7. *If the $\mathcal{O}(2)$ -edge intersects the $\mathcal{O}(3)$ -edge, or passes within less than δ_2 of it, and if $e_3 \leq 1.5152$, then there exists a geometric Mom-2 structure consisting only of $(1, 1, 2)$ -triples.*

Proof: Applying Lemma 6.5 we get that $e_h = e_j = e_k = e_l = e_1$. Hence $d(A, C) = d(A, D) = d(B, C) = d(B, D) = 0$ and therefore $\{A, B, C\}$ and $\{A, C, D\}$ are both $(1, 1, 2)$ -triples. They cannot be equivalent under the action of $\pi_1(N)$, because any element of the group which maps one triple to the other would have to map the pair $\{A, C\}$ to itself, and hence be either elliptic or the identity, a contradiction. Therefore these two triples are distinct and constitute a geometric Mom-2 structure, completing the proof. \square

Lemma 6.8. *If $e_n \leq 1.5152$ then the $\mathcal{O}(n)$ -edge neither intersects itself nor passes within δ_1 of itself.*

Proof: This is a direct corollary of Lemma 3.7. \square

The three preceding lemmas show that if N contains a geometric Mom-2 or Mom-3 structure of one of the types described in Theorem 5.9 then the $\mathcal{O}(n)$ -edges of the complex Δ are embedded in N and do not intersect one another, *unless* the Mom- n structure includes the $\mathcal{O}(3)$ -edge. In that case, either the $\mathcal{O}(n)$ -edges of the complex Δ are embedded and do not intersect or else there exists *another* geometric Mom- n structure, specifically a Mom-2 structure consisting only of $(1, 1, 2)$ -triples. In other words either the edges in question are embedded and do not intersect, or

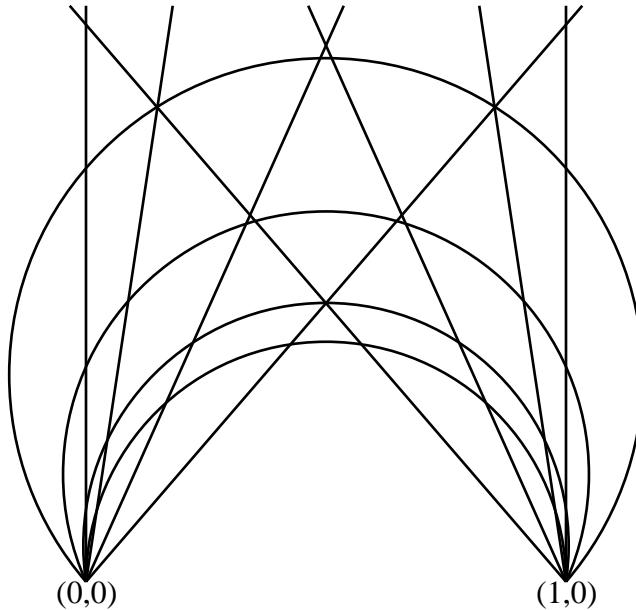


FIGURE 5. An ideal triangle, together with three equidistant curves for each side.

else we can find a simpler Mom- n structure; hence we may argue by induction that N must possess a geometric Mom- n structure with embedded edges. This type of induction argument will be repeated several times throughout this section.

To prove Theorem 6.1 we must also show that the 2-cells of Δ corresponding to (n, m, k) -triples are embedded and do not intersect one another. Since the 2-cells are simply connected and totally geodesic, and since the $\mathcal{O}(n)$ -edges in their boundary are geodesic arcs perpendicular to the horospheres at their endpoints, it is straightforward to show that if two 2-cells intersect then a $mc\mathcal{O}(n)$ -edge in the boundary of one 2-cell must intersect the other 2-cell. The previous lemmas imply that this intersection must occur in the interior of the 2-cell. Therefore to complete the proof of Theorem 6.1 it is sufficient to prove the following:

Proposition 6.9. *Suppose N has a geometric Mom- n structure of one of the types described in Theorem 5.9. If any of the $\mathcal{O}(n)$ -edges of the complex Δ intersect any of the 2-cells in Δ , then there must exist a simpler geometric Mom- n structure in N , which is also of one of the types described in Theorem 5.9.*

Proof: The idea of the proof is as follows: for every possible case, use the previous lemmas in this section to either demonstrate a contradiction or reduce the problem to a strictly previous case. But before enumerating the cases, consider the diagram in figure 5.

This diagram shows an ideal triangle in the upper half-space model of \mathbb{H}^2 with vertices at 0, 1, and ∞ , together with nine curves that are each equidistant from one of the three sides. For each of the three sides of the triangle there are three curves: one at a distance of δ_1 from the side, one at a distance of δ_2 , and one at a distance of δ_3 . (To construct this diagram it is sufficient to note the following.

A curve in the upper half-space model which is at a constant distance r from the line from 0 to ∞ is just a line passing through 0 with slope $\pm (\sinh r)^{-1}$; a curve which is equidistant from the line from 1 to ∞ is constructed similarly. A curve at constant distance r from the line from 0 to 1 is a circular arc passing from 0 to 1 through the point $(\frac{1}{2}, \frac{1}{2}e^{\pm r})$. See for example [Thu.]

From the diagram, the following result is immediate:

Lemma 6.10. *If p is a point in the interior of the ideal triangle with vertices at 0, 1, and ∞ , and if $\lambda_1, \lambda_2, \lambda_3$ are the sides of the triangle in any order, then:*

- (1) *If $d(p, \lambda_1) \geq \delta_3$ and $d(p, \lambda_2) \geq \delta_3$, then $d(p, \lambda_3) < \delta_1$.*
- (2) *If $d(p, \lambda_1) \geq \delta_3$ and $d(p, \lambda_2) \geq \delta_1$, then $d(p, \lambda_3) < \delta_3$.*
- (3) *(Corollary to the previous part) If $d(p, \lambda_1) \geq \delta_3$, then either one of $d(p, \lambda_2), d(p, \lambda_3)$ is $< \delta_2$ or else both of $d(p, \lambda_2), d(p, \lambda_3)$ are $< \delta_3$.*
- (4) *If $d(p, \lambda_i) \geq \delta_2$ for all $i \in \{1, 2, 3\}$ then $d(p, \lambda_j) < \delta_3$ for at least two $j \in \{1, 2, 3\}$.*

□

The conclusion in the third part of the above lemma is annoyingly weak. But note that the region in the ideal triangle where $d(p, \lambda_1) \geq \delta_3, d(p, \lambda_2) \geq \delta_2$, and $d(p, \lambda_3) \geq \delta_2$ is very small; if δ_3 is replaced with even a slightly smaller number then a stronger conclusion would result. This turns out to be useful enough that we do so now; the proof of the following lemma, while not following immediately from the diagram, is elementary enough that we omit it for brevity:

Lemma 6.11. *If $d(p, \lambda_1) \geq 0.9$, then one of $d(p, \lambda_2), d(p, \lambda_3)$ is $< \delta_2$.*

□

Now we begin enumerating the various cases to prove Proposition 6.9, according to which edge is involved and which triple corresponds to the 2-cell involved. In each case, let p be the point where the edge intersects the 2-cell, and note that the two-cell is contained in an ideal triangle which is isometric to that used in Lemma 6.10. We will refer to the sides of the triangle as λ_j for $j \in \{1, 2, 3\}$ and let $d_j = d(p, \lambda_j)$. Also we will let A_1, A_2 , and A_3 denote the horoballs such that λ_1 goes from A_1 to A_2 , λ_2 goes from A_2 to A_3 , and λ_3 goes from A_3 to A_1 . Finally let B_1 and B_2 be the horoballs at the ends of the $\mathcal{O}(n)$ -edge which passes through p . See figure 6.

Case 1: $\mathcal{O}(1)$ -edge, any triple. Part 1 of Lemma 6.10 implies that $d_j < \delta_3$ for some j . Now apply Lemma 6.6 to obtain a contradiction.

Case 2: $\mathcal{O}(2)$ -edge, $(1, 1, 2)$ -triple. Suppose that λ_1 and λ_2 contain lifts of the $\mathcal{O}(1)$ -edge while λ_3 contains a lift of the $\mathcal{O}(2)$ -edge. Part 1 of Lemma 6.10 implies that either $d_1 < \delta_3, d_2 < \delta_3$, or $d_3 < \delta_1$. In the first two cases Lemma 6.6 applies while in the third case Lemma 6.8 applies. Either possibility leads to a contradiction.

Case 3: $\mathcal{O}(2)$ -edge, $(1, 2, 2)$ -triple. Suppose that λ_1 contains a lift of the $\mathcal{O}(1)$ -edge while λ_2 and λ_3 contain lifts of the $\mathcal{O}(2)$ -edge. If $d_1 < \delta_3$ then Lemma 6.6 applies and we're done. If either d_2 or d_3 is less than δ_2 then the same argument used in the proof of Lemma 6.7 proves the existence of a geometric Mom-2 structure involving only $(1, 1, 2)$ -triples; either this new Mom-2 is embedded or else we reduce to case 2.

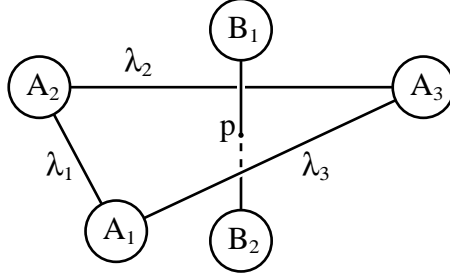


FIGURE 6. The horoballs and edges used in the proof of Proposition 6.9.

By part 3 of Lemma 6.10 the only remaining possibility is that $d_1 \geq \delta_3$, $d_2 < \delta_3$, and $d_3 < \delta_3$. Since $d_2 < \delta_3$, by Lemma 6.5, at least three of the four horoball pairs $\{A_2, B_1\}$, $\{B_1, A_3\}$, $\{A_3, B_2\}$, $\{B_2, A_2\}$ are elements of $\mathcal{O}(1)$. Similarly, since $d_3 < \delta_3$ at least three of the four pairs $\{A_3, B_1\}$, $\{B_1, A_1\}$, $\{A_1, B_2\}$, and $\{B_2, A_3\}$ lie in $\mathcal{O}(1)$. But $\{A_1, A_2\}$ is already in $\mathcal{O}(1)$; hence by Corollary 3.3 at least one of $\{A_1, B_1\}$, $\{A_2, B_1\}$ must not lie in $\mathcal{O}(1)$, and similarly one of $\{A_1, B_2\}$, $\{A_2, B_2\}$ must not lie in $\mathcal{O}(1)$. Up to symmetry, we must have $\{A_1, B_1\}$, $\{A_2, B_2\} \in \mathcal{O}(1)$, and $\{A_2, B_1\}$, $\{A_1, B_2\} \notin \mathcal{O}(1)$. Hence $d(A_2, B_1) \geq o(2)$ and $d(A_1, B_2) \geq o(2)$. Now apply Corollary 3.6 to A_1 , A_2 , B_1 , and B_2 to obtain

$$\begin{aligned} e_2^2 + 1 &\leq e_2 \cosh d_1 \\ \Rightarrow d_1 &\geq \cosh^{-1}(e_2 + e_2^{-1}) \\ &\geq \cosh^{-1} 2 \end{aligned}$$

Note that $\cosh^{-1} 2 = 1.3169\dots$ so Lemma 6.11 applies. Hence one of d_2 , d_3 is less than δ_2 , and a preceding argument applies. This completes the proof in this case.

We can save a bit of time at this point by noting that the argument in case 3 still works if we replace the $(1, 2, 2)$ -triple with a $(1, 2, 3)$ -triple, a $(1, 3, 3)$ -triple, or (with minor modifications) a $(1, 1, 3)$ -triple; henceforth we will assume that case 3 encompasses all of these possibilities.

Case 4: $\mathcal{O}(3)$ -edge, $(1, 1, 2)$ -triple. Suppose that λ_1 and λ_2 contain lifts of the $\mathcal{O}(2)$ -edge while λ_3 contains a lift of the $\mathcal{O}(2)$ -edge. If $d_1 < \delta_3$ or $d_2 < \delta_3$ then Lemma 6.6 applies, giving a contradiction. Otherwise by part 1 of Lemma 6.10 we have $d_3 < \delta_1$. Since $\delta_1 < \delta_2$, we can use the argument from Lemma 6.7 to find a geometric Mom-2 structure using only $(1, 1, 2)$ triples, reducing the problem to case 2.

Case 5: $\mathcal{O}(3)$ -edge, $(1, 1, 3)$ -triple. This case is nearly identical to the previous one, except we use Lemma 6.8 to obtain a contradiction instead of using Lemma 6.7 to reduce to case 2.

Case 6: $\mathcal{O}(3)$ -edge, $(1, 2, 2)$ -triple. Use the argument in case 3 with trivial modifications.

Case 7: $\mathcal{O}(2)$ -edge, $(2, 2, 3)$ -triple. Suppose that λ_1 and λ_2 contain lifts of the $\mathcal{O}(2)$ -edge while λ_3 contains a lift of the $\mathcal{O}(3)$ -edge. If $d_i < \delta_2$ for any $i \in \{1, 2, 3\}$ then we can use the argument from Lemma 6.7 as before to construct a geometric Mom-2 structure with just $(1, 1, 2)$ -triples and reduce to case 2. So suppose

$d_i \geq \delta_2$ for all $i \in \{1, 2, 3\}$; then by part 4 of Lemma 6.10 we have $d_j < \delta_3$ for at least two $j \in \{1, 2, 3\}$. Suppose that $d_1 < \delta_3$ and $d_2 < \delta_3$; the other possibilities can be handled similarly. Then by Lemma 6.5 at least three of the pairs $\{A_1, B_1\}$, $\{B_1, A_2\}$, $\{A_2, B_2\}$, and $\{B_2, A_1\}$ lie in $\mathcal{O}(1)$, and at least three of the pairs $\{A_2, B_1\}$, $\{B_1, A_3\}$, $\{A_3, B_2\}$, and $\{B_2, A_2\}$ lie in $\mathcal{O}(1)$.

Suppose all three of the pairs $\{A_1, B_i\}$, $\{A_2, B_i\}$, and $\{A_3, B_i\}$ were in $\mathcal{O}(1)$ for some $i \in \{1, 2\}$. Then both of the triples $\{A_1, A_2, B_i\}$ and $\{A_2, A_3, B_i\}$ are $(1, 1, 2)$ -triples. If there is an isometry $g \in \pi_1(N)$ taking one triple to another, then that isometry must be a parabolic isometry fixing B_i and taking A_1 to A_2 and A_2 to A_3 . But this would imply that the arc from A_1 to A_3 intersects either B_i , A_2 , or the arc from B_i to A_2 , contradicting our assumptions to this point. If there is no such isometry g , then we have a geometric Mom-2 structure with two distinct $(1, 1, 2)$ -triples and can reduce to case 2.

So suppose that at least one of the pairs $\{A_1, B_i\}$, $\{A_2, B_i\}$, and $\{A_3, B_i\}$ does not lie in $\mathcal{O}(1)$, both for $i = 1$ and for $i = 2$. Up to symmetry, the only way this can happen given our earlier assumptions is if the pairs $\{A_1, B_1\}$, $\{B_1, A_2\}$, $\{A_2, B_2\}$, and $\{B_2, A_3\}$ lie in $\mathcal{O}(1)$ while the pairs $\{A_1, B_2\}$ and $\{A_3, B_1\}$ do not. If $\{A_1, B_2\}$ lies in $\mathcal{O}(2)$, then the triples $\{A_1, A_2, B_2\}$ and $\{B_1, B_2, A_2\}$ form a geometric Mom-2 structure with one $(1, 1, 2)$ -triple and one $(1, 2, 2)$ -triple, allowing us to reduce the problem to either case 2 or case 3. So suppose $d(A_1, B_2) \geq o(3)$. For similar reasons we can suppose that $d(A_3, B_1) \geq o(3)$. Then applying Corollary 3.6 to A_1 , A_3 , B_1 , and B_2 we obtain

$$\begin{aligned} e_3^2 + 1 &\leq e_2 e_3 \cosh d_3 \\ &\leq e_3^2 \cosh d_3 \\ \Rightarrow d_3 &\geq \cosh^{-1}(1 + e_3^{-2}) \end{aligned}$$

Note that if $e_3 \leq 1.5152$ then this implies that $d_3 \geq 0.9$ and hence Lemma 6.11 applies. Therefore at least one of d_1 , d_2 is less than δ_2 , and a preceding argument applies. This completes the proof in this case.

Case 8: $\mathcal{O}(2)$ -edge, $(2, 3, 3)$ -triple. An almost identical argument to the one in case 7 applies, except that the problem may reduce to case 5 instead of case 2.

Case 9: $\mathcal{O}(3)$ -edge, $(2, 2, 3)$ -triple. Suppose that λ_1 and λ_2 contain lifts of the $\mathcal{O}(2)$ -edge while λ_3 contains a lift of the $\mathcal{O}(3)$ -edge. If $d_3 < \delta_1$ then Lemma 6.8 provides a contradiction, while if $d_i < \delta_2$ for $i = 1$ or 2 then the argument from Lemma 6.7 will produce a simpler geometric Mom-2 structure. Hence suppose that $d_3 \geq \delta_1$ and $d_i \geq \delta_2 > \delta_1$ for $i = 1$ and 2 . Part 2 of Lemma 6.10 then implies that $d_j < \delta_3$ for at least two different $j \in \{1, 2, 3\}$.

Suppose $d_1 < \delta_3$ and $d_3 < \delta_3$. Since $d_1 < \delta_3$, by Lemma 6.5 at least three of $\{A_1, B_1\}$, $\{B_1, A_2\}$, $\{A_2, B_2\}$, $\{B_2, A_1\}$ lie in $\mathcal{O}(1)$. Wolog assume that $\{A_1, B_1\}$ and $\{B_1, A_2\}$ lie in $\mathcal{O}(1)$. If $\{B_1, A_3\}$ lies in $\mathcal{O}(1)$ then the triples $\{A_1, B_1, A_2\}$ and $\{A_2, B_1, A_3\}$ are both $(1, 1, 2)$ -triples, and form a simpler geometric Mom-2 structure unless they are equivalent by the action of parabolic element $g \in \pi_1(N)$ which fixes B_1 ; but this would imply that the arc from B_1 to A_3 intersects the arc from A_1 to A_2 , a simpler case. If $\{B_1, A_3\}$ lies in $\mathcal{O}(2)$ then $\{A_1, B_1, A_2\}$ is a $(1, 1, 2)$ -triple while $\{A_2, B_1, A_3\}$ is a $(1, 2, 2)$ -triple and we have a simpler Mom-2. So suppose $\{B_1, A_3\} \notin \mathcal{O}(1) \cup \mathcal{O}(2)$. Then since $d_3 < \delta_3$, by Lemma 6.5 the pairs $\{B_2, A_1\}$ and $\{B_2, A_3\}$ must lie in $\mathcal{O}(1) \cup \mathcal{O}(2)$. Now consider $\{B_2, A_2\}$; if this pair lies in $\mathcal{O}(1) \cup \mathcal{O}(2)$ then the triples $\{A_1, A_2, B_1\}$ and $\{A_1, A_2, B_2\}$ will form

a simpler geometric Mom-2 structure, so suppose $\{B_2, A_2\} \notin \mathcal{O}(1) \cup \mathcal{O}(2)$. We have reached a point where $d(B_1, A_3) \geq o(3)$ and $d(B_2, A_2) \geq o(3)$, so applying Corollary 3.6 to the four horoballs A_2, A_3, B_1, B_3 we get

$$\begin{aligned} e_3^2 + 1 &\leq e_2 e_3 \cosh d_2 \\ &\leq e_3^2 \cosh d_2 \\ \Rightarrow d_2 &\geq \cosh^{-1}(1 + e_3^{-2}) \end{aligned}$$

Note that if $e_3 \leq 1.5152$ then this implies that $d_2 \geq 0.9$ and hence Lemma 6.11 implies that at least one of d_1, d_3 is less than δ_2 . But if $d_1 < \delta_2$ then by Corollary 3.6 we would have $\{B_2, A_2\} \in \mathcal{O}(1)$, while if $d_3 < \delta_2$ then $\{B_1, A_3\} \in \mathcal{O}(1) \cup \mathcal{O}(2)$; both conclusions contradict our assumptions to this point.

The argument when $d_2 < \delta_3$ and $d_3 < \delta_3$ is identical to this one by symmetry, and the the argument when $d_1 < \delta_3$ and $d_2 < \delta_3$ is similar.

Case 10: $\mathcal{O}(3)$ -edge, $(1, 2, 3)$ -triple. (This is by far the trickiest case.) Suppose that λ_1 contains a lift of the $\mathcal{O}(1)$ -edge, λ_2 contains a lift of the $\mathcal{O}(2)$ -edge, and λ_3 contains a lift of the $\mathcal{O}(3)$ -edge. If $d_1 < \delta_3$, then Lemma 6.6 leads to a contradiction. If $d_2 < \delta_2$, then as before we can repeat the argument of Lemma 6.7 to obtain a simpler Mom-2 and reduce to case 2. While if $d_3 < \delta_1$ then Lemma 6.8 provides a contradiction.

So suppose that $d_1 \geq \delta_3$, $d_2 \geq \delta_2$, and $d_3 \geq \delta_1$. Note that by part 2 of Lemma 6.10 this implies that $d_2 < \delta_3$ and $d_3 < \delta_3$. By Lemma 6.5, at least three of the four pairs $\{A_2, B_1\}$, $\{B_1, A_3\}$, $\{A_3, B_2\}$, and $\{B_2, A_1\}$ lie in $\mathcal{O}(1)$, while at least three of the four pairs $\{A_1, B_1\}$, $\{B_1, A_3\}$, $\{A_3, B_2\}$, $\{B_2, A_1\}$ lie in $\mathcal{O}(1) \cup \mathcal{O}(2)$. Unfortunately this is not quite enough information to construct a simpler Mom-2 or Mom-3, so we must dig deeper.

Let σ be the 2-cell spanning A_1, A_2, A_3 , and the arcs between them. Since $\{A_3, A_1\}$ and $\{B_1, B_2\}$ are both elements of the orthopair class $\mathcal{O}(3)$, there exists a group element $g \in \pi_1(N)$ which sends $\{A_3, A_1\}$ to $\{B_1, B_2\}$. Furthermore since σ is totally geodesic, $g(\sigma) \cap \sigma$ must contain a geodesic line segment with one endpoint p in the arc from B_1 to B_2 . Let q be the other endpoint of this line segment. What are the possible locations of q ?

If q lies in the interior of σ then either $g(A_j)$ intersects σ for some j , in which case we can apply Lemma 6.3, or else either $g(\lambda_1)$ or $g(\lambda_2)$ intersect σ , in which case the problem reduces to case 1 or case 3 respectively. So suppose q lies on the boundary of σ . If q lies in the interior of λ_1 , then $g^{-1}(\lambda_1)$ intersects σ and we can reduce to case 1. Similarly if q lies in the interior of λ_2 then we can reduce to case 3.

If q lies in $\sigma \cap A_2$, then g must be a parabolic element of $\pi_1(N)$ fixing A_2 . But note that the projections of λ_3 and $g(\lambda_3)$ onto the surface of A_2 clearly intersect; if g is parabolic fixing A_2 , then the only way this can happen is if λ_3 and $g(\lambda_3)$ themselves intersect. This contradicts Lemma 6.8.

If q lies in $\sigma \cap A_3$, then g must send the triple $\{A_1, A_2, A_3\}$ to the triple $\{A_3, B_1, B_2\}$; in particular $\{A_3, B_1, B_2\}$ is a $(1, 2, 3)$ -triple. Wolog, assume that $\{A_3, B_2\}$ lies in $\mathcal{O}(2)$ and $\{A_3, B_1\}$ lies in $\mathcal{O}(1)$. Since $d_2 \leq \delta_3$, this implies that $\{A_2, B_1\}$ and $\{A_2, B_2\}$ lie in $\mathcal{O}(1)$ as well. Thus $\{A_2, B_1, A_3\}$ is a $(1, 1, 2)$ -triple while $\{A_2, B_2, A_3\}$ is a $(1, 2, 2)$ -triple, forming a simpler geometric Mom-2 structure and reducing the problem to either case 1, 2, or 3.

There are two remaining possibilities. Suppose q lies in $\sigma \cap A_1$; then g must send $\{A_1, A_2, A_3\}$ to $\{A_1, B_1, B_2\}$. Note in particular that we must have $g(A_2) = A_1$. Wolog assume that $\{A_1, B_1\}$ lies in $\mathcal{O}(1)$ while $\{A_1, B_2\}$ lies in $\mathcal{O}(2)$. Applying Lemma 3.2 to the triple $\{A_1, A_2, B_1\}$ we see that $\{A_2, B_1\}$ cannot be in $\mathcal{O}(1)$. Since $d_2 < \delta_3$, we must have all of $\{B_1, A_3\}$, $\{A_3, B_2\}$, $\{B_2, A_2\}$ in $\mathcal{O}(1)$ instead. So $\{A_1, B_1, A_3\}$ is a $(1, 1, 3)$ -triple, while $\{A_1, B_2, A_3\}$ is a $(1, 2, 3)$ -triple. Now consider $g^{-1}(\sigma)$. Comparing σ and $g(\sigma)$, we see that $g^{-1}(\sigma)$ must intersect λ_3 , and must be bounded at one corner by $A_2 = g^{-1}(A_1)$. Specifically $g^{-1}(\lambda_1) \in \mathcal{O}(1)$ and $g^{-1}(\lambda_3) \in \mathcal{O}(3)$ must have one endpoint on A_2 . If any of the four arcs spanning the horoball pairs $\{A_1, B_1\}$, $\{B_1, A_3\}$, $\{A_3, B_2\}$, or $\{B_2, A_1\}$ (all of which are in $\mathcal{O}(1) \cup \mathcal{O}(2)$) intersect $g^{-1}(\sigma)$ then we can reduce to a previous case, so suppose this doesn't happen. We still must have $g^{-1}(\sigma)$ intersecting λ_3 . If either $g^{-1}(\lambda_1)$ or $g^{-1}(\lambda_2)$ intersect either of the two-cells spanning the triples $\{A_1, B_1, A_3\}$ or $\{A_1, B_2, A_3\}$, then we can still reduce the problem to a previous case. The only way $g^{-1}(\sigma)$ can intersect λ_3 without such an intersection occurring (and without $g^{-1}(\sigma)$ intersecting the interior of a horoball, which contradicts Lemma 6.3) is if $g^{-1}(A_1) = B_2$ and $g^{-1}(\lambda_3)$ intersects the two-cell spanning $\{A_1, B_1, A_3\}$. But this merely lets us reduce to case 5 instead.

Thus we come to the final possibility: suppose that q lies in the interior of λ_3 . Then the image of the arc from p to q under g^{-1} must be another geodesic line segment going from $g^{-1}(p)$ on λ_3 to $g^{-1}(q)$ in the interior of σ . Note that $p \neq g^{-1}(q)$, otherwise g would be elliptic of order 2. Since $g^{-1}(q)$ is the point where σ intersects the arc from $g^{-1}(A_1)$ to $g^{-1}(A_3)$, and since this arc is also a lift of the $\mathcal{O}(3)$ -edge, the same arguments that apply to p also apply to $g^{-1}(q)$. Specifically: if $d(g^{-1}(q), \lambda_1) < \delta_3$, $d(g^{-1}(q), \lambda_2) < \delta_2$, or $d(g^{-1}(q), \lambda_3) < \delta_1$, then we can apply Lemma 6.6, 6.7, or 6.8 respectively to get either a contradiction or a reduction to a simpler case. So assume that none of those three inequalities hold. Note that this implies that $d(g^{-1}(q), \lambda_3) < \delta_3$ just as our previous assumptions implied that $d_3 = d(p, \lambda_3) < \delta_3$.

Now consider the pairs $\{B_1, B_2\}$ and $\{g^{-1}(A_1), g^{-1}(A_3)\}$. These cannot be the same pair; if they were, then g^2 would fix the pair $\{B_1, B_2\}$ and hence be either elliptic or the identity, a contradiction. So the two pairs are either completely disjoint or else intersect in a single element. Suppose they intersect in a single element; wolog, assume in particular that $B_1 = g^{-1}(A_1)$. Note that we can't have $g(A_1) = B_1$; if we did then g^2 would fix A_1 and hence so would g , a contradiction. So $g(A_1) = B_2$, and $g(A_3) = B_1$. To summarize we have $g(B_1) = A_1$ and $g^2(B_1) = B_2$, while at the same time we must have $g^{-1}(B_1) = A_3$. Arrange the upper half-space model of \mathbb{H}^3 so that B_1 is centered at the point at infinity, $A_1 = g(B_1)$ and $A_3 = g^{-1}(B_1)$ are centered on the real line, and A_2 is centered at some point with positive imaginary part. By assumption, the arcs from B_1 to B_2 and from B_1 to $g^{-1}(A_3)$ both intersect the interior of the two-cell spanning the triple $\{A_1, A_2, A_3\}$; hence the centers of $B_2 = g^2(B_1)$ and $g^{-1}(A_3) = g^{-2}(B_1)$ also have positive imaginary part. But this is impossible: by direct calculation in $\text{PSL}(2, \mathbb{C})$, if g is an element such that $g(\infty)$ and $g^{-1}(\infty)$ are both real then either $g^2(\infty)$ and $g^{-2}(\infty)$ are also both real or else their imaginary parts have opposite signs. Hence, the pairs $\{B_1, B_2\}$ and $\{g^{-1}(A_1), g^{-1}(A_3)\}$ must be completely disjoint, or in other words the arcs $g(\lambda_3)$ and $g^{-1}(\lambda_3)$ do not share a horoball at their endpoints.

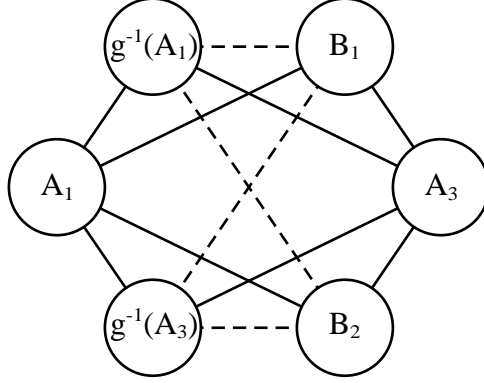


FIGURE 7. A possible collection of horoballs used in case 10; the solid lines (and three of the four dashed lines) indicate pairs which lie in $\mathcal{O}(1) \cup \mathcal{O}(2)$.

Now suppose that $d(\lambda_3, g(\lambda_3)) \geq \delta_2$ and $d(\lambda_3, g^{-1}(\lambda_3)) \geq \delta_2$. Examining figure 5 (and remembering our previous assumptions), this implies that both p and $g^{-1}(q)$ must lie in the small region bounded by the curves $d(\cdot, \lambda_1) = \delta_3$, $d(\cdot, \lambda_2) = \delta_2$, and $d(\cdot, \lambda_3) = \delta_2$. The diameter of this region can be readily computed (it's not a triangle but it is contained inside one), and it is far less than δ_1 . Hence $d(g(\lambda_3), g^{-1}(\lambda_3)) < \delta_1$, and hence Lemma 6.8 applies, producing a contradiction. So therefore one of $d(\lambda_3, g(\lambda_3))$ and $d(\lambda_3, g^{-1}(\lambda_3))$ must be less than δ_2 , and clearly if one is then both are by isometry. Furthermore, the diameter of the region in figure 5 bounded by the curves $d(\cdot, \lambda_1) = \delta_3$, $d(\cdot, \lambda_2) = \delta_2$, and $d(\cdot, \lambda_3) = \delta_1$ can also be computed to be less than δ_3 . Then by applying Lemma 6.5 repeatedly, we get the following:

- All four of the pairs $\{A_1, B_1\}$, $\{B_1, A_3\}$, $\{A_3, B_2\}$, and $\{B_2, A_1\}$ lie in $\mathcal{O}(1) \cup \mathcal{O}(2)$.
- All four of the pairs $\{A_1, g^{-1}(A_1)\}$, $\{g^{-1}(A_1), A_3\}$, $\{A_3, g^{-1}(A_3)\}$, and $\{g^{-1}(A_3), A_1\}$ lie in $\mathcal{O}(1) \cup \mathcal{O}(2)$.
- At least three of the pairs $\{B_1, g^{-1}(A_1)\}$, $\{g^{-1}(A_1), B_2\}$, $\{B_2, g^{-1}(A_3)\}$, and $\{g^{-1}(A_3), B_1\}$ lie in $\mathcal{O}(1) \cup \mathcal{O}(2)$.

So the six horoballs A_1 , A_3 , B_1 , B_2 , $g^{-1}(A_1)$, and $g^{-1}(A_3)$ form a picture like the one in figure 7, where each of solid edges and three of the four dashed edges is a lift of either the $\mathcal{O}(1)$ -edge or the $\mathcal{O}(2)$ -edge. Suppose, for sake of example, that the dashed edge from B_1 to $g^{-1}(A_1)$ is the only edge which is not a lift of either the $\mathcal{O}(1)$ -edge or the $\mathcal{O}(2)$ -edge. Then consider the triple $\{A_1, B_2, g^{-1}(A_3)\}$. This triple must be an (a, b, b) -triple where a and b are 1 and 2 in some order (i.e., it's either a $(1, 1, 2)$ -triple or a $(1, 2, 2)$ -triple). But this triple also has the property that each edge of the triple is shared by another triple in the diagram which must also be either a $(1, 1, 2)$ -triple or a $(1, 2, 2)$ -triple. It is straightforward to see that some pair of these four triples, specifically the pair which shares the edge which is a lift of the $\mathcal{O}(a)$ -edge, must be distinct under the action of $\pi_1(N)$ and hence form a geometric Mom-2 structure. A similar argument holds for each of the other dashed edges in the diagram.

This at last completes the proof in this case.

Case 11: $\mathcal{O}(3)$ -edge, $(1, 3, 3)$ -triple. Suppose λ_3 contains a lift of the $\mathcal{O}(1)$ -edge while λ_1 and λ_2 contain lifts of the $\mathcal{O}(3)$ -edge. If $d_3 < \delta_3$ then Lemma 6.6 provides a contradiction, while if $d_i < \delta_1$ for $i = 1$ or 2 then Lemma 6.8 provides a contradiction. So suppose $d_1 \geq \delta_1$, $d_2 \geq \delta_1$, and $d_3 \geq \delta_3$. Lemma 6.10 implies that d_1 and d_2 are both less than δ_3 .

Suppose that in fact $d_1 < \delta_2$. By Lemma 6.5, all four pairs $\{A_1, B_1\}$, $\{B_1, A_2\}$, $\{A_2, B_2\}$, $\{B_2, A_1\}$ and at least three of the four pairs $\{A_2, B_1\}$, $\{B_1, A_3\}$, $\{A_3, B_2\}$, $\{B_2, A_2\}$ lie in $\mathcal{O}(1) \cup \mathcal{O}(2)$. Wolog, assume that $\{B_1, A_3\} \in \mathcal{O}(1) \cup \mathcal{O}(2)$. We wish to construct a geometric Mom-2 or Mom-3 structure that only uses triples considered in previous cases. However we have to be careful to ensure that we do not construct a Mom-3 which is not torus-friendly; this can be ensured by not selecting any geometric Mom-3 structure that contains exactly two $(1, 2, 3)$ -triples. To start with, note that the triple $\{A_2, B_1, A_3\}$ is of type $(1, 1, 2)$ or $(1, 2, 2)$. Now consider the four triples $\{A_1, B_1, A_2\}$, $\{A_1, B_2, A_2\}$, $\{B_1, A_1, B_2\}$, and $\{B_1, A_2, B_2\}$. Each of these triples is of type $(1, 2, 3)$, $(1, 1, 3)$, or $(2, 2, 3)$. If all four of these triples are of type $(1, 2, 3)$, then no two triples can be equivalent under the action of $\pi_1(N)$ because each two of those triples share a common “edge”. So any three of those triples will form a geometric Mom-3 structure which is torus-friendly. If between one and three of these triples are of type $(1, 2, 3)$, choose one triple of type $(1, 2, 3)$, one triple not of type $(1, 2, 3)$, and $\{A_2, B_1, A_3\}$ to get a simpler geometric Mom-3 structure which is torus-friendly. And if none of the four triples is of type $(1, 2, 3)$, then all four must be of type $(a, a, 3)$ for a fixed $a \in \{1, 2\}$. Pick two such triples, say $\{A_1, B_1, A_2\}$ and $\{A_1, B_2, A_2\}$; these triples can’t be equivalent under the action of $\pi_1(N)$ since that group has no elliptic elements; hence they form a simpler geometric Mom-2 structure.

So instead suppose that $d_1 \geq \delta_2$ and for similar reasons suppose that $d_2 \geq \delta_2$. Suppose now that $\{A_2, B_1\} \notin \mathcal{O}(1) \cup \mathcal{O}(2)$. Since d_1 and d_2 are both less than δ_3 , Lemma 6.5 implies that all of $\{A_1, B_1\}$, $\{A_1, B_2\}$, $\{A_3, B_1\}$, $\{A_3, B_2\}$, and $\{A_2, B_2\}$ are in $\mathcal{O}(1) \cup \mathcal{O}(2)$. Then the triples $\{A_1, B_1, A_3\}$ and $\{A_1, B_2, A_3\}$ are each of type $(1, 1, 2)$ or $(1, 2, 2)$, and the triples $\{A_1, B_2, A_2\}$ and $\{A_2, B_2, A_3\}$ are each of type $(1, 1, 3)$, $(1, 2, 3)$, or $(2, 2, 3)$. If $\{A_1, B_2, A_2\}$ and $\{A_2, B_2, A_3\}$ are equivalent due to the action of $g \in \pi_1(N)$, then g must be parabolic fixing B_2 , and hence λ_3 must intersect the arc from B_2 to A_2 , a previous case. So $\{A_1, B_2, A_2\}$, $\{A_2, B_2, A_3\}$, and $\{A_1, B_2, A_3\}$ are all distinct triples and form a simpler Mom-3 which furthermore is torus-friendly, *unless* $\{A_1, B_2, A_2\}$ and $\{A_2, B_2, A_3\}$ are both of type $(1, 2, 3)$. That is only possible if $\{A_1, B_2\}$ and $\{B_2, A_3\}$ are both in $\mathcal{O}(2)$ (the other possibility, that they are both in $\mathcal{O}(1)$, makes $\{A_1, B_2, A_3\}$ a $(1, 1, 1)$ -triple, which is impossible). In this case $\{A_1, B_2, A_3\}$ is of type $(1, 2, 2)$ and cannot be equivalent to $\{A_1, B_1, A_3\}$ under the action of $\pi_1(N)$ since a group element sending one triple to another would have to fix the pair $\{A_1, A_3\}$; thus we have a simpler geometric Mom-2 structure.

So we may assume that $\{A_2, B_1\} \in \mathcal{O}(1) \cup \mathcal{O}(2)$, and by symmetry we may assume that $\{A_2, B_2\} \in \mathcal{O}(1) \cup \mathcal{O}(2)$. If the same holds for both $\{A_1, B_1\}$ and $\{A_1, B_2\}$ then we may proceed exactly as if $d_1 < \delta_2$. So assume that one of those two pairs is not in $\mathcal{O}(1) \cup \mathcal{O}(2)$, and similarly assume that one of the pairs $\{A_3, B_1\}$, $\{A_3, B_2\}$ is not in $\mathcal{O}(1) \cup \mathcal{O}(2)$. Now applying Corollary 3.6 to the four horoballs A_1 , A_3 , B_1 , and B_3 , we get either

$$e_3^2 + 1 \leq e_3 \cosh d_3$$

or

$$2e_3 \leq e_3 \cosh d_3$$

In either case, $d_3 \geq \cosh^{-1} 2 > 0.9$, and hence by Lemma 6.11 one of d_1, d_2 must be less than δ_2 , contradicting our assumptions to this point and completing this case.

Case 12: $\mathcal{O}(3)$ -edge, $(2, 3, 3)$ -triple. Suppose λ_1 and λ_2 contain lifts of the $\mathcal{O}(3)$ -edge while λ_3 contains a lift of the $\mathcal{O}(2)$ -edge.

Note that if $d_1 < \delta_1$ or $d_2 < \delta_1$ then we get a contradiction from Lemma 6.8, and if $d_3 < \delta_2$ then we can produce a simpler geometric Mom-2 structure just as in Lemma 6.7. So assume $d_1 \geq \delta_1, d_2 \geq \delta_1$, and $d_3 \geq \delta_2 > \delta_1$. By Lemma 6.10 this implies that at least two of d_1, d_2 , and d_3 must be less than δ_3 , and by Lemma 6.5 that in turn implies that at least two of the following statements are true:

- At least three of $\{A_1, B_1\}, \{B_1, A_2\}, \{A_2, B_2\}, \{B_2, A_1\}$ are in $\mathcal{O}(1) \cup \mathcal{O}(2)$.
- At least three of $\{A_2, B_1\}, \{B_1, A_3\}, \{A_3, B_2\}, \{B_2, A_2\}$ are in $\mathcal{O}(1) \cup \mathcal{O}(2)$.
- At least three of $\{A_3, B_1\}, \{B_1, A_1\}, \{A_1, B_2\}, \{B_2, A_3\}$ are in $\mathcal{O}(1) \cup \mathcal{O}(2)$.

Suppose for a moment that *all four* of $\{A_1, B_1\}, \{B_1, A_3\}, \{A_3, B_2\}, \{B_2, A_1\}$ are in $\mathcal{O}(1) \cup \mathcal{O}(2)$. I.e., suppose that $\{A_1, B_1, A_3\}$ and $\{A_1, B_2, A_3\}$ are both of type $(1, 1, 2)$ or $(1, 2, 2)$. If these two triples are not equivalent under $\pi_1(N)$ then they form a simpler geometric Mom-2 structure. If they are equivalent due to $g \in \pi_1(N)$, then either g is elliptic and fixes $\{A_1, A_3\}$ (a contradiction), g is parabolic fixing one of A_1 or A_3 (in which case the one-cell from B_1 to B_2 will intersect another one-cell, reducing the problem to a previous case), or else up to symmetry we may assume that $\{A_1, B_1\}, \{A_3, B_2\}$ are in $\mathcal{O}(1)$ and $\{A_1, B_2\}, \{A_3, B_1\}$ are in $\mathcal{O}(2)$. In the latter case, assume wlog that $\{B_1, A_2\} \in \mathcal{O}(1) \cup \mathcal{O}(2)$ (we know this must be true for one of $\{B_1, A_2\}, \{B_2, A_2\}$). Then one of the two triples $\{A_1, A_2, B_1\}$ and $\{A_2, A_3, B_1\}$ must be of type $(2, 2, 3)$ or $(1, 1, 3)$; this triple along with $\{A_1, B_1, A_2\}$ and $\{B_1, B_2, A_1\}$ form a simpler geometric Mom-3 structure where each triple is of a different type (and hence none are equivalent under $\pi_1(N)$) and exactly one triple is of type $(1, 2, 3)$ (so the Mom-3 is torus-friendly).

Hence we may assume that *at most* three of $\{A_1, B_1\}, \{B_1, A_3\}, \{A_3, B_2\}, \{B_2, A_1\}$ are in $\mathcal{O}(1) \cup \mathcal{O}(2)$.

If all four of $\{A_1, B_1\}, \{B_1, A_2\}, \{A_2, B_2\}, \{B_2, A_1\}$ lie in $\mathcal{O}(1) \cup \mathcal{O}(2)$ (or by symmetry all four of $\{A_2, B_1\}, \{B_1, A_3\}, \{A_3, B_2\}, \{B_2, A_2\}$) then by a similar argument to the one used in the beginning of the previous case we can also construct a simpler Mom-2 or a simpler geometric Mom-3 structure which is torus-friendly. So assume this does not happen either. In summary, we can assume that for none of the three sets of horoball pairs listed above do all four pairs in the set lie in $\mathcal{O}(1) \cup \mathcal{O}(2)$. (In particular this implies that $d_i \geq \delta_2$ for all $i \in \{1, 2, 3\}$, by Lemma 6.5.)

It is then straightforward to check that at least one of the three statements in the above list must in fact be *false*: you cannot choose three pairs from each set without choosing all four pairs from at least one set. In other words, exactly two of the three statements in the above list are true.

Now consider the three pairs $\{A_1, B_1\}, \{A_2, B_1\}$, and $\{A_3, B_1\}$. Suppose all three are in $\mathcal{O}(1) \cup \mathcal{O}(2)$. Then $\{A_1, B_1, A_3\}$ is of type $(1, 1, 2)$ or $(1, 2, 2)$, while $\{A_1, B_1, A_2\}$ and $\{A_2, B_1, A_3\}$ are each of type $(1, 1, 3)$, $(2, 2, 3)$, or $(1, 2, 3)$. Note that if these last two triples are equivalent due to $g \in \pi_1(N)$ then g must be parabolic fixing B_1 , in which case λ_3 must intersect the arc from B_1 to A_2 , a

previous case. So assume these triples are not equivalent under $\pi_1(N)$. Then the three triples $\{A_1, B_1, A_2\}$, $\{A_2, B_1, A_3\}$, and $\{A_3, B_1, A_1\}$ form a simpler geometric Mom-3 structure, which is torus-friendly unless $\{A_1, B_1, A_2\}$ and $\{A_2, B_1, A_3\}$ are both of type (1, 2, 3). Note that this is only possible if $\{A_1, B_1\}$ and $\{A_3, B_1\}$ are in $\mathcal{O}(1)$ while $\{A_2, B_1\}$ is in $\mathcal{O}(2)$. Now note that $\{A_j, B_2\}$ must be in $\mathcal{O}(1) \cup \mathcal{O}(2)$ for at least one $j \in \{1, 2, 3\}$. If $j = 2$ then $\{A_2, B_1, B_2\}$ is either a third (1, 2, 3)-triple (and not equivalent to either $\{A_1, B_1, A_2\}$ or $\{A_2, B_1, A_3\}$ since it shares an “edge” with both) or else it is of type (1, 1, 3) or (2, 2, 3); either way we get a geometric Mom-3 structure with either one or three triples of type (1, 2, 3), which therefore is torus-friendly. Suppose $j = 1$ or 3; by symmetry assume $j = 1$. Then by our previous assumptions $\{A_2, B_2\}$ and $\{A_3, B_3\}$ must not be in $\mathcal{O}(1) \cup \mathcal{O}(2)$. Now we can apply Corollary 3.6 to the horoballs A_2, A_3, B_1 , and B_2 to get

$$\begin{aligned} e_3 e_2 + e_3 &\leq e_3 e_2 \cosh d_2 \\ \Rightarrow d_2 &\geq \cosh^{-1}(1 + e_2^{-1}) \end{aligned}$$

Note that $e_2 \leq e_3 \leq 1.5152$ then implies $d_2 > 0.9$, and hence by Lemma 6.11 one of d_1, d_3 must be less than δ_2 , contradicting our assumptions up to this point.

So assume at least one of $\{A_1, B_1\}$, $\{A_2, B_1\}$, $\{A_3, B_1\}$ is not in $\mathcal{O}(1) \cup \mathcal{O}(2)$, and similarly for B_2 instead of B_1 . Now go back to the three statements listed above; we know exactly one of them is false. Suppose it is the first statement which is false and the others true (the other possibilities can be handled similarly). The only way to reconcile this with the sentence at the beginning of this paragraph, is if $\{A_1, B_1\}$ and $\{A_2, B_2\}$ are not in $\mathcal{O}(1) \cup \mathcal{O}(2)$, or the same but with the roles of B_1 and B_2 reversed. But in either case, applying Corollary 3.6 to A_1, A_2, B_1, B_2 yields

$$\begin{aligned} e_3^2 + 1 &\leq e_3^2 \cosh d_1 \\ \Rightarrow d_1 &\geq \cosh^{-1}(1 + e_3^{-2}) \end{aligned}$$

And once more, if $e_3 \leq 1.5152$ then this implies $d_1 > 0.9$, so by Lemma 6.11 one of d_2, d_3 must be less than δ_2 , contradicting our assumptions. This completes the proof in this case.

Case 13: $\mathcal{O}(4)$ -edge, (1, 1, 4)-triple. The exact same argument as in case 2 applies, with the obvious modifications.

This, finally, completes the proof of Proposition 6.9, and in turn the proof of Theorem 6.1. \square

7. TORUS-FRIENDLY MOM- n 'S

Having established Theorem 6.1, we now have an embedded cellular complex Δ corresponding to a geometric Mom- n structure in the cusped manifold N . The next step in upgrading Δ to an internal Mom- n structure of the type defined in [GMM2] is to prove the following:

Theorem 7.1. *Suppose N is a one-cusped hyperbolic 3-manifold with $\text{Vol}(N) \leq 2.848$ and suppose Δ is the embedded cellular complex corresponding to the geometric Mom- n structure produced by Theorems 5.9 and 6.1. Then the components of $N - \Delta$ each have torus boundary, or else there exists a simpler geometric Mom- n structure which is also of the type described in Theorems 5.9 and 6.1.*

Proof: From Theorem 5.9 we know that $n = 2$ or 3 and that the geometric Mom- n structure thus constructed is torus-friendly. Recall that this means either $n = 2$, or $n = 3$ and the Mom-3 structure does *not* have exactly two triples of type (p, q, r) where p, q , and r are distinct indices. We wish to prove that being torus-friendly implies that the boundary consists of a collection of tori.

Let M be a thickened neighborhood of Δ ; note that by construction $\chi(\partial M) = 2\chi(\Delta) = 0$. If ∂M consists of nothing but tori then we're done. If ∂M contains components which are not tori, then one of those components must be a sphere. Hence we wish to show that ∂M does not contain any spherical components. Since N is hyperbolic this is equivalent to showing that $N - \Delta$ does not have any components which are 3-balls.

Suppose $n = 2$ and that one of the components of $N - \Delta$ is a 3-ball B . In the universal cover \mathbb{H}^3 , B lifts to a collection of 3-balls; choose one of them and call it \tilde{B} . The boundary of \tilde{B} consists of two types of "faces". First, there are totally geodesic faces which are the lifts of 2-cells of Δ corresponding to triples. Note that no such 2-cell can possibly contribute more than two faces to $\partial\tilde{B}$. Second, \tilde{B} has horospherical faces which are lifts of pieces of the cusp torus. There can be an arbitrary number of such faces, but each such horospherical face must only be adjacent to totally geodesic faces. (In particular the number of totally geodesic faces of \tilde{B} must be greater than zero.) We can extrude \tilde{B} in the direction of the horospherical faces (or equivalently, extrude B in the direction of the cusp of N before lifting to \mathbb{H}^3) to obtain an ideal hyperbolic polyhedron which contains \tilde{B} , and whose ideal triangular faces each contain a unique totally geodesic face of \tilde{B} .

But a geometric Mom-2 structure only has two triples, and hence \tilde{B} cannot have more than four totally geodesic faces. Thus, the ideal polyhedron containing \tilde{B} must be an ideal tetrahedron. But the same argument can be made for any component of $N - \Delta$ which is not the cusp neighborhood, and we've already used up all of the available totally geodesic faces. Therefore B and the cusp neighborhood must be the only components of $N - \Delta$. This is impossible, since $\chi(\partial M) = 0$. Therefore $N - \Delta$ cannot contain any 3-ball components.

Now suppose $n = 3$ and that one of the components of $N - \Delta$ is a 3-ball B . As before, B lifts to a 3-ball \tilde{B} in \mathbb{H}^3 , which is in turn contained in an ideal hyperbolic polyhedron whose ideal triangular faces each contain a unique totally geodesic face of \tilde{B} . This time, there are six totally geodesic faces available; since a polyhedron with triangular faces must have an even number of faces, the number of totally geodesic faces of \tilde{B} must be either 4 or 6. If the number is 6, then as before this implies that B and the cusp are the only components of $N - \Delta$ which is impossible. Therefore \tilde{B} has 4 totally geodesic faces, and hence is contained in an ideal hyperbolic tetrahedron. This implies that B is a truncated ideal hyperbolic tetrahedron, where the faces arising from the truncation are horospherical instead of geodesic.

Now consider the 2-cells in Δ which correspond to triples from the geometric Mom-3 structure. Since there are only three of them, some pair of faces of B must arise from two sides of the same 2-cell. Suppose the corresponding triple is of type (a, b, c) where a, b , and c are distinct integers. The two faces of B must share a common edge. Lifting up to \mathbb{H}^3 , we see that \tilde{B} has two geodesic faces which project down to the same 2-cell in Δ , and that these two faces have a common edge. There must be a group element $g \in \pi_1(N)$ which sends one face to the other and fixes the

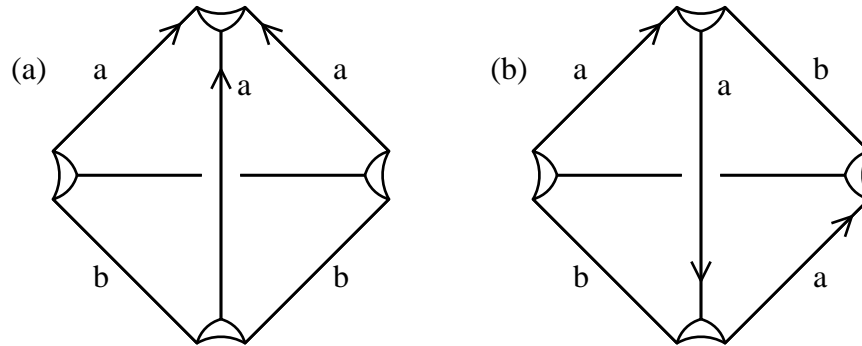


FIGURE 8. Configuration of the three-cell \tilde{B} , both (a) invalid and (b) valid

common edge, either fixing the horoballs at each end of the edge or swapping them. Therefore g is the identity or is elliptic of order 2; either result is a contradiction.

Therefore the triple in question is of type (a, a, b) for some distinct a and b . (Lemma 3.2 excludes the possibility that it is of type (a, a, a) .) Again, the two corresponding faces must share a common edge. This edge cannot be the $\mathcal{O}(b)$ -edge (as defined in Section 6) by the same argument as in the previous paragraph; hence the common edge must be a $\mathcal{O}(a)$ -edge. Choose an orientation for this edge; i.e., make the corresponding 1-cell a directed arc. Lifting to \mathbb{H}^3 again, this induces an orientation on at least three of the edges of \tilde{B} , namely all of those edges which project down to the $\mathcal{O}(a)$ -edge. There are now two possibilities; see figure 8. First, all three of those edges may be oriented toward the same horospherical face of \tilde{B} , as in figure 7(a). Note that there is a group element $g \in \pi_1(N)$ which sends one face of type (a, a, b) to the other, and that this group element preserves the orientation of the $\mathcal{O}(a)$ -edges; therefore g must be a parabolic element fixing the horosphere toward which those edges point. But a parabolic non-trivial element of $\pi_1(N)$ which takes a horosphere to itself must act on that horosphere by a translation. This would imply that all three edges labelled a in figure 7(a) lie in the same hyperbolic plane, and that \tilde{B} therefore is flat, which contradicts our assumption that Δ was embedded in N .

The other possibility is that some two of the edges are oriented toward different horospherical faces. Some thought will show that such a configuration must look like figure 7(b); we now assume that this is the situation that pertains.

In this situation, we have labelled all but one of the six geodesic edges of \tilde{B} ; now we turn our attention to the last edge. Suppose that this edge projects down to the $\mathcal{O}(a)$ -edge or the $\mathcal{O}(b)$ -edge. Then at least two of the three triples in our Mom-3 only incorporate the orthopair classes $\mathcal{O}(a)$ and $\mathcal{O}(b)$; throwing away the third triple will leave us with a simpler geometric Mom-2 structure as desired. Hence, suppose that the last edge projects down to the $\mathcal{O}(c)$ -edge, where c is the remaining index used in the Mom-3. Thus the other two geodesic faces of \tilde{B} both project to 2-cells corresponding to triples of type (a, b, c) . Note that these last two faces cannot project down to the same 2-cell, since that would imply the existence

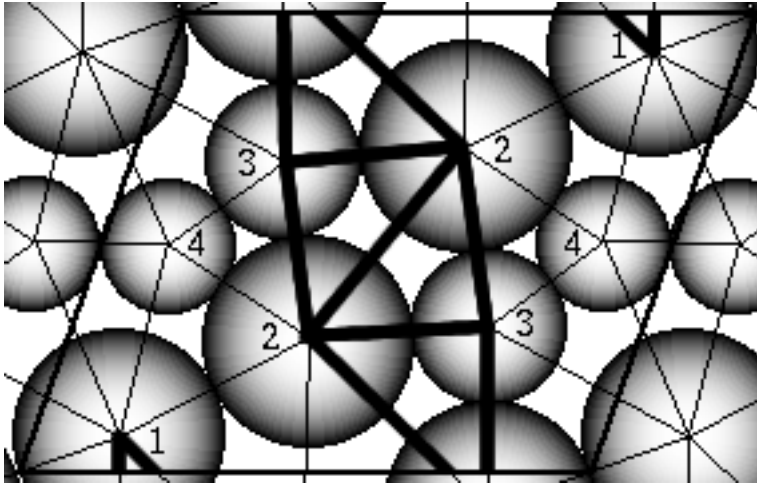


FIGURE 9. The manifold $m170$ with a Mom-3 structure which is not torus-friendly. The indices denote the orthoclasses of the horoballs.

of a non-trivial $g \in \pi_1(N)$ which is elliptic or the identity, just as before. Therefore the Mom-3 contains exactly two triples of type (a, b, c) where $a, b,$ and c are distinct, contradicting the assumption that the Mom-3 was torus-friendly. This proves the theorem. \square

It is worth pointing out that embedded geometric Mom-3's which are not torus-friendly, where some component of their complement is a 3-ball, do exist; we give an example below. It is also often the case that a manifold can have both a Mom-3 which is torus-friendly and a Mom-3 which is not. There are several hyperbolic manifolds which give rise to an embedded "Mom-like" cellular complex with three 1-cells and *four* 2-cells, such that two or three of the 2-cells are of type (a, b, c) . If one discards one of the four 2-cells, then the resulting geometric Mom-3 structure may or may not be torus-friendly depending on which 2-cell is discarded. It is easy to see, however, that if triples of type (a, b, c) , (a, b, c) , and (a, a, b) have already been found, then any possible fourth triple can be combined with some two of the first three triples to produce a Mom-3 which is torus-friendly.

As an example of a manifold containing a geometric Mom-3 which is not torus-friendly, consider the manifold known as $m170$ in the SnapPea census; a cusp diagram for this manifold is shown in figure 9. The highlighted triangles in the diagram are all corners of the same ideal hyperbolic simplex. This simplex is bounded by only three faces and three edges in the triangulation of $m170$; those faces and edges together make up a cellular complex corresponding to a geometric Mom-3 structure, with one triple of type $(2, 2, 3)$ and two distinct triples of type $(1, 2, 3)$. This is therefore a geometric Mom-3 which is not torus-friendly. At the same time the interior of the highlighted simplex is a component of the complement of this cellular complex, i.e. there is a component of $N - \Delta$ which is a 3-ball. Hence this geometric Mom-3 cannot be turned into a topological Mom-3 of the type described in [GMM2]. It is worthwhile to note, however, that this manifold does possess other

geometric Mom-3 structures which are torus-friendly. Specifically there exists a fourth triple, of type $(1, 1, 2)$, which can be used to construct such a Mom-3.

8. FULLNESS

Based on the result of Theorem 7.1, throughout this section we assume that N possesses a geometric Mom- n structure where $n = 2$ or 3 , that the corresponding cellular complex Δ is embedded, and that the components of $N - \Delta$ which are not cusp neighborhoods have torus boundary. At this point we switch from discussing cellular complexes to discussing handle structures as follows: thicken T to $T \times I$ where $T \times 0$ faces the cusp, thicken each 1-cell of Δ to a 1-handle from $T \times 1$ to itself, and thicken each 2-cell of Δ to a 2-handle which runs over $T \times 1$ and three 1-handles counting multiplicity. The reason for this change in focus is solely to take advantage of the language and conclusions of [GMM2]. Specifically, $T \times I$ and the newly constructed 1-handles and 2-handles form a handle decomposition of a submanifold $M \subset N$. We will abuse notation and allow Δ to also refer to the handle decomposition of M ; it should always be clear in context whether we are referring to a cellular complex or a handle decomposition. By Theorem 7.1 we may assume that the boundary of M is a collection of tori. Also, since M is a subset of a hyperbolic manifold and contains both a cusp torus and a geodesic arc from the cusp torus to itself, $i_*\pi_1(M)$ cannot be abelian, where $i : M \rightarrow N$ is the inclusion map. In other words, $i : M \rightarrow N$ is a *non-elementary* embedding. Therefore (M, T, Δ) is a topological internal Mom- n structure according to [GMM2]. As in [GMM2], we will adopt the terminology of Matveev and refer to the intersection of the 1-handles (resp. 2-handles) with $T \times 1$ as *islands* (resp. *bridges*), and the complement in $T \times 1$ of the islands and bridges will be called *lakes*. The *valence* of an island will be defined to be equal to the valence of the corresponding 1-handle, or equivalently the number of ends of bridges lying on the island. We assume that these valences are at least two; of any 1-handle has valence one simply remove both it and the 2-handle adjacent to it to obtain a simpler Mom- n structure.

Clearly each 1-handle in Δ contributes two islands while each 2-handle contributes three bridges. Suppose σ is a 2-handle corresponding to a triple of type (a, a, b) where the type is defined as in Section 6. In other words some lift $\tilde{\sigma}$ of σ in the universal cover of N is adjacent to three horoballs $\{A, B, C\}$ such that the orthopairs $\{A, B\}$ and $\{B, C\}$ are in $\mathcal{O}(a)$ while $\{C, A\} \in \mathcal{O}(b)$. Let a_0 and a_1 denote the islands which are the endpoints of the 1-handle around the $\mathcal{O}(a)$ -edge, and define b_0 and b_1 similarly. Then the intersection of $\tilde{\sigma}$ with ∂B projects down to a bridge whose endpoints both lie on islands in the set $\{a_0, a_1\}$.

Definition 8.1. *If this bridge described above joins a_0 to a_1 then we will say σ is a loxodromic 2-handle. If instead this bridge joins a_i to itself for $i = 1$ or 2 then we will say σ is a parabolic 2-handle.*

In either case there exists $g \in \pi_1(N)$ which sends $\{A, B\}$ to $\{B, C\}$ since those are in the same orthopair class; furthermore g is uniquely defined. If σ is a parabolic 2-handle then we must in fact have $g(A) = C$ and $g(B) = B$ (i.e. g is a parabolic group element), and the bridge from a_i to itself must follow a straight closed path in the cusp torus which corresponds to g . Wolog suppose $i = 0$, i.e. a_0 is joined to itself by a bridge. Then it is not hard to see that the other two bridges corresponding to σ must join a_1 to b_0 and b_1 respectively. These bridges have equal length by Lemma

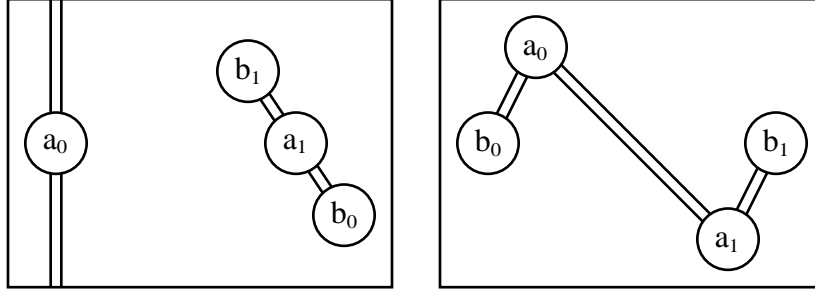


FIGURE 10. The islands and bridges belonging to a parabolic (left) and loxodromic (right) 2-handle. Note that the islands labelled a_0 and a_1 are endpoints of a single 1-handle (and similarly for b_0 and b_1).

3.4, and the angle between the bridges at a_1 must equal the angle between the two bridge ends at a_0 , since both angles are equal to the angle at which σ meets itself along the $\mathcal{O}(a)$ -edge. Clearly this angle is a straight angle at a_0 , and therefore at a_1 as well. Thus the configuration of islands and bridges resulting from σ is as in the left side of figure 10.

If σ is instead a loxodromic 2-handle then we must have $g(A) = B$ and $g(B) = C$ (which implies that g is a loxodromic group element although we will not prove that here). Again it is not hard to see that wolog the other two bridges corresponding to σ must join a_0 to b_0 and a_1 to b_1 respectively. Again, these two bridges must have the same length by Lemma 3.4 and the angle formed by the bridge ends at a_0 equals the angle formed by the bridge ends at a_1 . Given that N is orientable the configuration of islands and bridges arising from σ must look like the right side of figure 10.

The handle structure Δ is called *full* if all of the lakes are simply connected. If Δ is a full handle structure, then Theorem 4.1 of [GMM2] applies to N . Our goal in this section is to either prove that Δ is full or else construct a new topological internal Mom- n structure in N which is full. Lemma 4.5 of [GMM2] does this in the case that $n = 2$. Specifically it shows that a topological internal Mom-2 structure that is not full can be replaced with one which is. Hence we can combine Theorem 6.1, Theorem 7.1, Theorem 4.1 of [GMM2], and Lemma 4.5 of [GMM2] to conclude the following:

Proposition 8.2. *If N has a geometric Mom-2 structure, wolog that structure can be thickened into a topological internal Mom-2 structure (M, T, Δ) that is both full and not false. Consequently, M is hyperbolic and N can be recovered by Dehn surgery on M . \square*

The assertion in the second sentence is one of the primary results of [GMM2].

Unfortunately, the topological argument used in that proof does not extend easily to the $n = 3$ case. Fortunately in this context we may take advantage of the fact that our Mom-3 is more than just a topological object; by construction the 1-handles and 2-handles of Δ have geodesic cores, an assumption that is not made in [GMM2]. Call a Mom- n structure with this additional property a *geodesic* internal Mom- n structure. Our goal now is to prove the following:

Theorem 8.3. *If (M, T, Δ) is an embedded geodesic internal Mom-3 structure in a hyperbolic manifold N which is not false, with 1-handles corresponding to the orthopair classes $\mathcal{O}(1)$, $\mathcal{O}(2)$, and $\mathcal{O}(3)$, then N has a full topological internal Mom- k structure where $k \leq 3$.*

Proof: Suppose Δ is not full; i.e. suppose that Δ has one or more lakes which are not simply connected. The possible shapes for such a lake are the following: a torus with one or more holes, a disk with one or more holes, or an essential annulus with zero or more holes. In each case we wish to either find a contradiction or else construct a topological internal Mom-2 structure.

Suppose $T \times I$ contains a lake which is a torus with holes. Then there is a simple closed loop γ which bounds a disk in $T \times I$ containing all of the islands and lakes. Push that disk into $T \times I$ to obtain a compressing disk for M which separates M into two pieces M_1 and M_2 , such that M_1 is homeomorphic to $T \times I$ minus a 0-handle and M_2 consists of that 0-handle together with the 1-handles and 2-handles of Δ . Since (M, T, Δ) is not false all of the boundary components of M are tori; since M is the connected sum of M_1 and M_2 , and since M_1 has torus boundary, one of the boundary components of M_2 must be a 2-sphere. Since N is hyperbolic, the only possibility is that M_2 is contained inside a 3-ball. Therefore M is the connected sum of $T \times I$ and a sub-manifold of N contained inside a 3-ball; this is impossible if (M, T, Δ) is geodesic.

Now suppose $T \times I$ contains a lake which is a disk with holes. Then let γ be a simple closed curve parallel to the boundary of that disk, such that all of the islands and lakes inside the disk are also inside γ . Let T_0 denote the component of ∂M which contains γ . As before, γ bounds a disk in $T \times I$ which is a compressing disk for M although it may not separate M . If it does separate M , then arguing as in the previous case we may show that the component of the separated manifold which does not contain $T \times 0$ is contained in a 3-ball in N , which is a contradiction if (M, T, Δ) is geodesic. So suppose that the compressing disk does not separate M ; let M_1 be the manifold obtained after the compression. Note that the connectedness of M_1 implies that there must be a 1-handle in Δ with one endpoint inside γ and one endpoint outside; this implies that $T_0 - \gamma$ is connected, i.e. γ is essential in T_0 . Hence the compression turns T_0 into a connected 2-sphere boundary component of M_1 ; call this sphere S_1 . M_1 also has a handle structure Δ_1 consisting of the 0-handle carved out of $T \times I$ by the compressing disk, $T \times I$ minus that 0-handle (which is homeomorphic to $T \times I$), and the 1-handles and 2-handles of Δ . Now S_1 must bound a 3-ball in N , and that 3-ball must lie on the outside of M_1 since M_1 contains $T \times 0$. Add this 3-ball to M_1 as a 3-handle to obtain a new manifold $M_2 \subset N$ with torus boundary and new handle structure Δ_2 . Choose a 1-handle which connects the 0-handle of Δ_2 to $T \times I$; cancel that 1-handle with the 0-handle, and cancel the 3-handle with a 2-handle to obtain a handle structure Δ_3 with only 1-handles and 2-handles.

We need to know that the embedding $i_2 : M_2 \rightarrow N$ is non-elementary, but note that M_2 actually contains M : adding the 3-handle to M_1 restores the portion of M that was removed by the compression. So $i_2 : M_2 \rightarrow N$ is non-elementary since $i : M \rightarrow N$ is.

Now using the methods of [GMM2], (M_2, T, Δ_3) can be simplified to obtain a new topological internal Mom- k structure (M_4, T, Δ_4) on N . Since the construction of Δ_3 deleted 1-handles and 2-handles without adding new ones, the complexity

of (M_4, T, Δ_4) as defined in [GMM2] must be strictly less than the complexity of (M, T, Δ) ; hence $k \leq 2$. Then by Lemma 4.5 of [GMM2] we may assume that N contains a full topological internal Mom- k structure with $k \leq 2$.

If $T \times I$ contains a lake which is an essential annulus with one or more holes, let γ be a simple closed curve in the lake parallel to the boundary of one of those holes. Then proceed just as in the previous case.

The remaining possibility, and the one which will require the most analysis, is that $T \times 1$ contains lakes which are essential annuli without holes and lakes which are disks. Given six islands and nine lakes, for Euler characteristic reasons there must be three disk lakes and an unknown number of annulus lakes. Let A_1, A_2, \dots, A_r be the annulus lakes, and let B_1, B_2, \dots, B_r be the connected components of $T \times 1 - (\cup_{i=1}^r A_i)$. Each B_i is also an annulus, composed of islands, bridges, and disk lakes; furthermore each B_i must contain at least one island.

Lemma 8.4. *If $r > 1$ then N must contain a full topological internal Mom-2 structure.*

Proof: Suppose $r = 2$. Choose simple paths $\mu_i, i \in \{1, 2\}$, such that μ_i crosses B_i transversely for each i , μ_i does not cross any island for either i , and such that the total number of bridges crossed by μ_1 and μ_2 is minimal. Since there are only three disk lakes total in B_1 and B_2 , the number of bridges crossed by μ_1 and μ_2 combined is at most five.

Slice M open along $A_1 \times I$ and $A_2 \times I$ to obtain a new manifold $M_1 \subset N$; note that M_1 will still have torus boundary and the inclusion $i_1 : M_1 \rightarrow N$ will still be non-elementary. M_1 consists of two thickened annuli, namely $B_i \times I$ for $i \in \{1, 2\}$, and the 1-handles and 2-handles of Δ . Each μ_i forms part of the boundary of a disk in $B_i \times I$; thicken each disk to obtain a decomposition of $B_i \times I$ into a 1-handle (the thickened disk) and a 0-handle (the complement of the thickened disk). Thus we obtain a standard handle decomposition Δ_1 of M_1 consisting of two 0-handles, two new 1-handles (which we also refer to as μ_1 and μ_2), and the original 1-handles and 2-handles of Δ . By construction the total valence of the new 1-handles μ_1 and μ_2 is at most five.

Now let λ_1, λ_2 , and λ_3 denote the original 1-handles of Δ , ordered in such a way that the valence of λ_1 is maximal. Note that wlog we may assume that at least one of the λ_i 's connects the two B_i 's; otherwise we can simply throw away all of the 1-handles and 2-handles of Δ which are connected to B_1 to obtain a simpler geometric Mom- n and apply Proposition 8.2. Suppose that λ_1 has both endpoints on B_1 (or equivalently on B_2). Then in Δ_1 , λ_1 has both endpoints on the same 0-handle. Drill out the core of λ_1 and this 0-handle to obtain a new torus boundary component T_2 , and cancel the other 0-handle of Δ_1 with one of the λ_i 's which connects B_1 to B_2 . The result is a new manifolds $M_2 \subset N$ and an internal topological Mom- k structure (M_2, T_2, Δ_2) where Δ_2 consists of the remaining λ_i , μ_1 and μ_2 , and $T_2 \times I$. Since the sum of the valences of the λ_i 's equals 9, either the valence of λ_1 was at least 4 and the valence of the cancelled λ_i was at least 2, or else the valence of all the λ_i 's equals 3. Either way the total valence of the two λ_i 's removed to construct Δ_2 is at least 6, more than the total valence of the new 1-handles μ_1 and μ_2 . Therefore the complexity of (M_2, T_2, Δ_2) , as defined by [GMM2], is less than the complexity of (M, T, Δ) . Consequently $k \leq 2$. Then apply Lemma 4.5 of [GMM2] to complete the proof. If λ_1 connects B_1 to B_2 but λ_2 had both endpoints on B_1 , then drill out the core of λ_2 and one 0-handle and cancel λ_1

with the other 0-handle then proceed as above. If every λ_i connects B_1 to B_2 then drill out the cores of λ_1 , λ_2 , and both 0-handles at the same time to construct T_2 , then proceed as above. This completes the proof in this case.

If $r = 3$ the proof is similar. In this case we can construct μ_1 , μ_2 , and μ_3 as paths which cross at most six bridges in total, then split along each $A_i \times I$ and turn the μ_i 's into new 1-handles with total valence at most 6. The B_i 's must all be connected by 1-handles; if B_1 were not connected to the others then we could throw away all of the 1-handles and 2-handles of Δ connected to B_1 to obtain a simpler geometric Mom- n and apply Proposition 8.2. Hence at most one of the λ_i 's has both endpoints on the same B_i . Drill out the cores of enough λ_i 's and 0-handles to construct a new torus boundary component T_2 and cancel any remaining 1-handles with the remaining 0-handles to obtain an internal topological Mom- k (M_2, T_2, Δ_2) . Since the λ_i 's have total valence 9 and the μ_i 's have total valence at most 6, we have reduced complexity and hence $k \leq 2$. Then again apply Lemma 4.5 of [GMM2].

If $r \geq 4$ then at least two B_i 's contain only one island each. Since the bridges follow straight paths and since the B_i 's are annuli, if B_i contains only one island then it also contains only one bridge. Such an island is the endpoint of a 1-handle λ_1 which is connected to only a single 2-handle in Δ ; throw away the 1-handle and the 2-handle to obtain a simpler geometric Mom- n , then apply Proposition 8.2. This completes the proof of the lemma. \square

So suppose $r = 1$. Let $A = A_1$ and $B = B_1$ for simplicity. We wish to use an argument similar to the one in the above lemma to obtain a topological Mom-2 structure. Specifically we wish to find a path μ which crosses B transversely and which crosses no islands and as few bridges as possible. Suppose μ can be chosen to cross fewer than $v(\lambda_1)$ bridges, where $v(\lambda_1)$ is the valence of one of the 1-handles of Δ . Then as in the lemma we can split $T \times I$ along $A \times I$, then decompose $B \times I$ into a 0-handle and a 1-handle where the 1-handle is obtained by thickening a disk with μ in its boundary. Then by drilling out the cores of λ_1 and the 0-handle we will obtain a topological internal Mom-2 structure which is full by Lemma 4.5 of [GMM2].

Since B contains only three disk lakes we can always choose μ to cross at most four bridges. Thus we're done if there exists a 1-handle with valence five or more. However this is not always the case. Let λ_1 , λ_2 , and λ_3 be the 1-handles of Δ . We now consider the possible values for the valences of these three 1-handles. As previously stated the valence must add up to nine, so there are only three possibilities up to symmetry.

First suppose $v(\lambda_1) = 5$ and $v(\lambda_2) = v(\lambda_3) = 2$. Then we're done by the above argument.

Next, suppose $v(\lambda_1) = 4$, $v(\lambda_2) = 3$, and $v(\lambda_3) = 2$. If we can find a path μ connecting the boundary components of B which crosses no islands and no more than three bridges then the usual splitting-and-drilling procedure as will result in a strictly simpler internal Mom- k structure. However it is not immediately apparent that such a path must exist; further analysis is required.

Let $\partial_+ B$ and $\partial_- B$ denote the two boundary components of B . Consider the case where $\partial_+ B$ contains exactly one island, and hence there is a bridge connecting that island to itself along a straight path in $T \times 1$. That bridge must be one corner of a parabolic 2-handle, as in the left side of figure 10. The existence of a parabolic 2-handle turns out to have strong geometric consequences as follows:

Lemma 8.5. *Suppose $\partial_+ B$ contains exactly one island, and that consequently (M, T, Δ) contains a parabolic 2-handle. If (M, T, Δ) contains a second parabolic 2-handle, then N must have a full geodesic internal Mom-2 structure.*

Proof: As before, if λ is the 1-handle corresponding to $\mathcal{O}(a)$ then we will denote the islands at the end of λ by a_0 and a_1 in some order. Also for brevity we will say that a 2-handle is “of type (a, b, c) ” if it corresponds to a triple of type (a, b, c) . A parabolic 2-handle by definition of type (a, a, b) for some a and b ; assume the island in $\partial_+ B$ is the island a_0 . If the second parabolic 2-handle is of type (a, a, b) or (b, b, a) , then clearly we have a geometric Mom-2 structure which is necessarily full, torus-friendly, embedded, and so forth.

Suppose the second parabolic 2-handle is of type (a, a, c) , where $\mathcal{O}(c)$ is the remaining orthopair class in our Mom-3 structure. The only way this is possible is if the island a_1 is also connected to itself by a bridge. But then there would be two additional bridges which meet at the island a_0 at a straight angle, which is impossible if that is the only island in $\partial_+ B$.

Suppose the second parabolic 2-handle is of type (b, b, c) . Then wolog the island b_0 is also connected to itself by a bridge; since all of the bridges are contained in B , an annulus, the path this bridge follows must be in the same homotopy class as the bridge in $\partial_+ B$. Since the bridges are geodesic, this means they must have the same length. Recall that if e_a, e_b , and e_c are the elements of the Euclidean spectrum corresponding to $\mathcal{O}(a), \mathcal{O}(b)$, and $\mathcal{O}(c)$ then Lemma 3.4 implies that:

$$\frac{e_c}{e_b^2} = \frac{e_b}{e_a^2} \geq 1$$

Hence $e_c \geq e_b \geq e_a$, and wolog $c \geq b \geq a$. By the assumptions of Theorem 8.3 this implies that $a = 1, b = 2$, and $c = 3$. Now consider the island a_1 . While there is not necessarily a bridge connecting this island to itself, there is still a triple of horoballs corresponding to this island and the translation $g \in \pi_1(T)$ which acts in the direction of the annulus B . That triple must be of type (a, a, k) for some k . Therefore Lemma 3.4 implies that

$$\frac{e_k}{e_a^2} = \frac{e_b}{e_a^2}$$

So $e_k = e_b$. If $k \leq b = 2$ then this implies that N has a geometric Mom-2 structure. If $k \geq 3$ then $e_k \geq e_3 \geq e_2$, i.e. $e_3 = e_2$. But $e_3/e_2^2 = e_c/e_b^2 \geq 1$, so $e_3 = e_2$ implies that $e_3 = e_2 = 1$. This means that the horoballs centered at the islands b_0, b_1, c_0 , and c_1 are all *full-sized*, i.e. in the upper half-space model they appear as Euclidean spheres of diameter 1 and are tangent to the horoball at infinity. Since there is a parabolic 2-handle of type (b, b, c) , by Lemma 3.4 the distance from c_0 and c_1 to b_1 is at most $1/e_c = 1$, and hence the horoballs centered at c_0 and c_1 abut the horoball at b_1 . Similarly the length of the bridge from b_0 to itself is $e_c/e_b^2 = 1$ so the horoball at b_0 abuts itself; by symmetry, so does the horoball centered at b_1 . Note that this implies that shortest essential curve on the cusp torus has length at most one, so any full-sized horoball must abut itself. Finally the horoball at b_1 must abut the full-sized horoball centered at a_1 , thanks to the parabolic triple of type (a, a, b) . This is a contradiction, as there is no way to arrange all of these full-sized horoballs around the island b_1 while keeping their interiors disjoint.

If the second parabolic 2-handle is of type (c, c, a) then permute the variables, replacing c with a, a with b , and b with c . Then proceed as in the previous case.

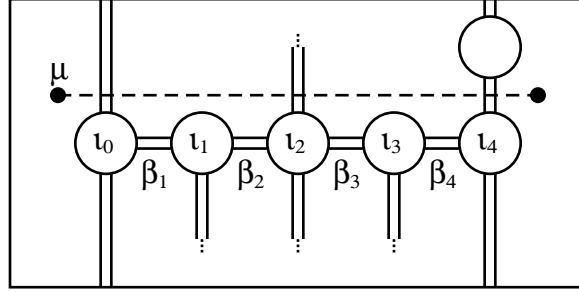


FIGURE 11. Finding a path μ from $\partial_+ B$ to $\partial_- B$ which crosses at most three bridges.

The remaining possibility is that the second parabolic 2-handle is of type (c, c, b) . Wolog the island c_0 is connected to itself by a bridge following a path in the same homotopy class (and hence of the same length) as the bridge in $\partial_+ B$. Define e_a , e_b , and e_c as before. Then

$$\frac{e_b}{e_c^2} = \frac{e_b}{e_a^2} \geq 1$$

Therefore $e_b \geq e_c = e_a$; hence $e_2 = e_1$ and wolog $a = 1$, $c = 2$, and $b = 3$. Note that the length of the bridge connecting a_0 to itself must be e_3 by Lemma 3.4. Now consider the island a_1 ; it is connected by bridges to the islands b_0 and b_1 , and these bridges both have length $1/e_3$ and meet at a_1 in a straight angle. Similarly, the island c_1 is connected to b_0 and b_1 by bridges of length $1/e_3$ which meet at c_1 in a straight angle. The only way this can happen inside the annulus B is if the island c_1 lies exactly halfway along the shortest geodesic path from a_1 to itself, and vice versa. In other words, there are two geodesic paths of length $e_3/2$ connecting a_1 to c_1 . These paths correspond to triples of horoballs of type $(1, 2, k)$ and $(1, 2, l)$ for some k and l such that $e_k = e_l = e_3/2 < e_3$; clearly $k, l \in \{1, 2\}$. These two triples cannot be equivalent under the action of $\pi_1(N)$; since they involve the same two islands, if $g \in \pi_1(N)$ mapped one triple to the other than g would also have to fix the cusp torus T , but the two bridges in question are clearly not equivalent under the action of $\pi_1(T)$. Therefore the triples $(1, 2, k)$ and $(1, 2, l)$ constitute a geometric Mom-2 structure. This completes the proof of the lemma. \square

So therefore we may assume that there is at most one island connected to itself by a bridge; in particular $\partial_- B$ contains more than one island.

Next consider a minimum-length sequence $\gamma = \{\iota_0, \beta_1, \iota_1, \dots, \beta_n, \iota_n\}$ of islands and bridges such that ι_0 is the sole island in $\partial_+ B$ and ι_n is an island in $\partial_- B$. Note there are at most four bridges in the sequence (otherwise there would be more than six islands in total). If γ contains only one or two bridges, it is straightforward to show that there must be a path μ from $\partial_+ B$ to $\partial_- B$ lying in a small neighborhood of γ which crosses at most three bridges. (Remember that at most two of the islands in the sequence γ can have valence 4.) Suppose γ contains four bridges; then together γ , $\partial_+ B$, and $\partial_- B$ contain all six islands and at least seven bridges. If the remaining bridges are not placed in such a way that at least two bridge ends meet the islands in γ on each side of the sequence (see figure 11), then there will be a path μ crossing at most 3 bridges. So suppose there are at least two

additional bridge ends on each side of the sequence; this implies that each of the two remaining bridges join an island in γ to another island in γ (and not, say to the other island in $\partial_- B$). Now note that up to reordering of the indices there are only two combinations of triples which result in a Mom-3 structure which is torus-friendly and with these valences, and without containing a Mom-2 structure as a subset: triples of type (a, a, b) , (a, a, c) , and (b, b, c) for some ordering $\{a, b, c\}$ of the indices $\{1, 2, 3\}$, or triples of type (a, a, c) , (a, b, b) , and (a, b, c) . There is no way to place the two remaining bridges that (a) preserves the minimality of γ (b) doesn't imply the existence of a second parabolic 2-handle, (c) ensures two islands of each valence $\{2, 3, 4\}$, and (d) ensures that the number of bridges between each island matches the numbers produced by one of the two combinations of triples described above. If γ contains 3 bridges then γ , $\partial_+ B$, and $\partial_- B$ contain at least five islands and at least six bridges. Again, if there are not two additional bridge ends on each side of γ then we can find an appropriate path μ ; so suppose there are at least two additional bridge ends on each side. This implies that the missing island is of valence 2, since otherwise it would account for too many bridge ends. Again, there is no way to place an island of valence 2 and the three remaining bridges that satisfies the conditions (a), (b), (c), and (d) above.

Therefore if $\partial_+ B$ contains exactly one island then we can find a path μ from $\partial_+ B$ to $\partial_- B$ crossing at most 3 bridges.

Suppose then that $\partial_+ B$ and $\partial_- B$ each contain at least two islands. Define γ as before; there are at most 3 bridges in the sequence. If there are only 1 or 2 bridges, then we can find μ crossing at most three bridges as before, so suppose γ contains exactly 3 bridges. Then γ , $\partial_+ B$, and $\partial_- B$ contain all six islands and at least seven bridges. Furthermore the two missing bridges must each join an island in γ to another island in γ or else there will be less than two additional bridge ends on one side of the sequence and hence a path μ crossing at most 3 bridges. But there is no way to place the two missing bridges that (a) preserves the minimality of γ , (b) ensures each missing bridge joins γ to itself, (c) ensures two islands of each valence $\{2, 3, 4\}$, and (d) ensures that the number of bridges between each island matches the numbers produced by one of the two combinations of triples described earlier.

This completes the proof in the case where $v(\lambda_1) = 4$, $v(\lambda_2) = 3$, and $v(\lambda_3) = 2$.

The remaining case is where $v(\lambda_1) = v(\lambda_2) = v(\lambda_3) = 3$. Define $\partial_+ B$ and $\partial_- B$ as before. If $\partial_+ B$ and $\partial_- B$ each contain exactly one island, then Lemma 8.5 above shows N has a full internal Mom-2 structure. Suppose $\partial_+ B$ and $\partial_- B$ each contain at least two islands. Consider the minimum-length sequence $\gamma = \{\iota_0, \beta_1, \iota_1, \dots, \beta_n, \iota_n\}$ as before; γ has no more than three bridges. If γ has two or fewer bridges, then since each island has valence 3 there must be a path μ from $\partial_+ B$ to $\partial_- B$ which crosses at most two bridges; split, thicken, and drill just as before to construct a topological internal Mom-2 structure. If γ has exactly three bridges, then γ , $\partial_+ B$, and $\partial_- B$ contain all six islands at at least seven bridges, but there is no way to place the missing two bridges that (a) preserves the minimality of γ and (b) ensures all six islands have valence 3, except for configurations like the one shown in figure 12, in which there is clearly a path μ which crosses only two bridges.

So wolog assume that $\partial_+ B$ contains exactly 1 island and $\partial_- B$ contains at least 2. If $\partial_+ B$ contains exactly one island then the Mom-3 structure must contain a

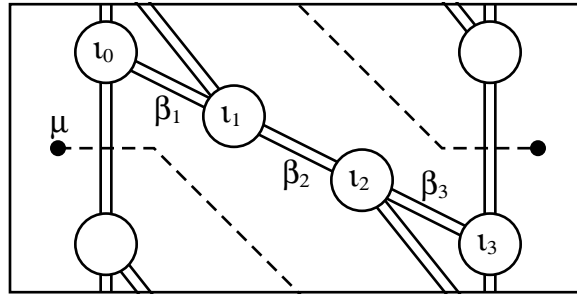


FIGURE 12. If each island has valence 3 and γ has exactly three bridges then we can always find a path μ crossing two bridges.

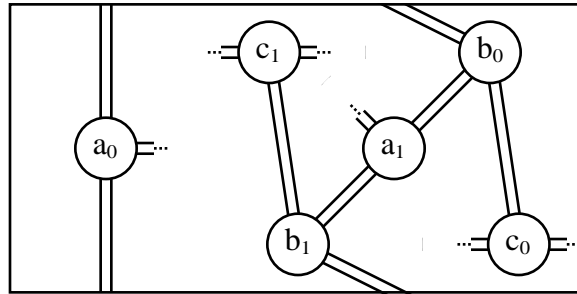


FIGURE 13. The islands and bridges corresponding to a parabolic 2-handle of type (a, a, b) and a loxodromic 2-handle of type (b, b, c) . Note there is no way to include a loxodromic 2-handle of type (c, c, a) as well.

parabolic 2-handle of type (a, a, b) for some a and $b \in \{1, 2, 3\}$, $a \neq b$. We may assume the island in $\partial_+ B$ is the one denoted a_0 , and hence the islands b_0 and b_1 are each joined to a_1 by bridges which meet at a_1 in a straight angle. Consider the other two 2-handles in the Mom-3 structure; we may assume they are either of type (b, b, c) and (c, c, a) , or of type (c, c, b) and (a, b, c) , where c is the remaining element of $\{1, 2, 3\}$, as no other combination produces 1-handles of the given valences without also including a geometric Mom-2 structure. Suppose the other 2-handles are of type (b, b, c) and (c, c, a) . If either of these 2-handles are parabolic then we can apply Lemma 8.5; so suppose neither is parabolic. A loxodromic 2-handle of type (b, b, c) implies that there is a bridge joining b_0 to b_1 ; this bridge together with the two-step path from b_0 to a_1 to b_1 must form a homotopically non-trivial loop in B . Also, wolog the island c_i is joined to b_i by a bridge for $i = 0, 1$. Moreover since (M, T, Δ) is a geodesic structure, the angles subtended by the three bridges at b_0 must equal the angles subtended by the three bridges at b_1 , albeit in opposite order since N is orientable. Thus the bridges and islands must be arranged as in figure 13; note in particular that c_0 and c_1 must be on opposite sides of the path $b_0 \rightarrow a_1 \rightarrow b_1 \rightarrow b_0$. Now the third 2-handle, which is of type cca and which we are assuming is non-parabolic, must imply the existence of a bridge in B which joins c_0 to c_1 without crossing any other bridge, which is clearly impossible.

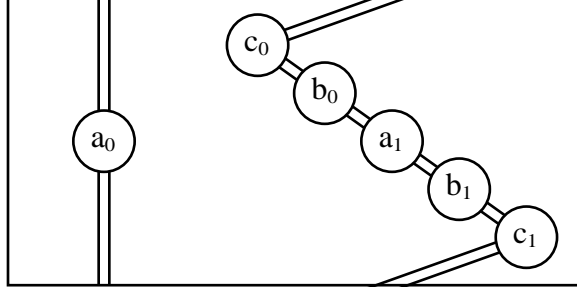


FIGURE 14. The islands and bridges corresponding to a parabolic 2-handle of type (a, a, b) and a loxodromic 2-handle of type (c, c, b) . Note there is no way to insert another bridge from a_1 to any of the b_i 's or the c_i 's.

Suppose then that the other 2-handles in the Mom-3 structure are of type (c, c, b) and (a, b, c) . Again, we may assume the 2-handle of type (c, c, b) is a loxodromic 2-handle, not a parabolic one. Wolog as in Figure 10 there exist bridges joining b_0 to c_0 , c_0 to c_1 , and c_1 to b_1 . These bridges together with the bridges joining b_0 and b_1 to a_1 must again follow a homotopically non-trivial loop in B . In order to form such a loop in the annulus B , the angle at b_0 between the bridge to a_1 and the bridge to c_0 must equal the angle at b_1 between the bridge to a_1 and the bridge to c_1 , and those angles must be in the same direction. But since (M, T, Δ) is a geodesic structure and since M is orientable those angles must in fact be equal in the opposite direction as well. Hence those two angles are both straight angles; see figure 14. The last 2-handle, of type (a, b, c) , implies that there are three more bridges: one joining a_i to b_j for some i and j , another joining b_k to c_l for some k and l where $k \neq j$, and a third joining c_s to a_t where $s \neq l$ and $t \neq i$. In particular there must be a another bridge joining a_1 to one of b_0, b_1, c_0, c_1 , but there is no way to lay such a bridge without intersecting a pre-existing bridge. This is a contradiction.

This completes the proof of the theorem. \square

9. PROOF OF THEOREM 1.1 AND APPLICATIONS

We begin by tying together the last three sections along with [GMM2] and completing the proof of Theorem 1.1.

Suppose N is a one-cusped hyperbolic 3-manifold with $\text{Vol}(N) \leq 2.848$. Then by Theorem 5.9 N contains a geometric Mom-2 or Mom-3 structure (which is not false), and by Theorem 6.1 we may assume that the corresponding cellular complex Δ is embedded in N . By Theorem 7.1, we may further assume that the components of $N - \Delta$ which are not cusp neighborhoods have torus boundary. Then by either Proposition 8.2 or Theorem 8.3 as appropriate, we may assume that the submanifold M and handle decomposition Δ obtained by thickening the geometric Mom-2 or Mom-3 structure satisfy the definition of a full topological internal Mom- k structure as given in [GMM2].

Then by Theorem 4.1 of [GMM2] we can conclude that there exists a full topological internal Mom- k structure (M, T, Δ) in N where $k \leq 3$ and M is hyperbolic. (Note this Mom- k structure may bear little to no resemblance to the structure we

started with!) This implies that N can be recovered from M by a hyperbolic Dehn filling on all but one of the cusps of M .

Finally Theorem 5.1 of [GMM2] enumerates the possible choices for M . There are only 21 hyperbolic manifolds M which can form part of a full topological internal Mom- k structure (M, T, Δ) for $k \leq 3$, and these are precisely the manifolds listed in the table in figure 1. This completes the proof of Theorem 1.1. \square

Theorem 1.1 does not in itself constitute an enumeration of all one-cusped manifolds with volume less than 2.848. However it is possible to analyze the Dehn surgery spaces of each of the 21 manifolds listed in figure 1. We use the following theorem from [FKP]:

Theorem 9.1. (*Futer, Kalfagianni, and Purcell*): *Let M be a complete, finite-volume hyperbolic manifold with cusps. Suppose C_1, \dots, C_k are disjoint horoball neighborhoods of some subset of the cusps. Let s_1, \dots, s_k be slopes on $\partial C_1, \dots, \partial C_k$, each with length greater than 2π . Denote the minimal slope length by l_{\min} . If $M(s_1, \dots, s_k)$ satisfies the geometrization conjecture, then it is a hyperbolic manifold, and*

$$\text{Vol}(M(s_1, \dots, s_k)) \geq \left(1 - \left(\frac{2\pi}{l_{\min}}\right)^2\right)^{3/2} \text{Vol}(M).$$

Therefore if M is one of the two-cusped manifolds listed in figure 1, and if we wish to enumerate all one-cusped manifolds with volume less than or equal to 2.848 that can be obtained by filling, then it is only necessary to examine surgery coefficients with slope less than or equal to

$$2\pi \left(\sqrt{1 - \left(\frac{2.848}{\text{Vol}(M)}\right)^{2/3}} \right)^{-1}$$

As an example, suppose M is the Whitehead link complement, known as m129 in the SnapPea census. This manifold admits a symmetry which exchanges the cusps, therefore it does not matter which cusp we choose to fill in. (This is true for all of the manifolds listed in figure 1 except s785.) Using SnapPea, we see that the volume of m129 is 3.6638..., which implies that we need only consider Dehn fillings along slopes of length less than 15.99. Given that a maximal cusp torus around one cusp has a longitude of length $2\sqrt{2}$ and a meridian of length $\sqrt{2}$ at right angles to the longitude, we need only consider Dehn fillings with coefficients (a, b) where a and b are relatively prime integers satisfying $2a^2 + 8b^2 \leq 256$, clearly a finite and manageable number of cases. For each such filling, we can use SnapPea, [Mos], and other such tools to confirm rigorously whether or not the resulting one-cusped manifold is hyperbolic and has volume less than 2.848.

For s776, the only three-cusped manifold in figure 1, we need to fill in two cusps to obtain a one-cusped manifold. However Theorem 9.1 only provides an upper bound on one of the corresponding slopes. Nevertheless, there are finite number of possibilities for that one slope and therefore filling in that one slope results in a finite number of two-cusped manifolds for which we can repeat the above analysis. Note that s776 admits symmetries which permute all of its cusps, so again it does not matter which cusps we fill. It should also be pointed out that Martelli and

Petronio have already determined the complete list of fillings on $s776$ that result in non-hyperbolic manifolds ([MP]).

The results of the above analysis will be presented in detail in an upcoming paper ([MM]); for now we present the results of that analysis without proof:

Theorem 9.2. *The only one-cusped orientable hyperbolic 3-manifolds with volume less than or equal to 2.848 are the manifolds known in the SnapPea census as $m003$, $m004$, $m006$, $m007$, $m009$, $m010$, $m011$, $m015$, $m016$, and $m017$.*

Theorem 1.1 can also be used to analyze closed hyperbolic 3-manifolds. Lemma 3.1 of [ACS] states the following:

Lemma 9.3. *Suppose that M is a closed orientable hyperbolic 3-manifold and that C is a shortest geodesic in M such that $\text{tuberad}(C) \geq (\log 3)/2$. Set $N = \text{drill}_C(M)$. Then $\text{Vol}(N) < 3.02 \text{Vol}(M)$.*

Here $\text{tuberad}(C)$ refers to the *tube radius*, i.e. the maximal radius of an embedded tubular neighborhood around C , while $\text{drill}_C(M)$ is simply the manifold $M - C$ equipped with a complete hyperbolic metric. In other words N is a one-cusped hyperbolic 3-manifold from which the closed manifold M can be recovered by Dehn filling. Two remarks are in order at this point. First, the Weeks manifold has volume less than $2.848/3.02$ and is the smallest known closed orientable hyperbolic 3-manifold. Second, according to [GMT] if the condition that $\text{tuberad}(C) \geq (\log 3)/2$ fails then M must have volume greater than that of the Weeks manifold anyway. Thus combining the above lemma with Theorem 9.2 yields the following:

Theorem 9.4. *Suppose that M is a closed orientable hyperbolic 3-manifold with volume less than that of the Weeks manifold. Then M can be obtained by a Dehn filling on one of the 10 one-cusped manifolds listed in Theorem 9.2.*

Clearly we can use this result to identify the minimum-volume closed hyperbolic 3-manifold, using [FKP] as before to limit the number of Dehn fillings that need to be considered. The results of such an analysis will also be presented in [MM].

Of future interest is the problem of strengthening the bound of 2.848 in Theorem 1.1, and thus improving the classification of both closed and cusped low-volume hyperbolic 3-manifolds. The SnapPea census of cusped manifolds suggests that the number 2.848 should be far higher: the smallest known manifold which does not possess an internal Mom-2 structure is $m069$, which has a volume greater than 3.4. It would not be unreasonable to attempt to prove a stronger version of Theorem 1.1 which would apply to all one-cusped manifolds with volume less than or equal to 3.7; with such a result it would be possible to determine the first infinite string of volumes of one-cusped hyperbolic manifolds limiting on the volume of the Whitehead link complement. Similarly it should be possible to determine the first infinite string of volumes of closed manifolds limiting on the figure-eight knot complement.

Finally we expect that Mom-technology can be applied directly to closed manifolds, and not just indirectly via the use of Lemma 9.3. The obstacles to this are primarily geometrical rather than conceptual. While the definition of a geometrical Mom- n structure can easily be extended to closed manifolds (by considering triples of geodesics rather than triples of horoballs), difficulties arise when we consider the lessvol and overlapArea functions defined in Lemmas 3.8 and 3.10. In the cusped case the lessvol function uses the fact that the equidistant surface between two

horoballs is a plane. In the closed case the equidistant surface between neighbourhoods of two geodesics is a more complicated surface. Similarly the overlapArea function uses the fact that the shadow of one horoball on the surface of another is a Euclidean circle; in the closed case, the shadow of one tubular neighbourhood on the surface of another is typically a not-quite-elliptical region whose shape depends on the both the distance and the angle between the corresponding core geodesics. Developing a Mom-based theory for closed manifold will require a more sophisticated analysis of these two geometrical problems.

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DEPARTMENT OF MATHEMATICS, PRINCETON UNIVERSITY, PRINCETON, NJ

DEPARTMENT OF MATHEMATICS, BOSTON COLLEGE, CHESTNUT HILL, MA

DEPARTMENT OF MATHEMATICS AND STATISTICS, UNIVERSITY OF MELBOURNE, MELBOURNE, AUSTRALIA

MOM TECHNOLOGY AND VOLUMES OF HYPERBOLIC 3-MANIFOLDS

DAVID GABAI, ROBERT MEYERHOFF, AND PETER MILLEY

0. INTRODUCTION

This paper is the first in a series whose goal is to understand the structure of low-volume complete orientable hyperbolic 3-manifolds. Here we introduce *Mom technology* and enumerate the hyperbolic Mom- n manifolds for $n \leq 4$. Our long-term goal is to show that all low-volume closed and cusped hyperbolic 3-manifolds are obtained by filling a hyperbolic Mom- n manifold, $n \leq 4$ and to enumerate the low-volume manifolds obtained by filling such a Mom- n .

William Thurston has long promoted the idea that volume is a good measure of the complexity of a hyperbolic 3-manifold (see, for example, [Th1] page 6.48). Among known low-volume manifolds, Jeff Weeks ([We]) and independently Sergei Matveev and Anatoly Fomenko ([MF]) have observed that there is a close connection between the volume of closed hyperbolic 3-manifolds and combinatorial complexity. One goal of this project is to explain this phenomenon, which is summarized by the following:

Hyperbolic Complexity Conjecture 0.1. (*Thurston, Weeks, Matveev-Fomenko*)
The complete low-volume hyperbolic 3-manifolds can be obtained by filling cusped hyperbolic 3-manifolds of small topological complexity.

Remark 0.2. Part of the challenge of this conjecture is to clarify the undefined adjectives *low* and *small*. In the late 1970's, Troels Jorgensen proved that for any positive constant C there is a finite collection of cusped hyperbolic 3-manifolds from which all complete hyperbolic 3-manifolds of volume less than or equal to C can be obtained by Dehn filling. Our long-term goal stated above would constitute a concrete and satisfying realization of Jorgensen's Theorem for "low" values of C .

A special case of the Hyperbolic Complexity Conjecture is the long-standing

Smallest Hyperbolic Manifold Conjecture 0.3. The Weeks Manifold M_W , obtained by $(5, 1)$, $(5, 2)$ surgery on the two components of the Whitehead Link, is the unique oriented hyperbolic 3-manifold of minimum volume.

Note that the volume of M_W is $0.942\dots$

All manifolds in this paper will be orientable and all hyperbolic structures are complete. We call a compact manifold *hyperbolic* if its interior supports a complete hyperbolic structure of finite volume.

Definition 0.4. A *Mom- n structure* (M, T, Δ) consists of a compact 3-manifold M whose boundary is a union of tori, a preferred boundary component T , and a handle decomposition Δ of the following type. Starting from $T \times I$, n 1-handles

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and n 2-handles are attached to $T \times 1$ such that each 2-handle goes over exactly three 1-handles, counted with multiplicity. Furthermore, each 1-handle encounters at least two 2-handles, counted with multiplicity. We say that M is a *Mom- n* if it possesses a Mom- n structure (M, T, Δ) .

Remarks 0.5. On a Mom- n , the handle decomposition Δ deformation retracts to an almost simple 2-complex which has $2n$ true vertices, in the sense of Matveev [Mv2]. Therefore Mom- n manifolds are a subset of those with Matveev complexity at most $2n$.

Here is the fundamental idea at the foundation of our project. Given a complete finite-volume hyperbolic 3-manifold N , start with either a slightly shrunken maximal horotorus neighborhood V of a cusp or slightly shrunken maximal tube V about a geodesic. After expanding V in the normal direction, it eventually encounters itself, thereby creating a 1-handle. Subsequent expansions give rise to the creation of 1, 2, and 3-handles. In the presence of low volume we expect that V will rapidly encounter 1 and 2-handles and ∂V together with a subset of these handles (perhaps somewhat perturbed to allow for the “valence-3 2-handle condition”) will create a Mom- n manifold M , for some $n \leq 4$. Furthermore, the complement of M will consist of cusp neighborhoods and tubular neighborhoods of geodesics. In practice, the handle structure may arise in a somewhat different manner; e.g., as a sub-complex of the dual triangulation of the Ford Domain (see [GMM3]).

The papers [GM] and [GMM] can be viewed as steps in this direction when V is a tubular neighborhood about a geodesic γ . Indeed, [GM] gives a lower bound on $\text{Vol}(N)$ in terms of the tube radius of γ and [GMM] gives a lower bound in terms of the first two ortholengths, or equivalently the radii of the expanding V as it encounters its first and second 1-handles.

Definition 0.6. If $i : M \rightarrow N$ is an embedding, then we say that the embedding is *elementary* if $i_*\pi_1(M)$ is abelian, and *non-elementary* otherwise.

In §1 we give the basic definitions regarding Mom- n manifolds embedded in hyperbolic 3-manifolds and state for later use some standard results about hyperbolic 3-manifolds and embedded tori in such. The end result of §2 - §4 is that if $n \leq 4$, then given a non-elementary Mom- n in a hyperbolic manifold one can find a non-elementary hyperbolic Mom- k , where $k \leq n$. Weaker results are given for general values of n . In §5 we enumerate the hyperbolic Mom- n manifolds for $n \leq 4$; Theorem 5.1 and Conjecture 5.2 together imply the following:

Theorem 0.7. *There are 3 hyperbolic Mom-2 manifolds, 21 hyperbolic Mom-3 manifolds (including the 3 hyperbolic Mom-2’s, which are also Mom-3’s).*

Conjecture 0.8. *There are 138 hyperbolic Mom-4 manifolds (including the hyperbolic Mom-2’s and Mom-3’s, which are also Mom-4’s).*

In §6 we show that any non-elementary embedding of a hyperbolic Mom- n manifold M , $n \leq 4$, into a compact hyperbolic manifold N gives rise to an *internal Mom- n structure* on N , i.e. every component of ∂M either splits off a cusp of N or bounds a solid torus in N .

In §7 we give examples of internal Mom-2 structures on cusped hyperbolic 3-manifolds, including in particular a detailed exposition of one of our key motivating examples.

In future papers we will use the Mom Technology to directly address the Hyperbolic Complexity Conjecture, and the Smallest Hyperbolic Manifold Conjecture. Indeed, in [GMM3] we will identify all 1-cusped hyperbolic 3-manifolds with volume less than 2.7 by showing that all such manifolds possess internal Mom- n structures with $n \leq 3$. This result, in combination with work of Agol-Dunfield (see [AST]), gives a lower bound of 0.86 for the volume of an orientable hyperbolic 3-manifold (but see the note below). The Agol-Dunfield result is an improvement of an earlier result of Agol which provides a tool for controlling the volume of a hyperbolic 3-manifold in terms of the volume of an appropriate cusped hyperbolic 3-manifold from which it is obtained via Dehn filling; the improved version utilizes recent work of Perelman.

This leads to three very promising directions towards the Smallest Hyperbolic Manifold Conjecture. Either, improve the technology of [GMM3] to identify the 1-cusped manifolds of volume less than 2.852, and then apply Agol-Dunfield. Or, extend the tube radius results of [GMT] from $\log(3)/2$ to 0.566 (the Agol-Dunfield volume bound involves the radius of a solid tube around a short geodesic in the original closed hyperbolic 3-manifold). Or, extend the method of [GMM3] to the closed case, thereby providing an essentially self-contained proof of the Smallest Hyperbolic Manifold Conjecture.

Note that all three approaches require an analysis of volumes of hyperbolic 3-manifolds obtained by Dehn filling a Mom-2 or Mom-3 manifold. These Dehn filling spaces have been extensively studied by J. Weeks and others, and it is highly likely that all low-volume manifolds in these Dehn filling spaces have been identified. However, some work will need to be done to bring these studies up to a suitable level of rigor.

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1. BASIC DEFINITIONS AND LEMMAS

Definition 1.1. Let M be a compact connected 3-manifold M with $B \subset \partial M$ a compact surface which may be either disconnected or empty. A *handle structure* Δ on (M, B) is the structure obtained by starting with $B \times I$ adding a finite union of 0-handles, then attaching finitely many 1 and 2-handles to $B \times 1$ and the 0-handles. We call $B \times I$ (resp. $B \times I \cup 0$ -handles) the *base* (resp. *extended base*) and say that the handle structure is based on B . The *valence* of a 1-handle is the number of times, counted with multiplicity, the various 2-handles run over it and the *valence* of a 2-handle is the number of 1-handles, counted with multiplicity, it runs over.

Following the terminology of Schubert [Sch] and Matveev [Mv1] we call the 0-handles, 1-handles and 2-handles *balls*, *beams* and *plates* respectively. We call *islands* (resp. *bridges*) the intersection of the extended base with beams (resp. plates) and the closed complement of the islands and bridges in $B \times 1 \cup \partial(0$ -handles) are the *lakes*. We say that Δ is *full* if each lake is a disc. If $B = \emptyset$, then we say that Δ is a *classical handle structure*.

Let M be a compact 3-manifold with ∂M a union of tori and let T be a component of ∂M . We say that (M, T, Δ) is a *weak Mom- n* if Δ is a handle structure based on T without 0-handles or 3-handles and has an equal number of 1 and 2-handles, such that each 1-handle is of valence ≥ 2 and each 2-handle is of valence

2 or 3. Furthermore, there are exactly n 2-handles of valence 3. A weak Mom- n with no valence-2 2-handles is a *Mom- n* . A weak Mom- n is *strictly weak* if there exists a valence-2 2-handle.

The following is a well-known existence result stated in our language.

Proposition 1.2. *A compact 3-manifold M has a weak Mom-structure if and only if ∂M is a union of at least two tori.*

Proof. If M has a weak Mom- n structure, then by definition all of its boundary components are tori and there is at least one such boundary component. Further, because there are no 3-handles in Δ , there must be another (torus) boundary component.

The converse is not much more difficult. In fact, if M has at least two boundary components, then it is standard to create it by starting with a thickened boundary component and adding 1 and 2-handles where the 1 and 2-handles are of valence ≥ 2 . By subdividing the 2-handles with 1-handles we satisfy the condition that the 2-handles have valence ≤ 3 . Since $\chi(M) = 0$, there are an equal number of 1 and 2-handles. \square

Definition 1.3. Call a torus that bounds a solid torus a *tube* and call a torus bounding a tube with knotted hole a *convolutube*. Recall that a *tube with knotted hole* is a $B^3 - \mathring{N}(\gamma)$, where γ is a knotted proper arc.

The following standard result follows from the loop theorem (see for example [Jac]).

Lemma 1.4. *If S is a torus in an irreducible 3-manifold N , then either S is incompressible or S is a tube or a convolutube. If $S \subset \mathring{N}$, ∂N is incompressible and S is compressible, and there exists an embedded essential annulus connecting S to a component of ∂N , then S is a tube.* \square

Proposition 1.5. *If M is a non-elementary compact, connected 3-manifold embedded in the compact hyperbolic 3-manifold N and ∂M is a union of tori, then, up to isotopy, N is obtained from M by first filling a subset of the components of ∂M by solid tori to obtain the manifold M_1 , then compressing a subset of the components of ∂M_1 to obtain the manifold M_2 , then attaching 3-balls to the 2-sphere components of ∂M_2 to obtain M_3 . Furthermore all of these operations can be performed within N .*

Proof. The components of ∂M that bound solid tori in N are exactly those boundary components which compress to the non- M side. Fill in all such tori to obtain the manifold M_1 . If P is a component of ∂M_1 which is not boundary parallel in N , then P is compressible in N and hence is a convolutube. These convolutubes can be isotoped to lie in pairwise disjoint 3-balls in N . Therefore we can compress all the compressible components of ∂M_1 (to obtain M_2) and cap the resulting 2-spheres with 3-cells to obtain M_3 which is isotopic to N .

Since M_3 must have all boundary components boundary parallel in N and M_3 is non-elementary, the result follows. \square

Corollary 1.6. *Let $M \subset \mathring{N}$ be a connected compact non-elementary submanifold in the compact hyperbolic 3-manifold N . If ∂M is a union of tori, then each component of ∂N is parallel to a component of ∂M via a parallelism disjoint from M .* \square

The following result is due to Kerckhoff (see [Koj]).

Lemma 1.7. *If γ is a simple closed geodesic in the complete, finite-volume hyperbolic 3-manifold N , then $N - \gamma$ has a complete finite-volume hyperbolic structure.* \square

Lemma 1.8. *Let M be a compact submanifold of the compact hyperbolic 3-manifold N .*

i) If $M_1 = M - \overset{\circ}{X}$, where $X \subset N$ is either a solid torus or 3-ball with $\partial X \subset M$, or M_1 is obtained by deleting a 2-handle or more generally deleting an open regular neighborhood of a properly embedded arc from M , then the inclusion $M \rightarrow N$ is a non-elementary embedding if and only if the inclusion $M_1 \rightarrow N$ is a non-elementary embedding.

ii) Suppose M is non-elementary, ∂M is a union of tori and A is an essential annulus in M . Split M along A ; then some component M_1 of the resulting manifold is non-elementary.

iii) Suppose $F \subset M$ is an embedded torus essential in M , and $M \subset N$ is non-elementary. Split M along F ; then exactly one component of the resulting manifold is non-elementary.

Proof. The conclusion is immediate in case (i) because both M_1 and M have the same π_1 -image.

Under the hypotheses of (ii) the boundary of each component of M split along A is also a union of tori. We consider the case where the split manifold connected, for the general case is similar. Since all tori in N separate, M is obtained from M_1 by attaching a thickened annulus A to a boundary parallel torus, a tube or a convolutube. In the first case M and M_1 have the same π_1 -image since N is anannular. If A is attached to the outside of a convolutube, then $\pi_1(M) \subset \mathbb{Z}$. If A is attached to the outside of a tube, then M can be enlarged to a Seifert fibered space in N and hence is elementary.

To prove (iii) note that F is either boundary parallel or is a tube or convolutube. In each case the result follows immediately. \square

Definition 1.9. Let N be a compact hyperbolic 3-manifold. An *internal Mom- n structure on N* consists of a non-elementary embedding $f : M \rightarrow N$ where (M, T, Δ) is a Mom- n and each component of ∂M is either boundary parallel in N or bounds a solid torus in N . We will sometimes suppress mention of the embedding and simply say that (M, T, Δ) is an internal Mom- n structure on N . In the natural way we define the notion of *weak internal Mom- n structure on N* .

Lemma 1.10. *A non-elementary embedding of the Mom- n manifold M into the compact hyperbolic 3-manifold N will fail to give an internal Mom- n structure on N if and only if some component of ∂M maps to a convolutube. In that case, a reembedding of M , supported in a neighborhood of the convolutubes gives rise to an internal Mom- n structure on N .* \square

Definition 1.11. A *general based Mom- n (M, B, Δ)* consists of a compact manifold M with ∂M a union of tori, $B \subset \partial M$ a compact codimension-0 submanifold of ∂M that is π_1 -injective in ∂M , and Δ a handle structure based on B without 0-handles such that every 1-handle is of valence- ≥ 2 , every 2-handle is of valence-3 and there are exactly n of each of them. A *weak general based Mom- n* is as above with Δ

perhaps having $k \geq 0$ extra valence-2 2-handles in addition to the n valence-3 2-handles and hence has $k + n$ 1-handles.

A *general based internal Mom- n structure on N* consists of a non-elementary embedding $f : M \rightarrow N$, where N is a compact hyperbolic 3-manifold and (M, B, Δ) is a general based Mom- n structure. Along similar lines we have the notion of *weak general based internal Mom- n structure on N* .

Notation 1.12. If X is a space, then let $|X|$ denote the number of components of X , $\overset{\circ}{X}$ denote the interior of X and $N(X)$ denote a regular neighborhood of X . If σ is a 2-handle, then let $\delta\sigma$ denote the lateral boundary, i.e. the closure of that part of $\partial\sigma$ which does not lie in lower index handles. If b is a bridge which lies in the 2-handle σ , then define $\delta b = b \cap \delta\sigma$.

2. HANDLE STRUCTURES AND NORMAL SURFACES

We slightly modify Haken's [Ha] theory of surfaces in handlebodies to our setting. We closely parallel the excellent exposition given by Matveev in [Mv1].

Definition 2.1. Let Δ be a handle structure on M based on $B \subset \partial M$. A compact surface $F \subset M$ is called *normal* if

- (1) F intersects each plate $D^2 \times I$ in parallel copies of the form $D^2 \times \text{pt}$.
- (2) Each component of the intersection of F with a beam $D^2 \times I$ is of the form $\alpha \times I$, where α is a proper arc whose endpoints are disjoint from $\delta(\text{bridges})$. Furthermore, each component of $D^2 \times 0 - \alpha$ intersects $\delta(\text{bridges})$ in at least two points.
- (3) Each component U of $F \cap B \times [0, 1] \cup 0$ -handles is π_1 -injective in $B \times [0, 1] \cup 0$ -handles. If $U \cap B \times 0 \neq \emptyset$, then U is a product disc or annulus, i.e. The inclusion $(U, U \cap (B \times 0), U \cap (B \times 1), U \cap (\partial B \times I)) \rightarrow (B \times I, B \times 0, B \times 1, \partial B \times I)$, can be relatively isotoped to a vertical embedding.
- (4) If U is a component of $F \cap B \times I$ with $F \cap B \times 0 \neq \emptyset$, then U is an essential vertical disc or annulus, i.e. $U = \alpha \times I$ where $\alpha \times 0$ is either an essential simple closed curve or essential proper arc in $B \times 0$.

Remark 2.2. i) For F closed, the second condition can be restated by requiring that α intersect distinct components of $D^2 \times 0 \cap (\text{bridges})$. When $\partial F \neq \emptyset$, the second condition implies that F is locally efficient in that it neither can be locally boundary compressed nor can its weight be reduced via an isotopy supported in the union of a 2-handle and its neighboring 1-handles.

ii) Note that ∂F lies in the union of the beams, lakes and $B \times 0$.

Lemma 2.3. (*Haken*) *If F is a compact, incompressible, boundary-incompressible surface in a compact irreducible 3-manifold, then F is isotopic to a normal surface.* \square

Definition 2.4. Let Δ be a handle structure on M based on $B \subset \partial M$. The *valence* $v(b)$ of a beam (resp. plate) is the number of plates (resp. beams) that attach to it, counted with multiplicity. Define the *complexity* $C(\Delta)$ to be $(\rho_1(\Delta), |\Delta^1|)$ lexicographically ordered, where $\rho_1(\Delta) = \sum_{\text{beams } b} \max(v(b) - 2, 0)$ and $|\Delta^1|$ is the number of 1-handles. In particular we have

$$\rho_1\text{-formula: } \rho_1(\Delta) = \sum_{\text{2-handles } \sigma} \text{valence}(\sigma) - 2|\Delta^1| + |\text{valence-1 1-handles}| + 2|\text{valence-0 1-handles}|$$

Lemma 2.5. (*Matveev*) *Let Δ be a handle structure on M based on B , $F \subset M$ a closed normal surface and let M' be M split along F . If each component of $M' \cap B \times [0, 1]$ disjoint from $B \times 0$ is a 3-ball, then M' has a handle structure Δ' based on B with $\rho_1(\Delta') = \rho_1(\Delta)$.*

Proof. This follows almost exactly as in §3 and §4 of [Mv1]: M' naturally inherits a handle structure Δ_1 from Δ as follows. The surface F splits $B \times I$ into various submanifolds one of which is homeomorphic to $B \times [0, 1]$ with $B \times 0 = B$. All of the other submanifolds which lie in M' are 3-balls. This new $B \times [0, 1]$ becomes the base and the 3-balls become 0-handles. The various 1 and 2-handles are split by F into 1 and 2-handles and as in [Mv1], $\rho_1(\Delta_1) = \rho_1(\Delta)$. \square

Lemma 2.6. *Given the handle structure Δ on (M, B) , if some 1-handle is valence-1, then there exists another structure Δ_1 on (M, B) with $C(\Delta_1) < C(\Delta)$.* \square

Lemma 2.7. *If (M, T, Δ) is a Mom- n , then $C(\Delta) = (n, n)$.* \square

Lemma 2.8. *Let Δ be a handle structure on (M, B) , $F \subset M$ a connected separating normal surface and M_1 be the component of $M - \overset{\circ}{N}(F)$ which does not contain B . If each component of $F \cap B \times [0, 1]$ is a disc, then M_1 has a classical handle structure Δ_1 with $\rho_1(\Delta_1) \leq \rho_1(\Delta)$.*

Proof. This follows as in the proof of Lemma 2.5 after noting that each component of $M_1 \cap B \times [0, 1]$ is a 3-ball and these 3-balls correspond to the 0-handles of the induced handle structure on M_1 . \square

Lemma 2.9. *If ∂M is a union of tori, and Δ is a handle structure on (M, T) with T a component of ∂M , then there exists a weak Mom- n (M, T, Δ_1) with $n \leq \rho_1(\Delta_1)$.*

Proof. First apply Lemma 2.6, then add 1-handles to subdivide the valence- k , $k \geq 4$, 2-handles into valence-3 2-handles. \square

Definition 2.10. If $B \neq \emptyset$ is a compact submanifold of ∂M , then define $\text{rank}_{\rho_1}(M, B)$ to be the least n such that there exists a handle decomposition Δ on (M, B) with $\rho_1(\Delta) = n$.

Problem 2.11. *Is there an example of a compact hyperbolic 3-manifold N with T a component of ∂N and A an essential annulus in T such that $\text{rank}_{\rho_1}(N, A) < \text{rank}_{\rho_1}(N, T)$?*

3. ESTIMATES FOR THE REDUCTION OF ρ_1 UNDER SPLITTING

The main result of these next two sections is Theorem 4.1 which shows that if a hyperbolic 3-manifold N has an internal Mom- n structure (M, T, Δ) with Δ full and $n \leq 4$, then it has an internal Mom- k structure (M_1, T_1, Δ_1) where $k \leq n$, Δ_1 is full, and M_1 is hyperbolic. If $n > 4$, we obtain the similar conclusion except that “full” is replaced by “general based” and hence T_1 can be a union of tori and annuli.

As far as we know, transforming a structure based on an annulus lying in T_1 to one based on the whole torus T_1 may require an increase in ρ_1 . This issue is responsible for many of the technicalities of this section and the next. See Problem 2.11.

Lemma 3.1. *Let $f : M \rightarrow N$ be a non-elementary embedding of a compact connected manifold into a compact irreducible 3-manifold. Suppose ∂M is a union of tori, T is a component of ∂M and that Δ is a handlebody structure on (M, T) without 0-handles such that each 2-handle is of valence ≥ 3 . Then there exists a non-elementary embedding $g : M' \rightarrow N$ with $\partial M'$ a union of tori and a handle structure Δ' on (M', T') with $T' = T$ such that*

$$\rho_1(\Delta') + 2|\text{valence-1 1-handles of } \Delta| + 3|\text{valence-0 1-handles of } \Delta| \leq \rho_1(\Delta).$$

If instead each 2-handle of Δ is of valence ≥ 2 then we have

$$\rho_1(\Delta') + |\text{valence-1 1-handles of } \Delta| + 2|\text{valence-0 1-handles of } \Delta| \leq \rho_1(\Delta).$$

Proof. Both assertions follow similarly by induction on the number of 1-handles of Δ . If η is a valence-1 1-handle, then cancelling η with its corresponding 2-handle creates a handle structure Δ_1 . The Lemma follows by applying the ρ_1 -formula to Δ_1 and induction. If η is a valence-0 1-handle, then the manifold M_1 obtained by deleting η is connected and $\chi(M_1) = 1$, hence has a 2-sphere boundary component S which bounds a 3-ball disjoint from $\overset{\circ}{M}_1$. Let (M_2, T, Δ_2) be obtained from (M, T, Δ) by deleting η as well as a 2-handle which faces S . M_2 is a non-elementary embedding by Lemma 1.8. Now apply the ρ_1 -formula and induction. \square

Lemma 3.2. *Let $f : M \rightarrow N$ be a non-elementary embedding of a manifold into a compact hyperbolic 3-manifold N , where ∂M is a union of tori. Suppose that M has a full handle structure Δ without 0-handles based on the component T of ∂M such that every 2-handle is of valence ≥ 3 . If either of the following are true then there exists a non-elementary embedding $f : M' \rightarrow N$ with handle structure Δ' on (M', T') such that $\rho_1(\Delta') + 2 \leq \rho_1(\Delta)$, $T' = T$ and $\partial M'$ is a union of tori:*

- i) There exists a valence-1 2-handle $\sigma \subset N - \overset{\circ}{M}$ that can be added to Δ .*
- ii) There exists a disc $D \subset \partial M$ such that ∂D is the union of two arcs $\alpha \cup \beta$, where β lies in a lake and α lies in a 2-handle λ . Furthermore, within $\lambda \cap \partial M$, α separates components of $\lambda \cap (1\text{-handles})$.*

Proof. By Lemma 3.1 we can assume that every 1-handle of Δ is of valence ≥ 2 . To prove (i) let Δ_1 be the handle structure on the manifold M_1 obtained by attaching σ to Δ along ∂M . Let η denote the 1-handle which σ meets. Let Δ_2 and M_2 be obtained by deleting a 2-handle $\lambda \neq \sigma$, which faces the resulting 2-sphere boundary component. Let Δ_3 be obtained by cancelling σ and η . Finally, in the usual way, reduce to a non-elementary M_4 with structure Δ_4 on (M_4, T) whose 1 and 2-handles are of valence ≥ 2 . Applying the ρ_1 -formula shows that $\rho_1(\Delta_4) + 2 \leq \rho_1(\Delta)$ unless, measured in Δ , $\text{valence}(\eta) = \text{valence}(\lambda) = 3$ and λ attaches to η at least twice. If λ attaches to η twice, then in the passage from Δ_1 to Δ_2 delete a 2-handle $\lambda_1 \neq \lambda$ which faces the 2-sphere. If λ attaches to η thrice, then either Δ is not full or η is the unique 1-handle of Δ . In the latter case $M_2 = T \times I$ which is elementary, a contradiction.

ii) Under these hypotheses we can attach a 2-handle $\sigma \subset N - \overset{\circ}{M}$ to Δ such that either $\text{valence}(\sigma)=1$ or $\text{valence}(\sigma) \leq \text{valence}(\lambda) + 2$ and λ faces the resulting 2-sphere boundary component. If $\text{valence}(\sigma) = 1$, then apply (i). Otherwise let Δ_1 be obtained by deleting λ and apply the ρ_1 -formula and if necessary Lemma 3.1. \square

Lemma 3.3. *Let M be a non-elementary embedding of a compact 3-manifold into the compact hyperbolic 3-manifold N with ∂M a union of tori, and let Δ be a handle structure of M based on $R \subset \partial M$. If there exists a valence ≥ 3 1-handle η of Δ which attaches to a 0-handle ζ , then there exists a non-elementary embedding $M' \rightarrow N$, and a handle structure Δ' based on $R' \subset \partial M'$ such that $\rho_1(\Delta') < \rho_1(\Delta)$. Here either $(M', R') = (M, R)$ or $M' = M - \overset{\circ}{V}$ and $R' = R \cup \partial V$ where V is an embedded solid torus in M .*

Proof. If η also attaches to either the base or a 0-handle distinct from ζ , then cancelling η with ζ gives rise to Δ' on (M, R) with $\rho_1(\Delta') < \rho_1(\Delta)$. If η attaches only to ζ , then let M' be obtained by deleting $\overset{\circ}{V}$, the open solid torus gotten by hollowing out ζ and η . Let $R' = R \cup \partial V$ and let Δ' be the induced structure on (M', R') . \square

Lemma 3.4. *Let N be a compact hyperbolic 3-manifold. If (M, T, Δ) is a full internal weak Mom- n structure on N and M is reducible then there exists (M_p, T_p, Δ_p) , a weak internal Mom- k structure on N such that $k + 2 = \rho_1(\Delta_p) + 2 \leq \rho_1(\Delta) = n$. If T does not lie in a 3-cell, then $T_p = T$.*

Proof. We first consider the case that no reducing sphere in M bounds a ball in N containing T . Let F be a least-weight normal reducing 2-sphere and note that $F \cap T \times I$ is a union of discs. Let M_0 and M'_0 be the components of M split along F . By Lemma 1.8 and the irreducibility of N exactly one of M_0, M'_0 is non-elementary. We let M_0 (resp. M'_0) denote the non-elementary (resp. elementary) component with Δ_0 (resp. Δ'_0) its induced structure. By hypothesis $T \subset M_0$.

We show that $\rho_1(\Delta'_0) > 0$. Let X denote the union of the islands and bridges of Δ and $Y' = X \cap M'_0$. If $\rho_1(\Delta'_0) = 0$, then each component A of Y' is an annulus. If some component A of Y' is disjoint from the lakes of Δ , then Y' would be disjoint from the lakes and so F would 2-fold cover a projective plane, which contradicts the fact that N is irreducible. Therefore each component of Y' has one boundary component in a lake and one component in $\overset{\circ}{X}$. Since Δ is full, this implies that F is a boundary parallel 2-sphere and contradicts the fact that ∂M is a union of tori.

If $Y = X \cap M_0$ then a similar argument shows that some component of Y is not an annulus and furthermore some 2-handle σ of Δ_0 faces F and attaches to a valence ≥ 3 1-handle. Delete σ from Δ_0 to obtain (M_1, T, Δ_1) with $\rho_1(\Delta_1) + 1 \leq \rho_1(\Delta_0)$. The standard simplifying moves as in Lemma 3.1 transform (M_1, T, Δ_1) to (M_2, T, Δ_2) , a weak Mom- k with $k = \rho_1(\Delta_2) \leq \rho_1(\Delta_1)$. Since $\rho_1(\Delta_0) + \rho_1(\Delta'_0) = \rho_1(\Delta) = n$ we have $k + 2 = \rho_1(\Delta_p) + 2 \leq \rho_1(\Delta) = n$. If M_2 is reducible, then split along an essential least-weight sphere F_2 which is normal with respect to Δ_2 . Retain the component which contains T and do the usual operations to obtain the weak Mom- k_2 (M_3, T, Δ_3) with $k_2 \leq k$ and M_3 non-elementary. By Haken finiteness, this procedure terminates in a finite number of steps, completing the proof in this case.

We now consider the case that T is compressible in M . Let F be a least-weight compressing disc for T . Note that $F \cap T \times I$ consists of discs and a single annulus. If M_1 is M split along F with induced handle structure Δ_1 and A is T split along $F \cap T$, then Δ_1 is based on A . Since M is obtained by attaching a 1-handle to a spherical component of ∂M_1 , it follows that M_1 is non-elementarily embedded

in N . Let M_2 be obtained by attaching 2-handles to ∂M_1 along the components of ∂A , then capping off the resulting 2-sphere which faces A with a 3-cell. The resulting manifold M_2 has two 2-sphere boundary components and the induced handle structure Δ_2 is classical. Since Δ is full some 2-handle of Δ_2 facing a 2-sphere of ∂M_2 attaches to a 1-handle of valence ≥ 3 . Delete this 2-handle to obtain the non-elementary M_3 with handle structure Δ_3 which satisfies $\rho_1(\Delta_3)+1 \leq \rho_1(\Delta)$. Delete another 2-handle to create M_4 and Δ_4 such that ∂M_4 is a union of tori. Now apply Lemma 3.3 to Δ_4 to create (M_5, T_5, Δ_5) so that $\rho_1(\Delta_5)+1 \leq \rho_1(\Delta_4)$. Finally cancel the extraneous 0-handles and valence-1 1-handles to create the desired weak internal Mom-k structure (M_6, T_6, Δ_6) .

Let F be a least-weight essential normal 2-sphere for Δ . Since we can assume that T is incompressible in M , $F \cap T \times I$ is a union of discs. If T does not lie in the ball bounded by $F \subset N$, then proceed as in the first part of the proof. Otherwise let M_0 and M'_0 be the components of M split along F , with M'_0 the component containing T . Since Δ is full and some component of $T \times I \cap M'_0$ is nonplanar, it follows that $\rho_1(\Delta'_0) \geq 1$ and hence $\rho_1(\Delta_0) + 1 \leq \rho_1(\Delta)$. To complete the proof apply Lemma 3.3 to Δ_0 , delete a 2-handle to create a manifold with torus boundary components and cancel low valence handles as in the previous paragraph. \square

Remark 3.5. If (M, T, Δ) is a weak internal Mom-n structure on the compact hyperbolic manifold N , and ∂M is compressible in M , then M is reducible. Therefore, if M is irreducible and Δ has an annular lake A that is homotopically inessential in $T \times I$, then the core of A bounds a disc in $T \times I$ which separates off a 3-cell in M . Absorbing this 3-cell into $T \times I$ simplifies Δ and transforms A into a disc lake.

From now on we will assume that if a homotopically inessential lake appears it is immediately removed via the above operation.

Definition 3.6. If Δ is a handle structure on M , then the *sheets* of Δ are the connected components of the space \mathcal{S} which is the union of the 2-handles and the valence-2 1-handles. So sheets are naturally thickened surfaces which are attached to a 3-manifold along their thickened boundaries. The valence of a sheet is the number of times the boundary runs over 1-handles counted with multiplicity.

Lemma 3.7. *Let N be a compact hyperbolic 3-manifold and $f : M \rightarrow N$ a non-elementary embedding where ∂M is a union of tori. Let Δ be a handle structure on M with no 0-handles based on a component T of ∂M such that the valence of each 2-handle is at least 3 and T does not lie in a 3-cell of N . If some sheet S of Δ is not a thickened disc then there exists a non-elementary $M' \subset N$ with handle structure Δ' based on a component T' of $\partial M'$ such that $\rho_1(\Delta') + 1 \leq \rho_1(\Delta)$. If equality must hold then S is a thickened annulus or Mobius band and if Δ is full then S is a thickened Mobius band and Δ' is full.*

Proof. By Lemma 3.1 and the proof of Lemma 3.4 we can assume that M is irreducible and each 1-handle of Δ has valence ≥ 2 . If M_1 denotes the manifold obtained by deleting the sheet S , then $\chi(M) = \chi(M_1) + \chi(S)$. Since $\chi(M) = 0$, if $\chi(S) < 0$, then ∂M_1 contains a 2-sphere. This implies that M is reducible. Note that if $\chi(S) = 0$, then M_1 is non-elementary.

Now assume that $\chi(S) = 0$. In this case S is either an annulus $\times I$ or a non-trivial I -bundle over a Mobius band. Since $\chi(S) = 0$, if S contains more than one 2-handle, then $\text{valence}(S) \geq 2$. If $\text{valence}(S) > 1$, then $\rho_1(\Delta_1) + 2 \leq \rho_1(\Delta)$ holds

where Δ_1 is the induced structure on M_1 . If S is a valence-1 annulus $\times I$, then Δ has an annular lake and $\rho_1(\Delta_1) + 1 = \rho_1(\Delta)$.

If S is a thickened Mobius band of valence-1, then $\rho_1(\Delta_1) + 1 = \rho_1(\Delta)$. If S attached to the component R of ∂M_1 and R had an annular lake, then R must be a tube with a compressing disc D whose boundary goes over a 1-handle η of Δ_1 exactly once. Let M_2 be obtained by attaching the 2-handle σ with core D to M_1 and Δ_2 the induced handle structure. Proceed as in the proof of Lemma 3.2 to show that exists a non-elementary embedding (M_3, T) with handle structure Δ_3 with $\rho_1(\Delta_3) + 2 \leq \rho_1(\Delta_1) + 1 \leq \rho_1(\Delta)$. \square

Lemma 3.8. *Let (M, T, Δ) be an internal Mom- n structure on the compact hyperbolic manifold N . Assume that every sheet of Δ is a disc and Δ is full. If there exists an embedded annulus A connecting the component S of $\partial M - T$ to T , then there exists a non-elementary embedding of a manifold M' into N such that $\partial M'$ is a union of tori and $\rho_1(\Delta') + 2 \leq \rho_1(\Delta)$. Here Δ' is a handle structure on M' based on T' . Either $T' = T$ or T' is the union of an essential annulus in T and possibly a component of $\partial M'$.*

Proof. By Lemma 3.4 we can assume that M is irreducible. Since all sheets are discs and letting $M_2 = M$, we obtain from Δ a full handle structure Δ_2 with no 0-handles based on the component $T_2 = T$ of ∂M_2 where each 1 and 2-handle of Δ_2 is of valence ≥ 3 . Finally $\rho_1(\Delta_2) \leq \rho_1(\Delta)$. We can assume that $A \subset M_2$ is a least-weight normal annulus connecting T_2 to the boundary component $S \neq T_2$. Since A is least weight, $A \cap T_2 \times I$ is a union of discs and a single annulus.

Let ∂A_0 (resp. ∂A_1) denote $A \cap T_2$ (resp. $A \cap S$). Our A has an induced handle structure Φ based on ∂A_0 as follows. The base consists of the annular component of $A \cap T_2 \times [0, 1]$, the 0-handles consist of the disc components of $A \cap T_2 \times [0, 1]$, the 1-handles (resp. 2-handles) consist of the intersections of A with the 1-handles (resp. 2-handles).

Let M_3 denote the manifold obtained by splitting M_2 along A and let Δ_3 denote the induced handle structure. As in §2, the ball components of $T_2 \times I$ split along A to become 0-handles of Δ_3 . The remaining component is $B' \times I$, where B' is T_2 split along ∂A_0 and hence Δ_3 is based on $B' \subset T_2$. By Lemma 1.8, M_3 is non-elementarily embedded in N .

If η is a 1-handle of Δ_2 and if $\{\eta_i\}$ denotes the 1-handles of Δ_3 which descended from η , then $\sum_i (\text{valence}(\eta_i) - 2) \leq \text{valence}(\eta) - 2$ with equality if and only if A_1 does not run over η . In fact, counting with multiplicity, if A_1 runs over the 1-handles of Δ_2 more than once then by operations as in Lemma 3.1 we can pass to M_4 and Δ_4 with $\rho_1(\Delta_4) + 2 \leq \rho_1(\Delta)$. We will now assume that A_1 ran over a unique 1-handle η and it did so with multiplicity one. This implies that $\rho_1(\Delta_3) + 1 = \rho_1(\Delta_2)$. Also A_1 is the union of two arcs α and β , where α lies in a 1-handle of Δ_2 and β lies in a lake. β lies either in the base of Φ or in a 0-handle v^* of Φ .

Give A a transverse orientation. Call a 0-handle v of Φ *plus* (resp. *minus*) if the transverse orientation of the disc $v \subset T_2 \times I$ points away from (resp. towards) T_2 . Each such disc v separates off, in $T_2 \times I$, a 3-ball v_B . Let v_D denote $v_B \cap T_2 \times 1$.

If β lies in a 0-handle v^* , then Lemma 3.3 applies to Δ_3 and the Lemma is proved. Indeed, since A is least weight, the disc v_D^* contains a bridge b in its interior and this bridge is not parallel to β . In other words, there is no embedded disc in a lake whose boundary is a concatenation of four arcs $\beta_1, \beta_2, \beta_3$ and β_4 such that $\beta_1 = \beta, \beta_3$ lies in ∂b , and β_2 and β_4 are arcs lying in islands. Otherwise, since A_1

runs over a unique 1-handle, this implies that Δ_2 had a valence-1 2-handle. If v_D^* is disjoint from the other 0-handles of Φ , then Lemma 3.3 applies to the 0-handle v_B^* of Δ_3 . Otherwise, v_B^* is split into balls by the various 0-handles of Φ and Lemma 3.3 applies to one of these balls.

From now on we assume that β lies in the base of Φ . Let $X \subset T_2 \times 1$ denote the union of the islands and bridges of Δ_2 . A similar but easier argument to the one given in the previous paragraph shows that for each 0-handle v of Φ either ∂v is boundary parallel in X or Lemma 3.3 applies. This implies that if $v \neq w$ are 0-handles of Φ , and $v_B \subset w_B$, then ∂v and ∂w are normally parallel in X . Furthermore, no 1-handle of Φ connects a plus 0-handle to a minus 0-handle of Φ . Also, if v^0 and v^1 are two 0-handles of Φ that are connected by a 1-handle, then $v_B^0 \cap v_B^1 = \emptyset$. Finally, there do not exist 0-handles w^0, w^1, \dots, w^n of Φ such that for $i = 1, 2, \dots, n-1$, w^i is connected to w^{i+1} by a 1-handle and $w_B^n \subset w_B^0$.

It follows that there exists a disc $E \subset A$ whose boundary is the union of two arcs ϕ and ψ where ϕ is a proper arc in a 2-handle of Φ and ψ lies in the $T \times 1 \cap (\text{base}(A))$. Furthermore $E \cap (0, 1\text{-handles } \Phi) \neq \emptyset$ and is connected. By the previous paragraph if v, w are 0-handles of Φ lying in E , then they are of the same parity and $v_B \cap w_B = \emptyset$.

Therefore, E can be normally isotoped, with respect to Δ_2 to a disc $G \subset \partial M$ such that ∂G is a union of two arcs, one lying in a lake and the other in a 2-handle. Now apply (ii) of Lemma 3.2 to Δ_2 . \square

Lemma 3.9. *Let (M, T, Δ) be a full internal Mom- n structure on the compact hyperbolic 3-manifold N such that every sheet is a disc. Suppose that there exists an essential embedded annulus A with $\partial A \cap T = \emptyset$. Then either there exists a full internal Mom- k structure (M', T', Δ') on N with $k < n$ or there exists a non-elementary embedding $M' \rightarrow N$ with handle structure (M', T', Δ') where $\partial M'$ is a union of tori, T' is a component of $\partial M'$, and $\rho_1(\Delta') + 2 \leq \rho_1(\Delta)$.*

Proof. Assume that M is irreducible. By Lemma 3.8 we can assume that no essential annulus connects T to a component of $\partial M - T$. Let A be a least-weight essential normal annulus with $A \cap T = \emptyset$. By the second sentence, $A \cap T \times I$ is a union of discs. Let M_1 be the non-elementary component of M split along A and let (M_1, T_1, Δ_1) be the induced handle structure. As in the previous proof, $\rho_1(\Delta_1) + 1 \leq \rho_1(\Delta)$. Indeed, since ∂A traverses at least two 1-handles of Δ , counted with multiplicity, the inequality will be strict unless A normally double covers a Mobius band. If M_1 is disjoint from T , then the result follows from Lemma 3.3. Assume now that $T \subset \partial M_1$ and Δ_1 is full.

Cancel the 0-handles with 1-handles to obtain (M_2, T_2, Δ_2) . If Δ_2 is not full, then some 1-handle of valence ≥ 3 was cancelled and hence ρ_1 is reduced. Next cancel valence-1 2-handles to obtain (M_3, T_3, Δ_3) . Again fullness is preserved or ρ_1 is reduced. If Δ_3 has a valence-2 2-handle σ connecting distinct 1-handles, then cancel one of these 1-handles with σ . If the 2-handle σ attaches to the same 1-handle α , then the core of $\sigma \cap \alpha$ can be viewed as an embedded annulus or Mobius band C in M_3 . Splitting M_3 along C we obtain (M_4, T_4, Δ_4) where T_4 is the newly created boundary component of M_4 and Δ_4 is the induced structure. Note that $\rho_1(\Delta_4) < \rho_1(\Delta_3)$ since Δ_3 is full. After finitely many such operations the lemma is proved. \square

Lemma 3.10. *Let (M, T, Δ) be a full internal Mom- n structure on the compact hyperbolic 3-manifold N such that every sheet is a disc. Suppose that there exists an*

essential embedded annulus A with $\partial A \subset T$. Then either there exists a full internal Mom- k structure (M', T', Δ') on N with $k < n$ or there exists a non-elementary embedding $M' \rightarrow N$ with handle structure (M', T', Δ') where $\partial M'$ is a union of tori, T' is a component of $\partial M'$, and $\rho_1(\Delta') + 2 \leq \rho_1(\Delta)$.

Proof. If M is reducible or there exists an essential annulus with some boundary component disjoint from T , then apply Lemma 3.4, Lemma 3.8 or Lemma 3.9. Let $(M, \partial M - T, \Sigma)$ be the dual handle structure. I.e. the 1-handles (resp. 2-handles) of Σ are in 1-1 correspondence with the 2-handles (resp. 1-handles) of Δ and for each handle σ , the core of σ is the cocore of the dual handle and vice versa. Note that $\rho_1(\Sigma) = \rho_1(\Delta)$ and each 1-handle of Σ is of valence 3, but Σ may have valence-2 2-handles.

Let A be an essential annulus, least weight with respect to Σ . Since each essential annulus has its entire boundary in T , $A \cap (\partial M - T) \times I$ is a union of discs. Let M_1 be the non-elementary component of M split along A with (M_1, R_1, Σ_1) the induced structure. As before $\rho_1(\Sigma_1) < \rho_1(\Sigma)$. Eliminate the 0-handles and low valence-1 and 2-handles of Σ_1 to obtain (M_2, R_2, Σ_2) and note that either Σ_2 is full or $\rho_1(\Sigma_2) + 2 \leq \rho_1(\Sigma) = \rho_1(\Delta)$. If Σ_2 is full, then (M_3, T_3, Δ_3) the handle structure dual to (M_2, R_2, Σ_2) is full. Let (M_4, T_4, Δ_4) be obtained by eliminating the valence-2 2-handles as in the previous proof. As before $\rho_1(\Delta_4) + 1 \leq \rho_1(\Delta)$ with equality holding only if Δ_4 is full. \square

Lemma 3.11. *Let (M, T, Δ) be a full internal Mom- n structure on the compact hyperbolic 3-manifold N . If M is not hyperbolic, then either there exists a full internal Mom- k structure (M', T', Δ') on N with $k < n$ and M' hyperbolic or there exists a general based internal Mom- l , $l + 2 \leq n$, structure (M', B', Δ') on N .*

Proof. By Lemma 3.4, 3.7, 3.8, 3.9 or 3.10 it suffices to consider the case that M is irreducible and anannular. By Thurston [Th2], if M is not hyperbolic, then it contains an essential torus. Let F be a least-weight essential torus. Since there are no essential annuli, $F \cap T \times I$ is a union of discs.

Let M_1 denote the component of M split along F which contains T and let M'_1 denote the other component. Let Δ_1 and Δ'_1 denote the induced handle structures. Note that Δ_1 is based on T , Δ'_1 is a classical structure and $\rho_1(\Delta_1) + \rho_1(\Delta'_1) = \rho_1(\Delta)$. By Lemma 1.8 one of M_1 or M'_1 is non-elementary. Let $X \subset T \times I$ be the union of the islands and bridges of Δ , $Y = M'_1 \cap X$ and $Z = M_1 \cap X$.

If M_1 is elementary, $\rho_1(\Delta_1) \geq 1$ since Z is nonplanar. In the usual way obtain a handle structure Σ on M'_1 such that $\rho_1(\Sigma) \leq \rho_1(\Delta')$, Σ has exactly one 0-handle and each 1-handle is of valence ≥ 2 . Since M'_1 is non-elementary, some 1-handle is of valence ≥ 3 . To complete the proof apply Lemma 3.3.

From now on assume that M_1 is non-elementary. We first show that $\rho_1(\Delta'_1) \geq 1$. If $\rho_1(\Delta'_1) = 0$, then Y is a union of annuli, each component of which intersects ∂X in ≤ 1 circle, since Δ is full. If $Y \cap \partial X \neq \emptyset$, then each component must have this property and hence F is boundary parallel in M . If $Y \cap \partial X = \emptyset$, then M'_1 is an I-bundle over a Klein bottle, which is impossible in an orientable hyperbolic 3-manifold.

In the usual way pass from (M_1, T_1, Δ_1) to a Mom structure (M_2, T_2, Δ_2) with $\rho_1(\Delta_2) \leq \rho_1(\Delta_1)$. Since M is anannular, Δ_2 has no annular lakes. If some component of ∂M_2 is a convolutube, then reimbed M_2 in N to get a full internal Mom- k structure where $k < n$. \square

4. FROM MOM- n TO HYPERBOLIC MOM- n

The following is the main result of this section:

Theorem 4.1. *If (M, T, Δ) is a full internal Mom- n structure on the compact hyperbolic 3-manifold N and $n \leq 4$, then there exists a full internal Mom- k structure (M', T', Δ') on N where M' is hyperbolic and $k \leq n$.*

For general n , either there exists an internal Mom- k structure (M', T', Δ') in N with $k \leq n$ and M' hyperbolic or there exists a general based internal Mom- k structure (M', B', Δ') on N such that M' is hyperbolic and each component of B' is either a component of $\partial M'$ or an annulus which is essential in a component of ∂M . If $M \neq M'$, then $k + 2 \leq n$.

We first prove some preliminary lemmas about complexity (see Definition 2.4).

Lemma 4.2. *(Clean-Up Lemma) Let M be a compact 3-manifold with ∂M a union of tori and N a compact hyperbolic 3-manifold. If $M \rightarrow N$ is a non-elementary embedding and Δ is a handle structure on (M, B) , then there exists (M', B', Δ') a weak general based internal Mom- n structure on N with $C(\Delta') \leq C(\Delta)$.*

If M is hyperbolic, then either $\rho_1(\Delta') < \rho_1(\Delta)$ or $M' = M$.

Proof. Proof by induction on $C(\Delta)$. By the usual cancellation operations pass to a handle structure Δ_2 on (M, B) without 0-handles or 1 and 2-handles of valence 1. If Δ has a valence-0 1-handle, then delete it and proceed as in Lemma 3.1 to obtain Δ_3 on (M_3, B_3) with M_3 non-elementary and ∂M_3 a union of tori. All of these operations reduce $C(\Delta)$.

If Δ_2 has a valence-0 2-handle σ , then compress M_3 along a disc D which passes once through σ and intersects $B \times I$ in an annulus. Let Δ_4 be the resulting handle structure on (M_4, B_4) where M_4 is the non-elementary component of M_3 split along D and B_4 is $B_3 \cap M_4$. This splits σ into two 2-handle components, at least one of which lies in Δ_4 . Absorb such 2-handles into $B_4 \times I$. This creates (M_5, B_5, Δ_5) where $M_5 = M_4$ and B_5 is obtained by attaching one or two discs to ∂B_4 . If some component of B_5 is a 2-sphere S , then create (M_6, B_6, Δ_6) by filling in S with a 3-cell E and identifying $S \times I \cup E$ as a 0-handle of Δ_6 . Now cancel with a 1-handle to obtain (M_7, B_7, Δ_7) with $C(\Delta_7) < C(\Delta_3)$. If one or two components of B_5 are discs and there exists a non-simply connected component of B_5 , then the ball components of $B_5 \times I$ are now re-identified as 0-handles and cancelled with 1-handles to obtain (M_7, B_7, Δ_7) . If all the components of B_5 are discs, then transforming the components of $B_5 \times I$ to 0-handles produces a classical handle structure $\Delta_{5.5}$ on M_5 . Now apply the proof of Lemma 3.3 to create (M_6, B_6, Δ_6) with $C(\Delta_6)$ reduced. If some component of ∂M_6 is a 2-sphere, then delete a 2-handle facing it to obtain (M_7, B_7, Δ_7) .

By repeatedly applying the above operations we can assume that our Δ_7 has no 0-handles, and each 1-handle and 2-handle is of valence ≥ 2 . Furthermore, no component of B_7 is a disc. If some component G of B_7 is not π_1 -injective in ∂M_7 then attach a 2-handle to a component of $\partial G \times I$ whose restriction to ∂M_7 bounds a disc, to create (M_8, B_8, Δ_8) . Delete a 2-handle which faces the resulting 2-sphere boundary component and simplify as in the previous paragraphs to create (M_9, B_9, Δ_9) with $C(\Delta_9) < C(\Delta_7)$. We can now assume that each component of B_9 is essential in ∂M_9 , Δ_9 is 0-handle free and all 1 and 2-handles are valence ≥ 2 .

If ∂M_9 contains convolutubes, then reimbed M_9 in N to eliminate them. If M is not hyperbolic, then the resulting $(M_{10}, B_{10}, \Delta_{10})$ satisfies the conclusions of our clean up lemma.

If M is hyperbolic, then the proof also follows as above. However, note that ∂M is incompressible so in the above process valence-0 1-handles or lakes compressible in $B \times I$ can be eliminated without changing M and with reducing complexity. The topology of M may change if we delete a 2-handle or apply Lemma 3.3; however, if such operations must be done, they can be done in such a manner that reduces ρ_1 . \square

Lemma 4.3. *If (M, B, Δ) is a strictly weak general based Mom- n structure with M hyperbolic, then there exists a general based Mom- k structure (M, B', Δ') with $C(\Delta') < C(\Delta)$.*

Proof. Proof by induction on $C(\Delta)$. If σ is a valence-2 2-handle which goes over distinct 1-handles, then cancelling a 1-handle with σ creates (M, B, Δ_1) with $C(\Delta_1) < C(\Delta)$.

We now assume that no valence-2 2-handle σ goes over distinct 1-handles. Suppose that σ goes over the same 1-handle α twice. Then $\sigma \cup \alpha$ can be viewed as an embedded annulus or Mobius band A with boundary on B . Since M is hyperbolic and orientable, A is either a boundary parallel annulus or a compressible annulus.

If A is boundary parallel, then it together with an annulus on T cobound a solid torus $V \subset B$ such that A wraps longitudinally around V exactly once. Melting V into $B \times [0, 1]$ eliminates both α and σ , together with all the 2-handles and 1-handles inside of V . The resulting manifold (M, B_1, Δ_1) is a weak provisional Mom- k with $k \leq n$ whose induced handle structure Δ_1 satisfies $C(\Delta_1) < C(\Delta)$.

If A is not boundary parallel, then M is reducible, a contradiction. \square

Lemma 4.4. *If (M, B, Δ) is a general based Mom- n and M is hyperbolic, then $n \geq 2$.*

Proof. This follows by direct calculation. \square

Lemma 4.5. *If the compact hyperbolic 3-manifold N has a general based internal Mom-2 structure (M, B, Δ) , then it has a full internal Mom-2 structure.*

Proof. By splitting along annuli in $B \times I$ we can assume that every non-peripheral lake of B is a disc. If B is the union of two tori T_1 and T_2 , then each 1-handle must connect T_1 to T_2 . This implies that each 2-handle is of even valence which is a contradiction. If B consists of a single torus, then (M, B, Δ) is a full Mom-2 structure. We finally assume that B contains an annulus. We only discuss the case that B is connected; the case where B is either the union of an annulus and torus or two annuli is similar and easier.

First suppose that Δ has a single sheet of valence four. Let η denote the valence-4 1-handle of Δ . There exists an essential compressing disc D for $B \times I$ which cuts across the bridges in at most three components. View $N(D)$ as a 1-handle and $B \times I - \overset{\circ}{N}(D)$ as a 0-handle to obtain a classical handle structure with two 1-handles respectively of valence 4 and ≤ 3 . Now as in Lemma 3.3 hollow out the 0-handle and η to get a non-elementary M_1 with handle structure Δ_1 based on a torus with a single 1-handle of valence ≤ 3 . Therefore we obtain a Mom-1 structure on a hyperbolic 3-manifold, which is a contradiction.

Now consider the case that Δ has two valence-3 1-handles. If some essential disc D in $B \times I$ cuts the bridges in ≤ 2 components, then as above, we obtain a handle structure with one 0-handle and three 1-handles. Hollowing out a valence-3 1-handle and the 0-handle produces a handle structure Δ_1 on a non-elementary manifold M_1 with $\rho_1(\Delta_1) = 1$, which is a contradiction. If no essential disc D exists as above, then we can find one which cuts the bridges in exactly three components. One readily enumerates the possible handle structures that satisfy our assumptions of valence-3 1-handles and non-peripheral disc lakes. After applying the hollowing out procedure, a Mom-2 handle structure is created. Of the two choices of which 1-handle is to be hollowed out, one will produce a full Mom-2 structure. \square

Remark 4.6. As an example of the type of non-full Mom-2 discussed in the previous lemma, note that the figure-8 knot complement contains a non-full Mom-2 structure with two 1-handles of valence 3, in addition to having a full Mom-2 structure.

Proof of Theorem 4.1 The proof is by induction on $\rho_1(\Delta)$. If (M, T, Δ) is not hyperbolic, then apply Lemma 3.11 to obtain (M_1, T_1, Δ_1) where $M_1 \rightarrow N$ is a non-elementary embedding, ∂M_1 is a union of tori and either (M_1, T_1, Δ_1) is a full internal Mom- k structure, $k < n$, or $\rho_1(\Delta_1) + 2 \leq \rho_1(\Delta)$. In the former case the proof follows by induction. In the latter case as seen in the next paragraph we will produce a general based internal Mom- k structure (M_4, B_4, Δ_4) with $k \leq n - 2$ and M_4 hyperbolic. Lemma 4.4 then implies that $k \geq 2$ and hence $n \geq 4$. If $k = 2$, then Lemma 4.5 implies that N has a full internal Mom-2 structure (M_5, B_5, Δ_5) . M_5 must be hyperbolic, or else the above argument for $n = 2$ will give a contradiction to Lemma 4.4, completing the proof.

So assume that (M_1, T_1, Δ_1) is not a full internal Mom structure. If M_1 is reducible, then split along a normal reducing 2-sphere, retain the non-elementarily embedded component, cap off the resulting 2-sphere boundary component with a 3-handle and cancel that 3-handle with a 2-handle. After a sequence of such operations we obtain (M_2, T_2, Δ_2) with $\rho_1(\Delta_2) \leq \rho_1(\Delta_1)$ and M_2 irreducible. If M_2 contains an embedded essential torus R , then split along R and retain the non-elementarily embedded component. After a sequence of such operations we obtain (M_3, T_3, Δ_3) with $\rho_1(\Delta_3) \leq \rho_1(\Delta_2)$ and M_3 is irreducible and geometrically atoroidal. Since M_3 is non-elementarily embedded in N , it is not a Seifert fibered space and hence is anannular and so by Thurston [Th2] it is hyperbolic. After repeatedly applying Lemmas 4.2 and 4.3 we obtain a general based internal Mom- k structure (M_4, T_4, Δ_4) on N where M_4 is hyperbolic and $\rho_1(\Delta_4) \leq \rho_1(\Delta_3)$. \square

5. ENUMERATION OF HYPERBOLIC MOM- n 'S FOR $2 \leq n \leq 4$

Let (M, T, Δ) be a full hyperbolic Mom- n , with $2 \leq n \leq 4$. The handle structure Δ collapses to a cellular complex K in the following fashion. Each 1-handle collapses to the arc at its core, and each 2-handle collapses to the disc at its core (expanded as necessary so that it is still attached to the cores of the appropriate 1-handles). Also, $T \times I$ collapses to $T \times 1$, subdivided into 0-cells, 1-cells, and 2-cells corresponding to the islands, bridges, and lakes of (M, T, Δ) . (Note that if (M, T, Δ) were not full, we might have a non-simply connected lake and K would not be a proper cellular complex.)

The resulting complex K is a *spine* for M in the sense of [MF]. If all of the 1-handles of Δ are of valence 3, then it is also a *special spine* in the sense of [MF]; however K is not a special spine in general. In particular, in a special spine the link of each point is either a circle or a circle with two or three radii, but if Δ has a 1-handle of valence n then the endpoints of the corresponding arc in K will have links which are a circle with n radii. This, however, is the only way in which K fails to be a special spine.

In section 2 of [MF] Matveev and Fomenko describe how a manifold with a special spine can be reconstructed by gluing together truncated or ideal simplices dual to the vertices of the spine. This construction is easily generalized to our situation, and shows that M can be reconstructed from K by gluing together ideal polyhedra dual to the vertices of K . The result is an ideal cellulation of M which is dual to the cellular complex K .

The 3-cells of this cellulation will be dual to the elements of K^0 , which consist of the endpoints of the cores of the 1-handles of Δ . In addition, since we've assumed each 1-handle of Δ meets at least two 2-handles, each point $v \in K^0$ will be the endpoint of at least two curves in $T \times 1 \cap K^1$. Hence if n_v is the valence of v in the 1-skeleton of K then $n_v \geq 3$. If $n_v \geq 4$ then v is dual to an $(n_v - 1)$ -sided pyramid: the base of the pyramid is dual to the core of a 1-handle while the sides are dual to curves in $T \times 1 \cap K^1$. If $n_v = 3$ then v is dual to a "digonal pyramid", which we eliminate from the cellulation by collapsing it to a face in the obvious fashion. Thus K is dual to a cellulation of M by ideal pyramids. Since the bases of these pyramids correspond to the ends of the 1-handles of Δ , we can pair them up into a collection of ideal dipyrramids.

We can say more concerning the possible types and combinations of dipyrramids. On one hand, each vertex v is adjacent to $n_v - 1$ edges in $T \times 1 \cap K^1$, and each such edge has two endpoints; on the other hand, the core of each 2-cell of Δ contributes three edges to $T \times 1 \cap K^1$, and there are n such cores in a Mom- n . Therefore $\sum_v (n_v - 1) = 6n$ in a Mom- n . Furthermore, $n_v - 1$ must be at least 2 and (if it's greater than 2) equals the number of sides of the pyramid dual to v . Finally the vertices v occur in pairs since each one corresponds to an end of a 1-handle, and the vertices in each pair have the same valence. Therefore for a Mom-2, there are only two possibilities: four three-sided pyramids, which glue together to form two three-sided dipyrramids, or two four-sided pyramids and two "digonal pyramids", which (after eliminating the "digonal pyramids") glue together to form a single ideal octahedron. Similarly, there are only three possibilities for a Mom-3: three three-sided dipyrramids, a three-sided dipyramid together with an octahedron, or a five-sided dipyramid by itself. The five possibilities for a Mom-4 are: four three-sided dipyrramids, two three-sided dipyrramids and an octahedron, one three-sided dipyramid and one five-sided dipyramid, two octahedra, or one six-sided dipyramid.

Thus, if (M, T, Δ) is a hyperbolic Mom-2, Mom-3, or Mom-4 then M can be obtained by gluing together the faces of one of these ten sets of ideal polyhedra. Enumerating the possibilities for M then becomes a matter of enumerating the ways in which the faces of these polyhedra can be glued together to form a hyperbolic 3-manifold.

This task is simplified somewhat by the following observation: the faces of each dipyramid always have exactly one vertex which is dual to the cusp neighborhood $T \times [0, 1)$. When gluing the polyhedra together to form N , all such vertices must be

identified with one another and with no other vertices. Thus given any two faces, there is only one possible orientation-preserving way that those two faces could be glued together.

Hence it is sufficient to enumerate the number of ways in which the faces of one of the ten sets of polyhedra can be identified in pairs. Although it is almost trivial to program a computer to do this, care must be taken as the number of possibilities is a factorial function of the number of faces, and a naive approach can rapidly exhaust a computer's memory. To reduce the demands on the computer, a refinement to the naive approach was employed. First, for each possible set of polyhedra a symmetry group was computed. Each dipyrmaid has dihedral symmetry, while if a given set of polyhedra contains two dipyramids with the same number of sides then they can be exchanged to provide an additional symmetry. Secondly, an ordering was chosen for the set of all possible pairings of faces, namely the lexicographic ordering of the pairings when represented as permutations. Our computer program considered the set of pairings in order, and any pairing was immediately rejected if it was conjugate to a previous pairing via an element of the symmetry group. This considerably reduced the running time of the program.

The next step in the process is to eliminate pairings which result in obviously non-hyperbolic manifolds. While the program SnapPea can in principle handle this, for reasons of speed our program checked one necessary criterion itself: whether the link of every ideal vertex was Euclidean. Computing the Euler characteristic of the link of each ideal vertex in the cellular complex resulting from a pairing was easy to do and eliminated many cases from consideration. Our program also eliminated any pairing in which the vertices supposedly dual to the original cusp neighborhood or solid torus in fact glued together to form two or more ideal vertices.

The above considerations resulted in a list of gluing descriptions of 4,231 manifolds which might be hyperbolic Mom-2's or Mom-3's. At this point, SnapPea was employed to try and compute hyperbolic metrics for each of these manifolds, and to find further hyperbolic symmetries among the manifolds which admitted such metrics. SnapPea claimed to find hyperbolic metrics in 164 cases. In three of those cases SnapPea had experienced an obvious floating-point error and "found" a hyperbolic metric with an absurdly low volume. In those three cases the programs Regina and GAP (which, unlike SnapPea, do not rely on floating-point arithmetic) were used to calculate the fundamental groups of the corresponding manifolds. In all three cases the fundamental group was isomorphic to the group $\langle a, b | [a, b^n] \rangle$ where $n = 3$ or 5 , which has a non-trivial center. Therefore, these three manifolds could not possibly be hyperbolic and were rejected. That left 161 cases, which SnapPea identified as belonging to a total of 21 isometry classes of hyperbolic manifolds.

Some comments about rigor are in order here. Since SnapPea relies on floating-point arithmetic, some of its results are unavoidably inexact. In particular, there is no guarantee that SnapPea will find a hyperbolic metric on a manifold even if one exists, or that SnapPea will correctly discern the absence of a hyperbolic metric in cases where it doesn't exist. In practice it is our experience that if one is careful to allow SnapPea to simplify a triangulation before attempting to find a metric, then if a metric exists SnapPea will either find it or fail to make a determination, while if a metric doesn't exist SnapPea will either correctly say so or on rare occasions "find" a metric with absurdly low volume due to floating-point error. Still, from a standpoint of rigor this is problematic. Fortunately there is at least one task

which SnapPea does perform exactly, and that is finding isometries between two different cusped manifolds: SnapPea will only report that an isometry exists if it finds identical triangulations of the two manifolds. (See [We]; in particular see the comments in the source code file *isometry.c*.) This is a combinatorial operation, not a floating-point one, and hence we are confident that SnapPea performs this operation rigorously.

Those familiar with SnapPea's source code may object that SnapPea re-triangulates each manifold before determining if an isometry exists, and that SnapPea uses floating-point information to choose the re-triangulation. To this objection we would reply that while floating-point information is used to *choose* the re-triangulation, the actual re-triangulating is still a combinatorial operation, i.e. it uses integer arithmetic. The new triangulation is guaranteed to have the same topological type (see the comments in *canonize_part_1.c* from [We]), and hence the possibility of floating-point error does not invalidate the result when SnapPea reports that it has found an isometry.

Thus while we are trusting SnapPea when it says that the 161 manifolds mentioned above are all in the isometry class of one of 21 manifolds from the SnapPea census, we are confident that we are not sacrificing rigor in so doing. Furthermore, the census manifolds were recently confirmed to be hyperbolic by Harriet Moser in [Mos], establishing that we have found 21 different hyperbolic Mom-2's and Mom-3's.

Unfortunately, we still can't trust SnapPea when it fails to find a hyperbolic metric for a given manifold, as that result is not guaranteed to be rigorous. This means that there are 4,067 manifolds from the above list of 4,231 which may still be hyperbolic despite SnapPea evidence to the contrary. These manifolds were analyzed separately in the same way as the three manifolds for which SnapPea claimed to have found an absurdly low hyperbolic volume. Namely, we used Regina and GAP as before to compute the fundamental groups of the manifolds in question, and then examined the list of groups to see if any of them might be the fundamental group of a hyperbolic manifold. The vast majority of the groups on the list either had a non-trivial center, or else had two rank-2 Abelian subgroups which intersected in a rank-1 Abelian subgroup (also impossible in the fundamental group of a hyperbolic 3-manifold). Some of the groups required further analysis but were still eventually rejected; for example, many groups had an index-two subgroup with one of the above properties even when it was not clear that the whole group had such properties.

In the end the hand analysis did not reveal any new hyperbolic 3-manifolds in the list of gluing descriptions. This completes the proof of the following:

Theorem 5.1. *There are 3 hyperbolic manifolds M such that (M, T, Δ) is a Mom-2 for some T and Δ : the manifolds known in SnapPea's notation as $m125$, $m129$, and $m203$. There are 18 additional hyperbolic manifolds M such that (M, T, Δ) is a Mom-3 structure for some T and Δ : the manifolds known in SnapPea's notation as $m202$, $m292$, $m295$, $m328$, $m329$, $m359$, $m366$, $m367$, $m391$, $m412$, $s596$, $s647$, $s774$, $s776$, $s780$, $s785$, $s898$, and $s959$.*

Some comments about this list are in order. The manifold $m129$, better known as the complement of the Whitehead link, is the only manifold on this list which is obtained by gluing together the faces of an ideal octahedron. Also, all but one of these manifolds have two cusps. The exception is the three-cusped $s776$, which

m357	s579	s883	v2124	v2943	v3292	v3450
m388	s602	s887	v2208	v2945	v3294	v3456
s441	s621	s895	v2531	v3039	v3376	v3468
s443	s622	s906	v2533	v3108	v3379	v3497
s503	s638	s910	v2644	v3127	v3380	v3501
s506	s661	s913	v2648	v3140	v3383	v3506
s549	s782	s914	v2652	v3211	v3384	v3507
s568	s831	s930	v2731	v3222	v3385	v3518
s569	s843	s937	v2732	v3223	v3393	v3527
s576	s859	s940	v2788	v3224	v3396	v3544
s577	s864	s941	v2892	v3225	v3426	v3546
s578	s880	s948	v2942	v3227	v3429	

FIGURE 1. Conjectured list of SnapPea manifolds which are strict Mom-4's.

is the complement in S^3 of a three-element chain of circles (the link 6_1^3 in Rolfsen's notation).

Enumerating hyperbolic Mom-4's was more difficult: merely enumerating the possible gluing descriptions resulted in a list of 1,033,610 possibilities (compared to 4,231 possibilities in the previous case). From this list, SnapPea identified 138 different hyperbolic manifolds. In another 493 cases, SnapPea was either unable to make a determination or else experienced an obvious floating-point error. In each such case, the fundamental group of the corresponding manifold was again computed by Regina and GAP, and in each case the fundamental group was isomorphic to $\langle a, b | [a^n, b] \rangle$ for some n , or else had two rank-2 Abelian subgroups which intersected in a rank-1 Abelian subgroup. Therefore these exceptional cases do not correspond to hyperbolic manifolds. Note that all of the Mom-2's and Mom-3's appear in the Mom-4 list; the same manifold can admit multiple handle structures.

Based on the above result, we propose the following:

Conjecture 5.2. *There are 138 hyperbolic manifolds M such that (M, T, Δ) is a Mom-2, Mom-3, or Mom-4 for some T and Δ . Of these, 117 are strict Mom-4's, i.e. Mom-4's which are not Mom-2's or Mom-3's.*

Of the 117 strict Mom-4's, SnapPea was successfully used to identify 83 of them as manifolds from the SnapPea census. Those manifolds appear in Figure 1. SnapPea was not able to identify the remaining 34 manifolds, and in fact 33 of those manifolds have volumes which do not appear anywhere in the SnapPea census, presumably because the Matveev complexity of the corresponding manifolds is greater than 7 (see [MF]). The remaining manifold has the same volume and homology as the census manifold v3527; it is conceivable that SnapPea was simply unable to find a corresponding isometry.

The unidentified manifolds are listed in Figure 2. The notation used can be interpreted as follows: the numbers before the semi-colon describe the type of ideal polyhedra used to construct the manifold. For example, the first entry in the figure has the numbers “3, 3, 4” to the left of the semi-colon; each “3” indicates an ideal triangular dipyrmaid, while each “4” indicates an ideal square dipyrmaid (i.e. an ideal octahedron). Each ideal dipyrmaid has two “polar” vertices and either three

(3, 3, 4 ; 3, 6, 8, 0, 13, 19, 1, 15, 2, 17, 14, 18, 16, 4, 10, 7, 12, 9, 11, 5)
(3, 3, 4 ; 3, 6, 11, 0, 10, 9, 1, 15, 14, 5, 4, 2, 16, 18, 8, 7, 12, 19, 13, 17)
(3, 3, 4 ; 3, 6, 11, 0, 9, 18, 1, 19, 13, 4, 15, 2, 16, 8, 17, 10, 12, 14, 5, 7)
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(3, 3, 3, 3 ; 3, 6, 12, 0, 17, 9, 1, 11, 18, 5, 23, 7, 2, 20, 15, 14, 21, 4, 8, 22, 13, 16, 19, 10)
(3, 3, 4 ; 3, 4, 6, 0, 1, 18, 2, 11, 16, 10, 9, 7, 15, 19, 17, 12, 8, 14, 5, 13)
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(3, 3, 4 ; 3, 4, 6, 0, 1, 18, 2, 14, 16, 13, 19, 17, 15, 9, 7, 12, 8, 11, 5, 10)
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(4, 4 ; 15, 5, 13, 7, 9, 1, 11, 3, 14, 4, 12, 6, 10, 2, 8, 0)
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(3, 3, 4 ; 3, 6, 11, 0, 8, 19, 1, 15, 4, 17, 14, 2, 16, 18, 10, 7, 12, 9, 13, 5)
(3, 3, 4 ; 6, 7, 10, 8, 13, 17, 0, 1, 3, 15, 2, 19, 16, 4, 18, 9, 12, 5, 14, 11)
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(3, 3, 3, 3 ; 3, 6, 12, 0, 9, 16, 1, 18, 11, 4, 23, 8, 2, 19, 15, 14, 5, 21, 7, 13, 22, 17, 20, 10)

FIGURE 2. Conjectured Mom-4's not identified by SnapPea.

or four “equatorial vertices”. Number the faces of all the polyhedra sequentially in such a way that the faces “north” of each equator are numbered before the faces “south” of each equator. For example, in the first entry the first triangular dipyramid has faces 0, 1, and 2 next to one polar vertex, and faces 10, 11, and 12 next to the other polar vertex. The next triangular dipyramid has faces 3, 4, and 5 as well as faces 13, 14, 15, and the square dipyramid has faces 6 through 9 and 16 through 19. (This somewhat unintuitive numbering scheme was chosen for convenience when writing the computer software for this part of the paper.) Then the numbers to the right of the semi-colon form a permutation which describes how to glue together the faces of the ideal polyhedra. For example, in the first entry the string of numbers which begins with “3, 6, 8, 0, . . .” imply that face 0 is glued to

face 3, face 1 is glued to face 6, and so on. Since we are requiring “polar” vertices to be identified solely with other “polar” vertices, no other information is needed to reconstruct the polyhedral gluing.

One additional point of information: all but eight of the manifolds in the list satisfy $|\partial M| = 2$; seven satisfy $|\partial M| = 3$ and one satisfies $|\partial M| = 4$. Thanks to the timely assistance of Morwen Thistlethwaite, the authors were able to positively identify all eight of these manifolds:

Conjecture 5.3. *There are 8 hyperbolic manifolds M such that (M, T, Δ) is a Mom- n for some $2 \leq n \leq 4$ and $|\partial M| > 2$. All eight manifolds are complements of links in S^3 : the links $6_1^3, 6_2^3$ (the Borromean rings), $8_1^3, 8_9^3, 8_2^4$, and the links with Gauss codes $jccddEGHiJBFca, jbecceaHbIJDGF$, and $mccdfiEhAjKLcmbdFG$.*

At the time of writing we are still searching for an efficient way to verify Snap-*Pea*’s computations in the Mom-4 case; clearly, examining over a million fundamental groups by hand is not a practical solution. Until a better way is found, our enumeration results in the Mom-4 case should properly be considered speculative.

6. HYPERBOLIC MOM- n ’S IN HYPERBOLIC 3-MANIFOLDS ARE INTERNAL MOM- n STRUCTURES FOR $n \leq 4$

Let R be a convolotube in the interior of a compact hyperbolic 3-manifold N and let V be the cube with knotted hole bounded by R . By drilling out solid tori from $N - \overset{\circ}{V}$, we can create a manifold M which is non-elementarily embedded in N and whose boundary contains a convolotube. We call such an embedding *knotted*. The goal of this chapter is to show that if $n \leq 4$, any embedding of a Mom- n manifold (M, T) into a compact hyperbolic manifold (N, T) is unknotted.

Definition 6.1. Let M be a compact 3-manifold and T a possibly empty union of components of ∂M . We say that (M, T) is *hereditarily unknotted*, if every non-elementary embedding into a compact hyperbolic 3-manifold N , taking T to components of ∂N , has the property that each component of ∂M is either boundary parallel or bounds a solid torus.

Remark 6.2. If (M, T) is hereditarily unknotted and M_1 is obtained by filling a component of $\partial M - T$, then (M_1, T) is hereditarily unknotted.

Lemma 6.3. *If (M, T) is a hereditarily unknotted Mom- n manifold non-elementarily embedded in the hyperbolic 3-manifold N such that T bounds a tubular neighborhood of a geodesic, then (M, T) is an internal Mom- n structure.*

Proof. Let V be the solid torus bounded by T . By Lemma 1.7, if $N_1 = N - \overset{\circ}{V}$ with cusp neighborhoods deleted, then N_1 is compact hyperbolic. Therefore $(M, T) \subset (N_1, T)$ is a non-elementary embedding and hence any component of $\partial M - T$ either bounds a solid torus or is boundary parallel in N_1 . Therefore similar properties hold in N and hence (M, T) is an internal Mom- n structure on N . \square

Remark 6.4. The condition that T bounds a neighborhood of a geodesic is essential.

Lemma 6.5. *Let M be a compact hyperbolic 3-manifold with T a union of components of ∂M . If $\partial M - T$ is connected, (M, T) is hereditarily unknotted.*

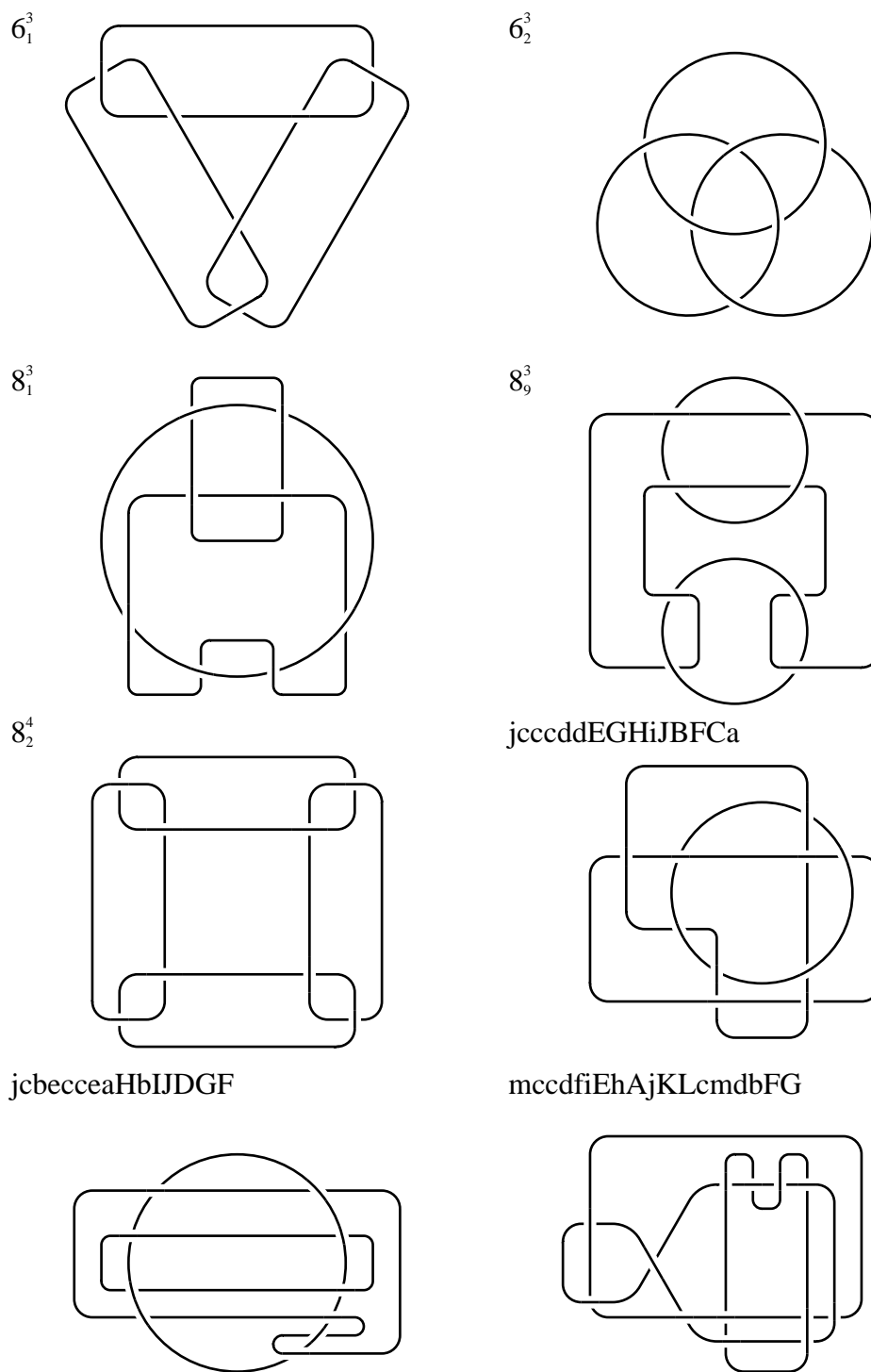


FIGURE 3. Eight links whose complements are Mom-4's with 3 or more cusps.

Proof. If under a non-elementary embedding $(M, T) \rightarrow (N, T)$, $\partial M - T$ was a convolutube, then M would be reducible. \square

The following result establishes criteria for showing that (M, T) is hereditarily unknotted.

Lemma 6.6. *Let M be a compact hyperbolic 3-manifold with V_1, \dots, V_n components of ∂M and T a nonempty union of some other components. If any of the following hold, there exists no non-elementary embedding $(M, T) \rightarrow (N, T)$ such that N is compact hyperbolic and $\{V_1, \dots, V_n\}$ is exactly the set of convolutubes of $\partial M \subset N$.*

i) *The manifold obtained by some filling of M along V_1, \dots, V_n is a 3-manifold without any hyperbolic part. (That is, after applying sphere and torus decompositions there are no hyperbolic components.)*

ii) *After some filling of M along V_1, \dots, V_n , the surface T is compressible.*

iii) *For every filling on a non-empty set of components of $\partial M - T \cup V_1 \cup \dots \cup V_n$, either $V_1 \cup \dots \cup V_n$ is incompressible or the filled manifold has no hyperbolic part.*

Proof. Suppose that (M, T) embeds in (N, T) , where among the components of ∂M , V_1, \dots, V_n are the set of convolutubes and W_1, \dots, W_m are the tubes. Let W_i^* denote the solid torus bounded by W_i and V_i^* denote the cube with knotted hole bounded by V_i . Let B_1, \dots, B_n be pairwise disjoint 3-balls in N such that for each i , $V_i \subset B_i$.

i) Let \hat{M} be a manifold obtained by filling the V_i 's. Let \hat{N} be obtained by deleting the V_i^* 's and doing the corresponding fillings along the V_i 's. Therefore \hat{N} is obtained from \hat{M} by Dehn filling and \hat{N} is a connected sum of N with $S^2 \times S^1$'s and/or lens spaces and/or S^3 's. This implies that \hat{M} has a hyperbolic part.

ii) If T is compressible in \hat{M} it is compressible in \hat{N} and hence in N , which is a contradiction.

iii) First observe that V_i compresses in the manifold M' obtained by filling M where each W_i is filled with W_i^* . Topologically, M' is homeomorphic to N with n open unknotted and unlinked solid tori removed and so has a hyperbolic part. \square

Theorem 6.7. *If the Mom- n manifolds for $n \leq 4$ with three or more boundary components are exactly those listed in Figure 3 (i.e. if Conjecture 5.1 is true), then any hyperbolic Mom- n manifold (M, T) with $n \leq 4$ is hereditarily unknotted.*

Proof. By Lemma 6.5 it suffices to consider the case where M is one of the eight Mom-4 manifolds with at least three boundary components listed in Figure 3. If M is any of the first six manifolds and T is any component of ∂M , then (M, T) is hereditarily unknotted by criterion (i) of Lemma 6.6. For manifolds 7 and 8, depending on which boundary component is used for T , applications of (i) and (iii) imply that they are hereditarily unknotted. \square

7. EXAMPLES OF MOM- n STRUCTURES

In this section we give some representative examples of hyperbolic manifolds N which contain an internal Mom-2 or Mom-3 structure (M, T, Δ) . Our goal in this section is to give the reader an intuitive feel for how these particular cell complexes arise inside hyperbolic manifolds. All of the manifolds in this section involve manifolds N with torus boundary, with the base torus of the Mom-structure (M, T, Δ) being ∂N . To obtain Mom- n structures on closed manifolds, note that if

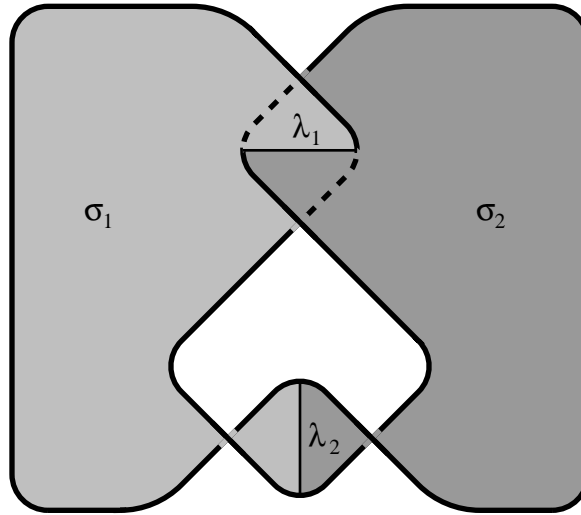


FIGURE 4. The figure-8 knot complement equipped with a Mom-2.

$T = \partial N$ then a Mom- n structure (M, T, Δ) on N passes to a Mom- n structure on any manifold obtained by filling ∂N .

Example 7.1. The first example is the figure-8 knot complement. We construct a Mom-2 (M, T, Δ) inside this manifold as follows. The torus T is just the boundary of the manifold. The 1-handles λ_1 and λ_2 span the two tangles which make up the standard diagram of this knot, as seen in figure 4. Finally the 2-handles σ_1 and σ_2 are symmetrically placed as shown in the diagram. Note that, as required, each 2-handle meets three 1 handles counting multiplicity. Specifically, each 2-handle meets λ_1 twice and λ_2 once. Also, one can see from the diagram that the complement of $T \cup \{\lambda_i\} \cup \{\sigma_j\}$ consists of a solid torus, and that the solid torus retracts onto a homotopically non-trivial simple closed curve (which is a geodesic in N). Thus this is a valid hyperbolic Mom-2 structure on N .

Moreover, we can quickly determine the nature of the ideal triangulation of M described in the Section 5. The ends of λ_1 are each dual to a four-sided pyramid in this triangulation, and the two endpoints of λ_2 are each dual to a “digonal pyramid”, so that each get eliminated. Thus the figure-8 knot complement possesses a Mom-2 structure (M, T, Δ) where M is a two-cusped hyperbolic manifold which is in turn obtained by gluing together the faces of an ideal octahedron. By the comments after Theorem 5.1, M must be the complement of the Whitehead link. And indeed, it is easy to verify that if one drills out the core of the solid torus in the complement of M one obtains a manifold homeomorphic to the complement of the Whitehead link.

Example 7.2. Next we will let N be the manifold known as m003 in the SnapPea census. This manifold has first homology group $\mathbb{Z} + \mathbb{Z}/5$, and hence is not a knot complement; instead, we will present this manifold as the union of two regular ideal hyperbolic tetrahedra; see figure 5. Note that in the diagram each face is glued to the corresponding face on the other tetrahedron, in such a way that the edges match

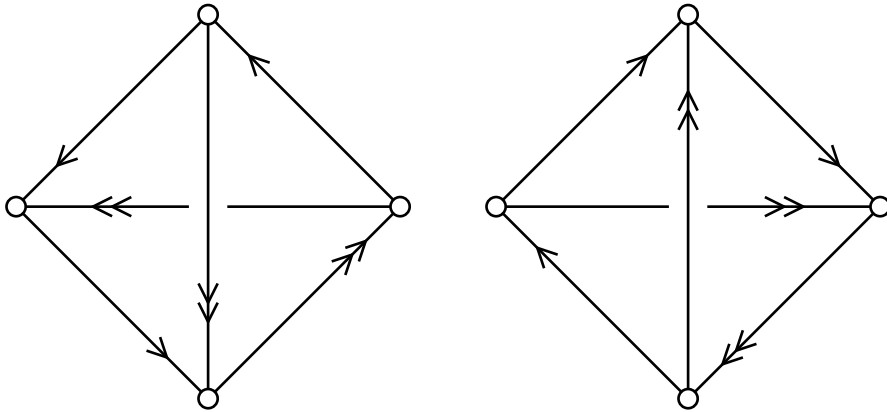


FIGURE 5. The two ideal tetrahedra making up the manifold m003.

up into two equivalence classes as shown. To make N a compact manifold with torus boundary, assume the ideal tetrahedra are truncated. Now suppose that we construct (M, T, Δ) in this case as follows. For the 1-handles, we use neighborhoods of the two edges shown in the diagram, truncated by the torus $T = \partial N$. And for the 2-handles, we use neighborhoods of the two truncated triangles which are formed by gluing together the faces on the front of each tetrahedron in the diagram. It is a simple exercise to confirm that the complement of the resulting embedded manifold M consists of a solid torus, and that the solid torus retracts onto a simple closed geodesic curve, and that therefore this manifold possesses a valid hyperbolic Mom-2 structure. Each of the 1-handles in this Mom-2 meets three of the 2-handles, counting multiplicity; therefore we can conclude that m003 contains a Mom-2 (M, T, Δ) where M is obtained by gluing together two ideal three-sided dipyrramids. From Theorem 5.1 and the comments following it we know this must be either m125 or m203. Further investigation with SnapPea shows that it must in fact be m125.

It is instructive to get another view of this Mom-2 by constructing a cusp diagram for this manifold. Specifically, consider the triangulation induced on T by the given ideal triangulation of m003. The two ideal tetrahedra in m003 will appear as eight triangles, the four ideal triangles will appear as twelve edges, and the two edges will appear as four vertices. The resulting cusp diagram is shown in figure 6; keep in mind this is a diagram of a torus, so the edges of the parallelogram are identified with one another. (The labels inside each triangle indicate which of the ideal simplices contributes that triangle to the cusp diagram.) The highlighted edges in the cusp diagram are those that correspond to the 2-handles of the handle structure Δ ; in other words, they along with the four vertices of the diagram comprise $\Delta^1 \cap T$.

Example 7.3. As another example in this vein, consider the manifold $N = \text{m017}$. This manifold has first homology group $\mathbb{Z} + \mathbb{Z}/7$, so again it is not a knot complement in S^3 , but for brevity's sake we only present a cusp diagram here. In figure 7, the corners of the three ideal hyperbolic tetrahedra which make up m017 can be seen. And again, the highlighted edges in the cusp diagram correspond to two faces of those tetrahedra which provide the 2-handles for an internal Mom-2 in this manifold. Note that we can determine from the cusp diagram alone that

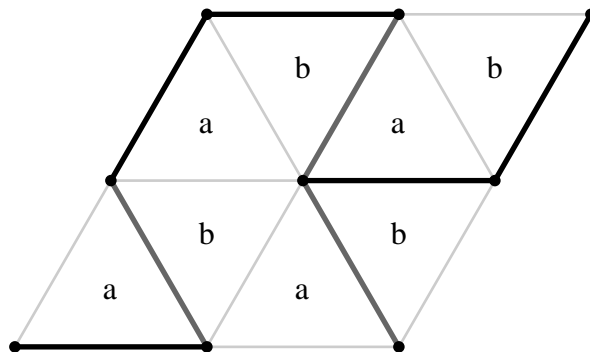


FIGURE 6. The cusp diagram for m003, with the components of the Mom-2 highlighted.

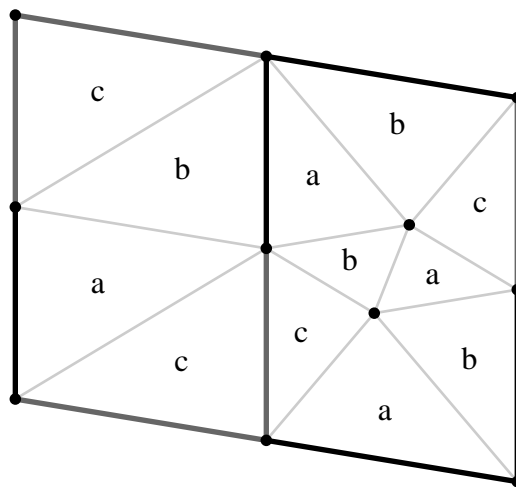


FIGURE 7. The cusp diagram for m017, with the components of a Mom-2 highlighted.

the 1-handles of this Mom-2 meet four and two 2-handles respectively, counting multiplicity, and that therefore in the resulting Mom-2 structure (M, T, Δ) the manifold M is obtained by gluing together an ideal octahedron. As before, this implies that M must be homeomorphic to the complement of the Whitehead link. Some further work with SnapPea confirms this: m017 is obtained by $(-7, 2)$ Dehn surgery on either component of the link.

Example 7.4. Finally, we include the motivating example for this paper. Figure 8 shows the maximal cusp diagram of the 1-cusped manifold m011 as provided by Weeks' SnapPea program. Unlike the previous cusp diagrams in this section, it also shows all the horoballs at hyperbolic distance at most 0.51 from the maximal horoball at infinity. The parallelogram shows a fundamental domain for the $\mathbb{Z} \oplus \mathbb{Z}$ action. Note that the ideal triangulation presented in this diagram is dual to the Ford decomposition of the manifold. In particular the 1-simplices of the triangulation are geodesics orthogonal to pairs of horoballs; these 1-simplices appear either

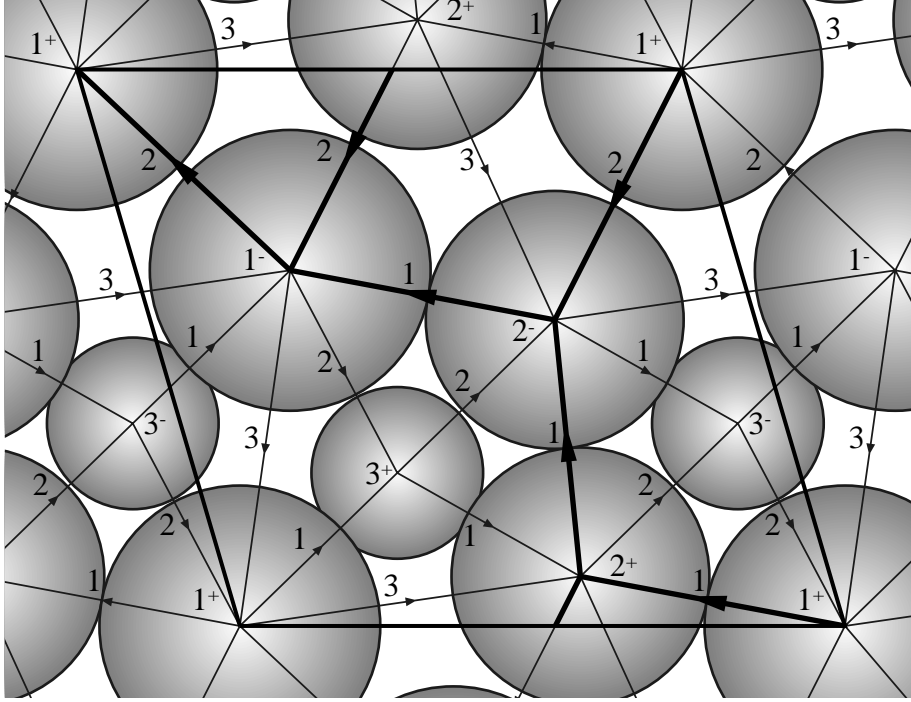


FIGURE 8. SnapPea's cusp diagram for $m011$, with the components of a Mom-2 highlighted.

as edges in the figure joining the endpoints of the simplex in S_∞^2 , or as the vertical geodesics passing from the center of each horoball to the horoball at infinity.

Let H_∞ denote the horoball at infinity. There are six horoballs in the diagram up to the $\mathbb{Z} \oplus \mathbb{Z}$ -action, labelled $1-$, $1+$, $2-$, $2+$, $3-$, and $3+$. This notation means that if $\gamma \in \pi_1(m011)$ takes horoball $n\pm$ to H_∞ , the horoball at infinity, then H_∞ is transformed to one labelled $n\mp$. The geodesic from $n-$ to H_∞ is oriented to point into H_∞ and hence the geodesic from $n+$ to H_∞ is oriented out of H_∞ . These orientations induce, via the $\pi_1(m011)$ -action, the indicated orientations on the edges of the diagram. We explain, by example, the meaning of the edge labels. The edge 3 from $2+$ to $2-$ corresponds to a geodesic σ with the property that when $2+$ is transformed to H_∞ , then $2-$ is transformed to $3+$ and σ is transformed to the vertical geodesic oriented from H_∞ to $3+$. (Had the edge been oriented oppositely, then $2-$ would have been transformed to $3-$.) SnapPea did not provide the orientation information, however such information can be derived from the SnapPea data.

By staring at this diagram we can see how $m011$ contains a Mom-2. Let V_0 be the maximal horotorus neighborhood of the cusp, slightly shrunken. By expanding V_0 , the expanded V_0 touches the (expanded) horoballs labeled 1, thereby creating a 1-handle denoted E_1 . Let V_1 denote this expanded V_0 . Further expansion creates V_2 which is topologically V_1 together with another 1-handle E_2 , this 1-handle occurring between horoball 2 and H_∞ . The edge labelled 1 between horoballs $2-$ and $2+$ corresponds to a valence three 2-handle which goes over E_1 once and E_2 twice.

Similarly the edge labelled 2 between horoballs $1-$ and $1+$ gives rise to a valence three 2-handle going twice over E_1 and once over E_2 . The parallelogram of Figure 8 can also be viewed as ∂V_0 , with the centers of $1-, 1+, 2-, 2+$ as the attaching sites of the 1-handles and the thick black lines corresponding to where the 2-handles cross over ∂V_0 .

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DEPARTMENT OF MATHEMATICS, PRINCETON UNIVERSITY, PRINCETON, NJ 08544

DEPARTMENT OF MATHEMATICS, BOSTON COLLEGE, CHESTNUT HILL, MA 02467

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF CALIFORNIA RIVERSIDE, RIVERSIDE, CA 92521