QUANTUM STRUCTURES FOR LAGRANGIAN SUBMANIFOLDS

YASHA FEST

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Joint work with Octav Cornea, University of Montreal

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Our goal is <u>NOT</u> Lagrangian intersections!!!

Quick review of HF and QH

The pearl complex (suggested by Fukaya, Oh).

- $L \subset M$ monotone.
  $\omega(A) > 0$ iff $\mu(A) > 0$, $A \in \pi_2(M, L)$.
  $N_L =$ min. Maslov number ≥ 2.
  Put $\bar{\mu} = \frac{1}{N_L}\mu : \pi_2(M, L) \to \mathbb{Z}$.

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- $f : L \to \mathbb{R}$ Morse. $\mathcal{C}_*(f) = \mathbb{Z}_2\langle \text{Crit}(f)\rangle \otimes \Lambda_*$.
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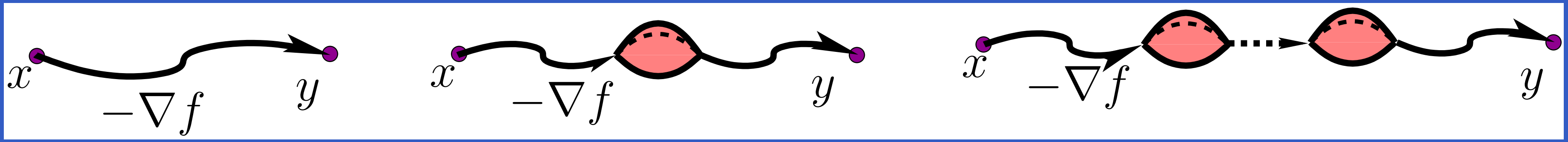
  $d(x) = \sum_{\mathbf{A},y} n_I(x, y; J, \mathbf{A}) y\, t^{\bar{\mu}(\mathbf{A})}$.



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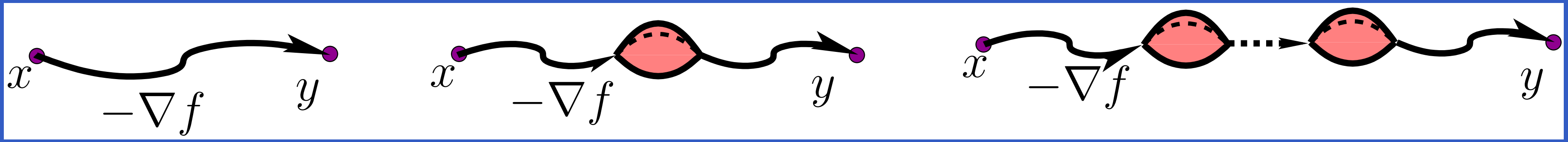

- PSS-Albers argument $\Longrightarrow H_*(\mathcal{C}(f), d) \cong HF_*(L, L)$.

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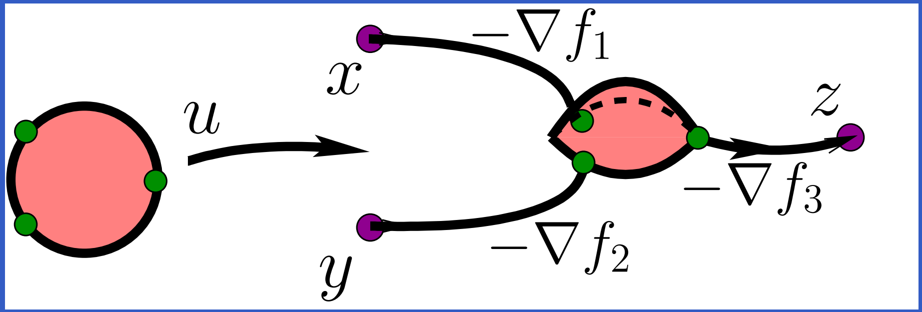

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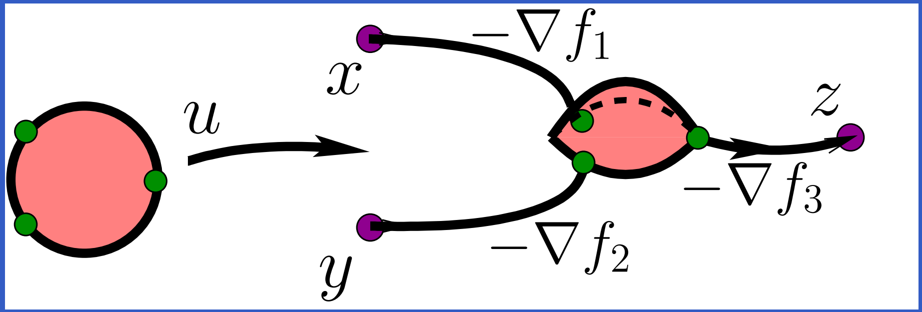

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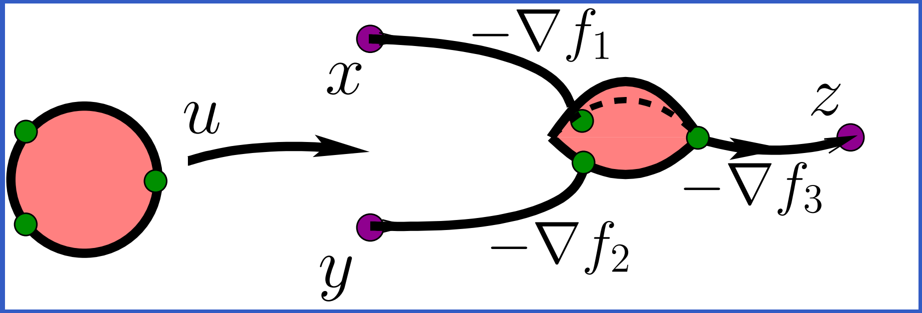

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- $HF(L)$ becomes a (NON COMMUTATIVE) ring with a unity $w \in HF_n(L)$. Actually, $w = [\text{max}]$.

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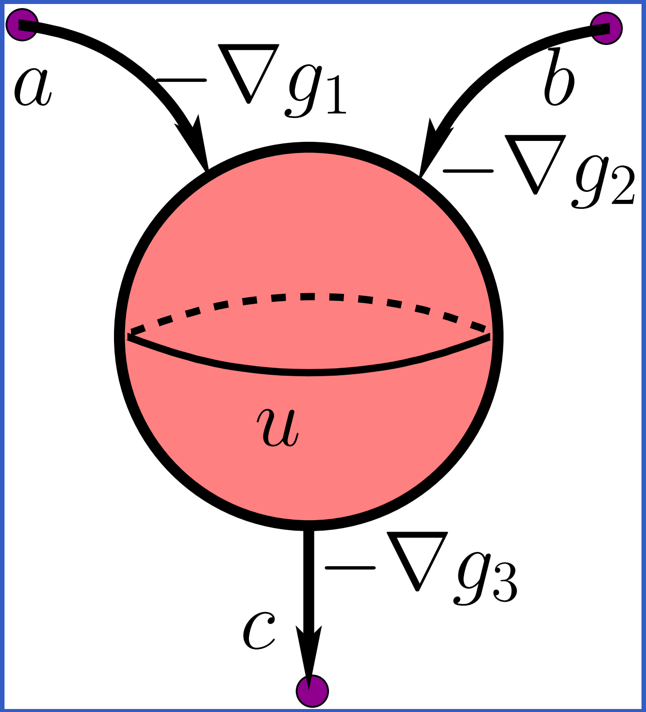

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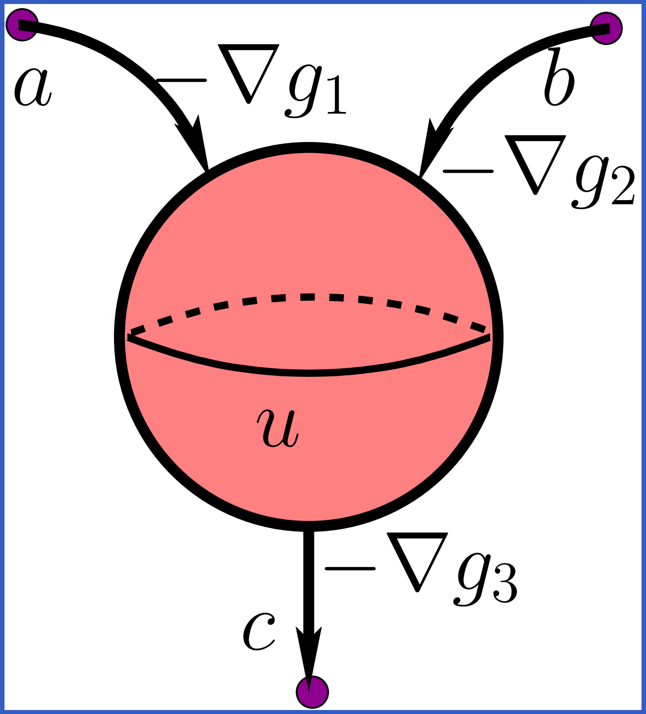

- $QH_*(M)$ becomes a (commutative) ring.
  Unity = fundamental class $u = [M] \in QH_{2n}(M)$.

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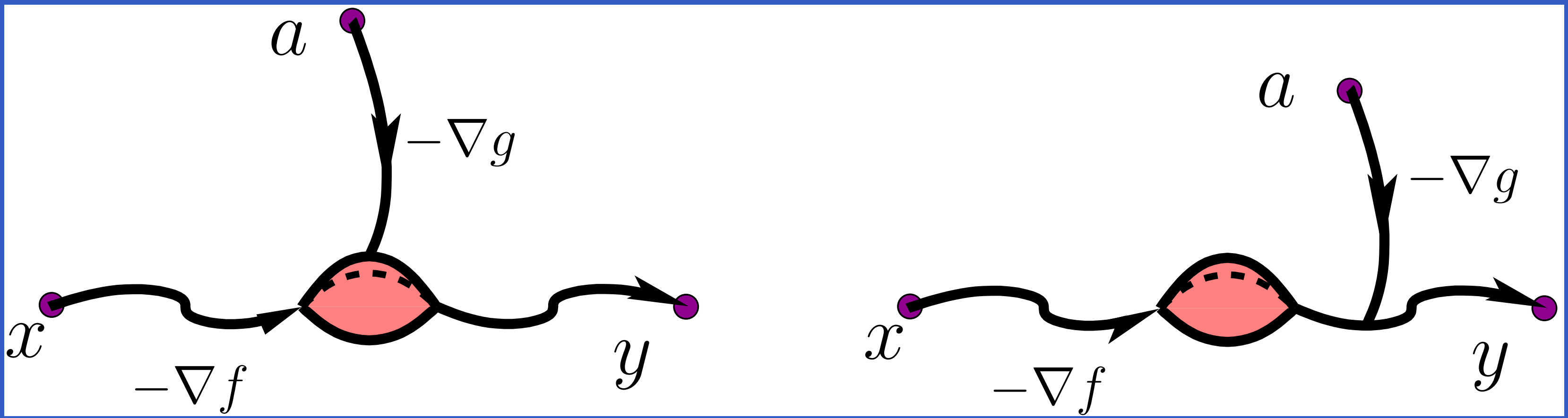

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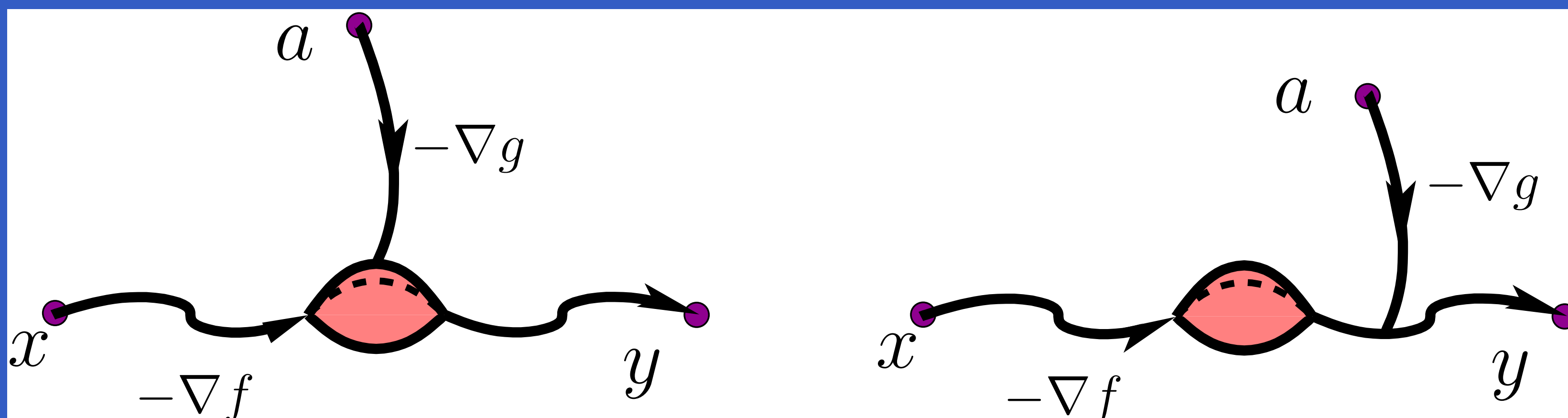
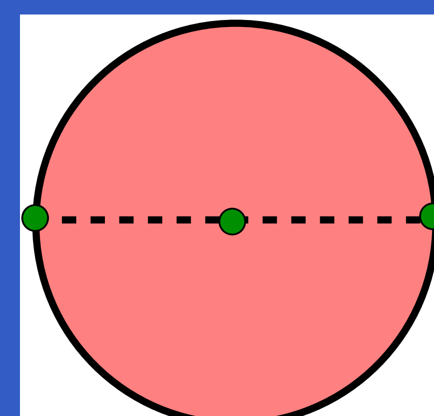


<u>IMPORTANT</u>: The 3 points on the disk marked by the $-\nabla g$ and $-\nabla f$ trajectories must lie on the same hyperbolic geodesic.



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$\forall \quad a, b \in QH(M)$, $\gamma, \delta \in HF(L)$:

$a * (b * \gamma) = (a * b) * \gamma$,

$a * (\gamma * \delta) = (a * \gamma) * \delta = \gamma * (a * \delta)$,

$u * \gamma = \gamma$ etc.

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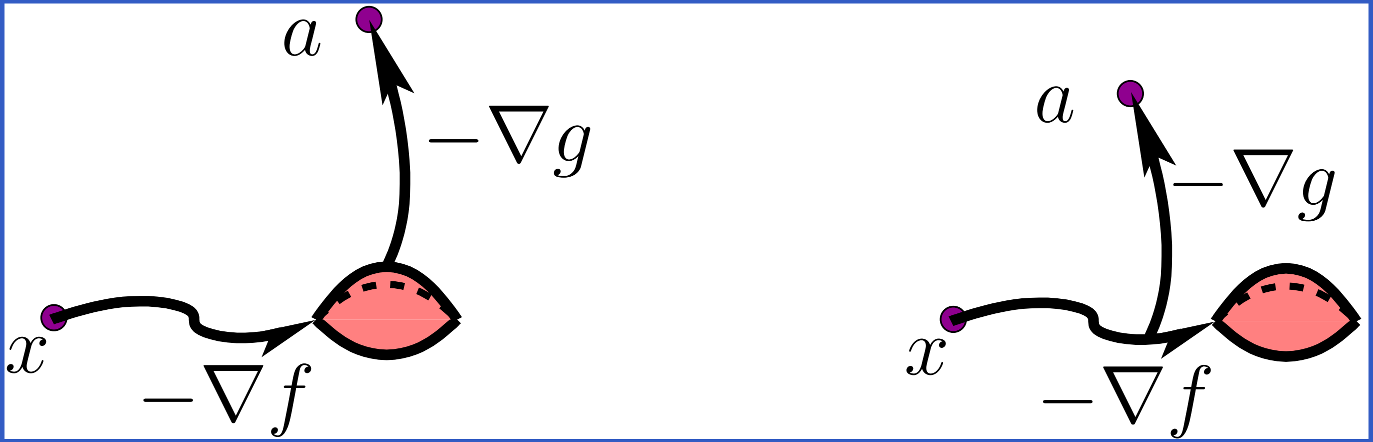

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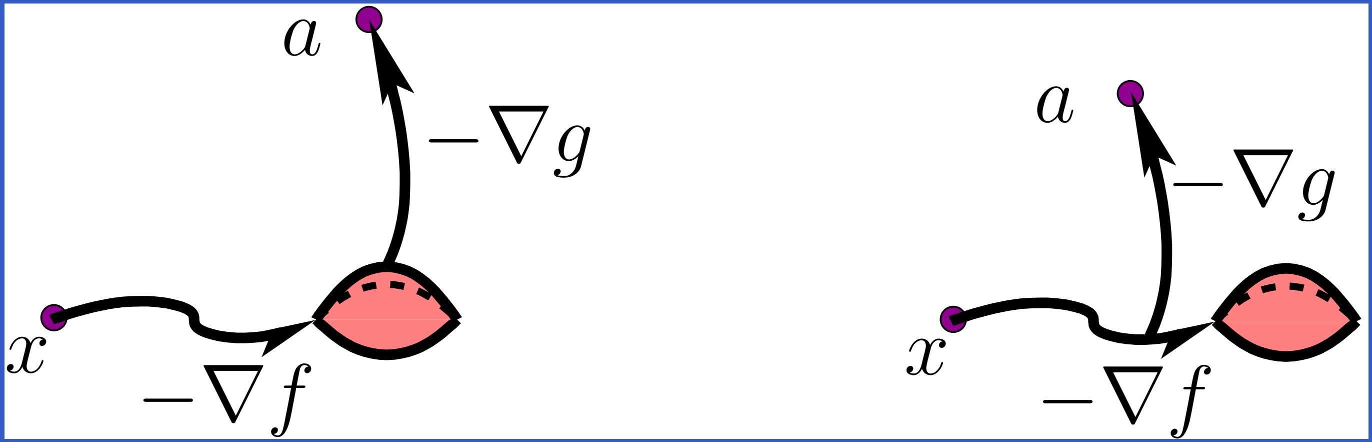

Everything is compatible with duality:
$\forall\, h \in H_*(M)$, $\alpha \in HF_*(L)$: $\langle PD(h), i_L(\alpha)\rangle = \epsilon_L(h * \alpha)$.

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(Compatibility with the quantum product was previously noticed by Buhovsky and by Fukaya-Oh-Ohta-Ono).

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The restriction of λ to $Symp_0(M) \cap Symp(M, L)$ gives automorphisms of $HF_*(L)$ as an algebra over $QH_*(M)$.

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(A similar object has been studied in the context of Lagrangian intersections by Fukaya-Oh-Ohta-Ono).

Main blocks of the proof

- Transversality.
  We need all 0-dim & 1-dim moduli spaces of pearly trajectories to be smooth and of expected dimensions.
  Transversality for holomorphic disks requires them to be *absolutely distinct* + somewhere injective.
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- Compactification of the 1-dim moduli spaces of pearls.
- Gluing.
  Existence: we followed Fukaya-Oh-Ohta-Ono.
  Uniqueness: we proved surjectivity of the gluing map for 0 and 1-dim moduli spaces.

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So $h \in QH_{2n-2}(\mathbb{C}P^n)$ is invertible !

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$\Longrightarrow a * (-) : HF_*(L) \xrightarrow{\cong} HF_{*+i-2n}(L)$ isomorphism.

Ex. $M = \mathbb{C}P^n$. $h \in H_{2n-2}(\mathbb{C}P^n)$ hyperplane,
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So $h \in QH_{2n-2}(\mathbb{C}P^n)$ is invertible !

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Very useful if we know that $HF_i(L) \neq 0$ for some i.

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Statement 1 was proved before by Seidel by other methods. An alternative proof by B.

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Let α_i be the generator of HF_i. So that $\alpha_{-1} = \alpha_n t$, $\alpha_{i+l(n+1)} = \alpha_i t^{-l}$ etc.

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In other words for L as before we have:

i	...	-1	0	1	...	n-1	n	n+1	...
HF_i	...	$\mathbb{Z}_2\alpha_n t$	$\mathbb{Z}_2\alpha_0$	$\mathbb{Z}_2\alpha_1$	...	$\mathbb{Z}_2\alpha_{n-1}$	$\mathbb{Z}_2\alpha_n$	$\mathbb{Z}_2\alpha_0 t^{-1}$	...

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Thm: If $n =$ even or $L \approx \mathbb{R}P^n$ then

$$\alpha_j * \alpha_k = \alpha_{j+k-n} \quad \forall k, j \in \mathbb{Z}.$$

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$$i_L(\alpha_{2k}) = a_{2k} + a_{2k+n+1}t, \quad \forall\, 0 \leq 2k \leq n,$$
$$i_L(\alpha_{2k+1}) = 0, \quad \forall\, k.$$

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<u>Thm:</u> Let $L \subset \mathbb{C}P^n$ with $2H_1(L;\mathbb{Z}) = 0$. If $n =$even or $L \approx \mathbb{R}P^n$ then $\forall\, x', x'' \in L$ and $\forall J$, $\exists\, J$-holomorphic disk $u : (D, \partial D) \to (\mathbb{C}P^n, L)$ with $u(-1) = x'$, $u(1) = x''$ & $\mu([u]) = n + 1$. The $\#$ of such disks (upto parametrization) is even ≥ 2.

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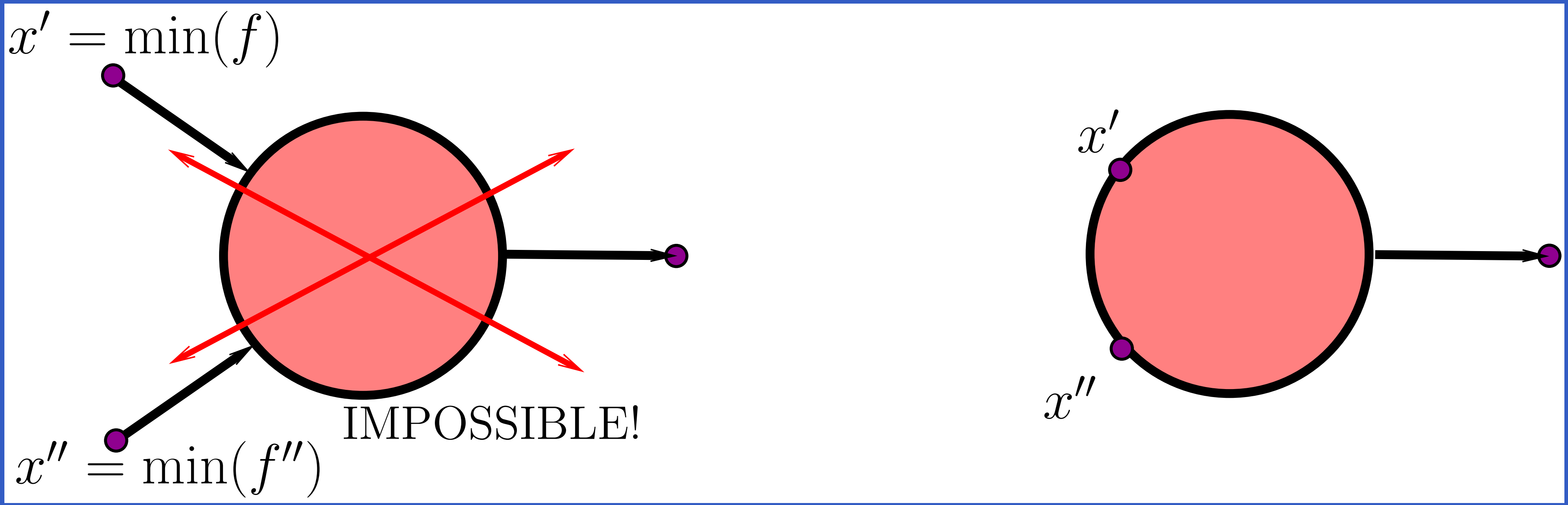

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3. Suppose $n = 2$ and $L \approx \mathbb{R}P^2$. Let $x', x'' \in L$ two distinct points, $p \in \mathbb{C}P^2 \setminus L$. Then for generic J, $\exists$ a J-holomorphic disk $u : (D, \partial D) \to (\mathbb{C}P^2, L)$ with $\mu([u]) = 6$ and $u(-1) = x'$, $u(1) = x''$ and $u(0) = p$. The number of such disks is odd.

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Results in the same spirit hold for Fano complete intersections.

(The point is that we know QH by work of Beauville.)

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Remark: This notion of semi-simplicity is somewhat different than semi-simplicity in the sense of Dubrovin (we work with different coefficient ring $\mathbb{F}$).

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This happens for example, if $M \subset \mathbb{C}P^{n+1}$ is complex hypersurface of degree $3 \leq d \leq \frac{n}{2} + 1$. For $d = 1$ and $d = 2$, $QH_*^{ev}(M; \mathbb{F})$ *is semi-simple*.

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Proof. A criterion of Abrams says that $QH_*^{ev}(M; \mathbb{F})$ is semi-simple iff the *quantum Euler class* $\mathcal{E}$ is invertible. But $\mathcal{E} \in QH_0(M)$. Let $S^n \approx L \subset M$. Under above assumptions, $HF_*(L) = H_*(L) \otimes \mathbb{F}$. Now use the module structure to deduce that $\mathcal{E} * (-)$ gives iso's $HF_*(L) \cong HF_{*-2n}(L)$ ... contradiction. $\square$

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This is a monotone Lagrangian torus with $N_L = 2$.

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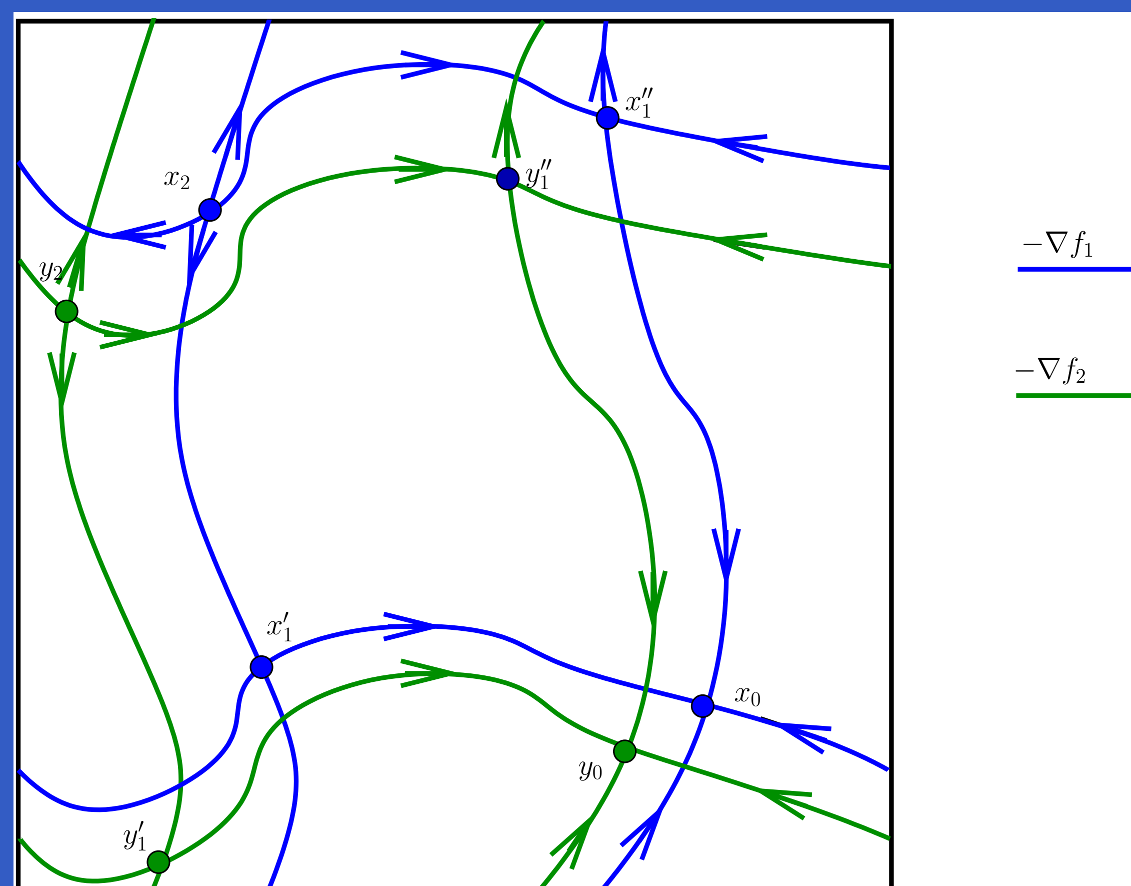

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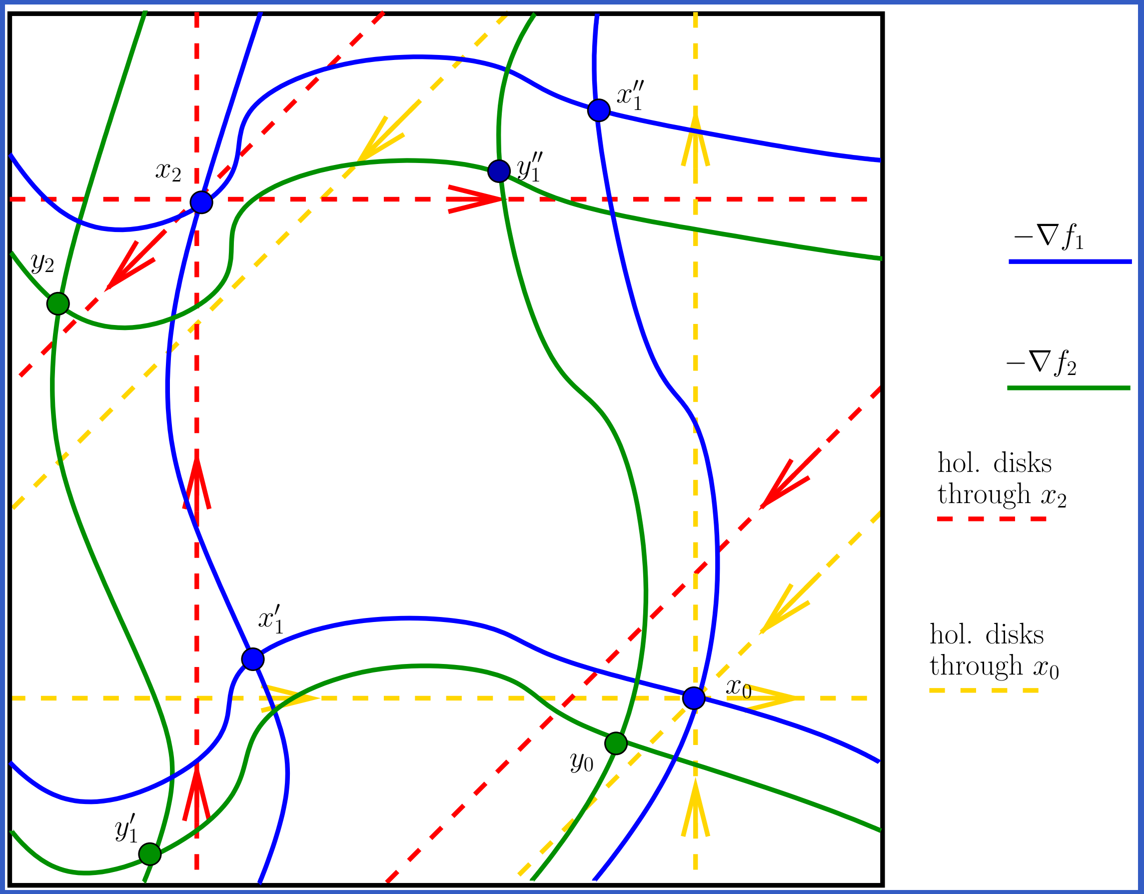

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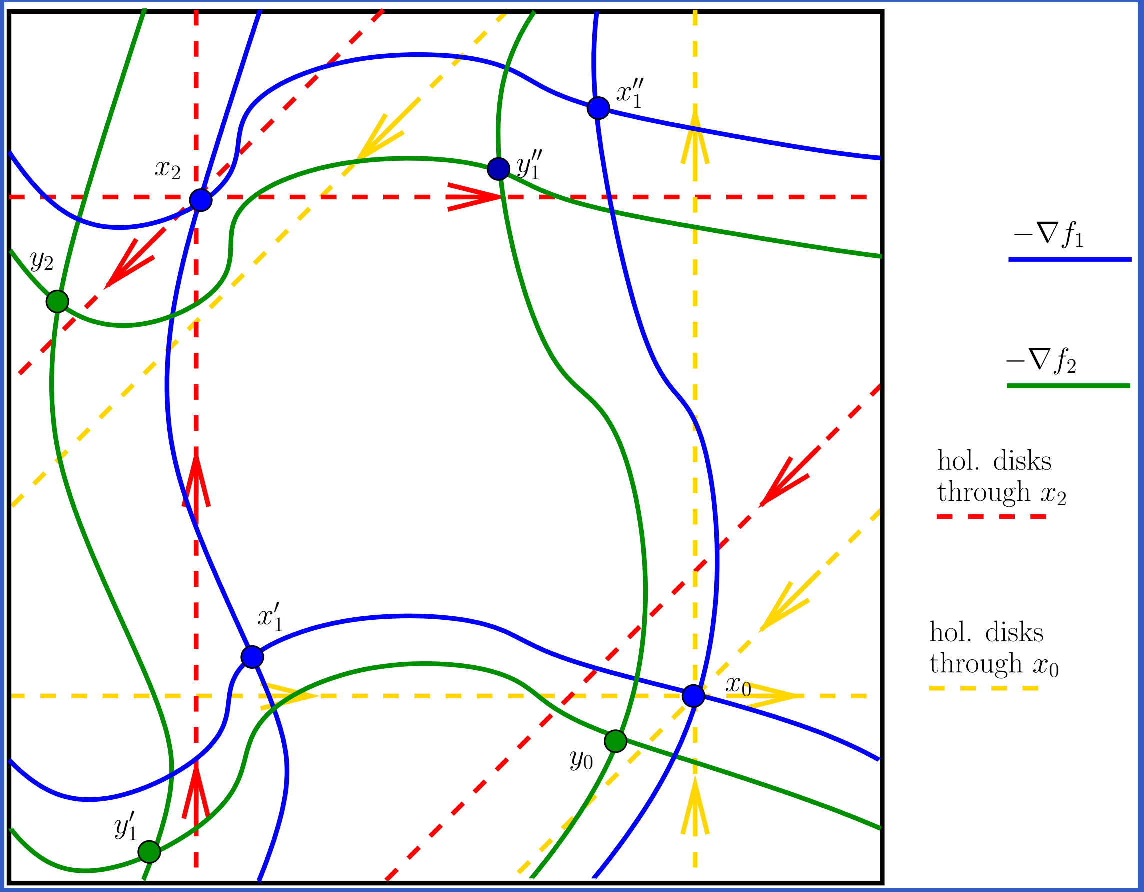
Computation of $m * m$ follows from associativity.



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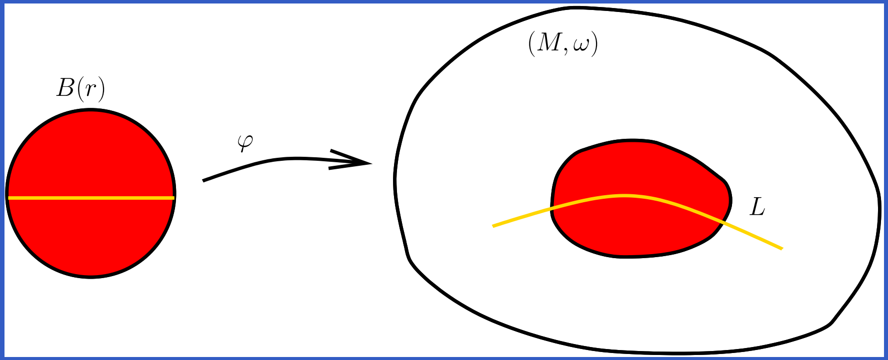

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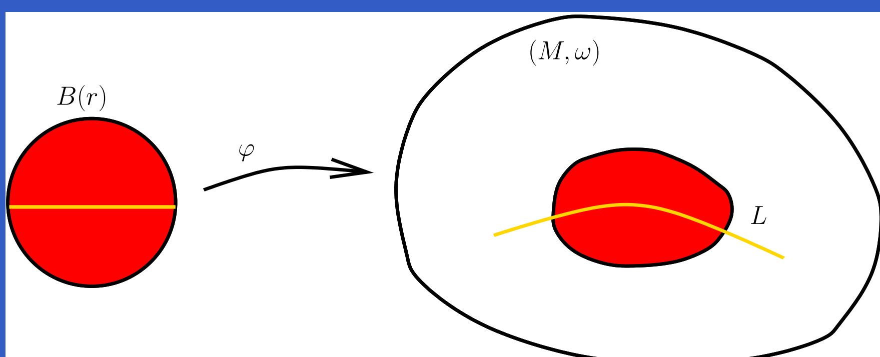

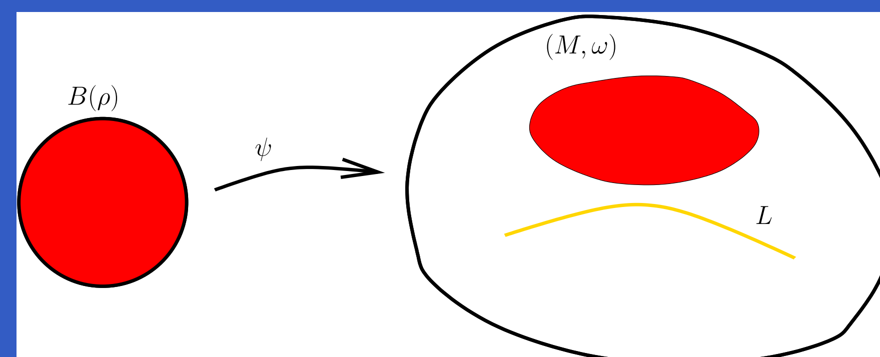

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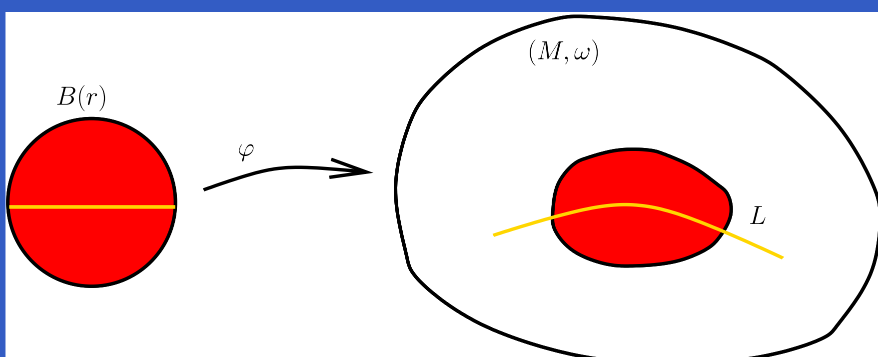

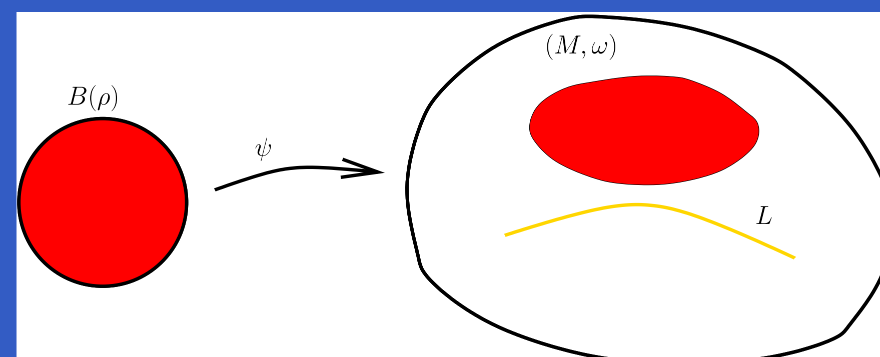

Absolute symplectic packing introduced by Gromov, studied further by McDuff-Polterovich, Karshon, Traynor, B. etc.

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Proof. Dichotomy for tori: either $HF_*(\mathbb{T}) \cong H_*(\mathbb{T}) \otimes \Lambda$ or $HF_*(\mathbb{T}) = 0$. In the latter case $\exists$ a J-holomorphic disk with $\mu = 2$ through $\forall\, pt \in \mathbb{T}$. $\square$

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Cor: $Gr(\mathbb{T}^n_{\text{clif}})^2 \leq \frac{2}{n+1}$, $\quad Gr(\mathbb{C}P^n \setminus \mathbb{T}^n_{\text{clif}})^2 = \frac{n}{n+1}$.

Cor: $L \subset \mathbb{C}P^n$ Lagrangian with $2H_1(L;\mathbb{Z}) = 0$. Then $Gr(\mathbb{C}P^n \setminus L)^2 \leq \frac{1}{2}$. (Previously known for $L = \mathbb{R}P^n$ [B.]).

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Enumerative geometry for $\mathbb{T}_{\text{clif}} \subset \mathbb{C}P^2$

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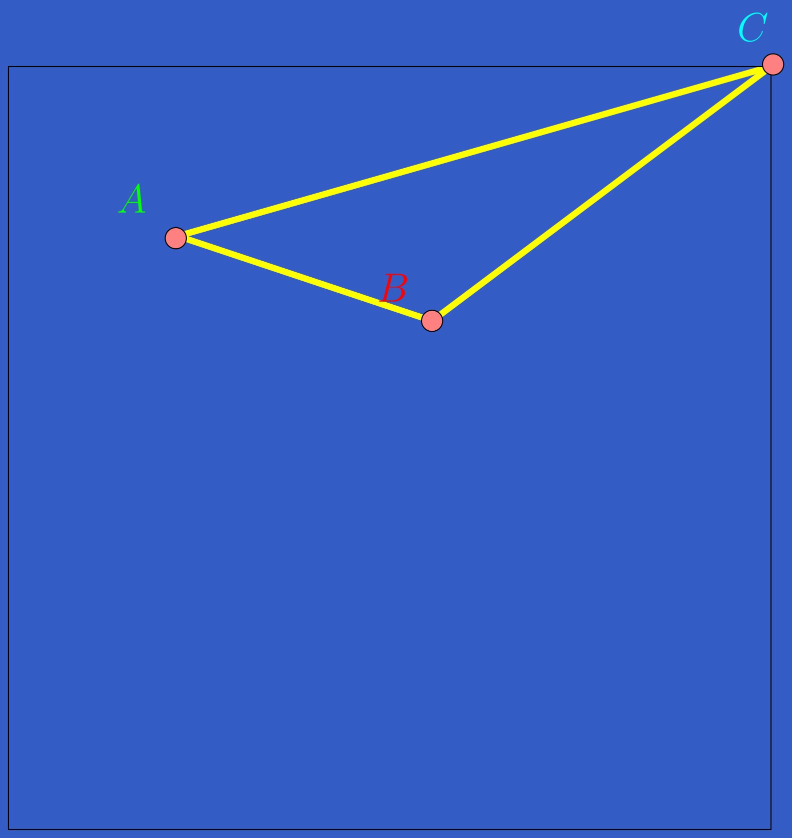
$\exists$ related work of Cho with other identities by different approach.

Holomorphic disks through 3 points on $\mathbb{T}_{\text{clif}}$



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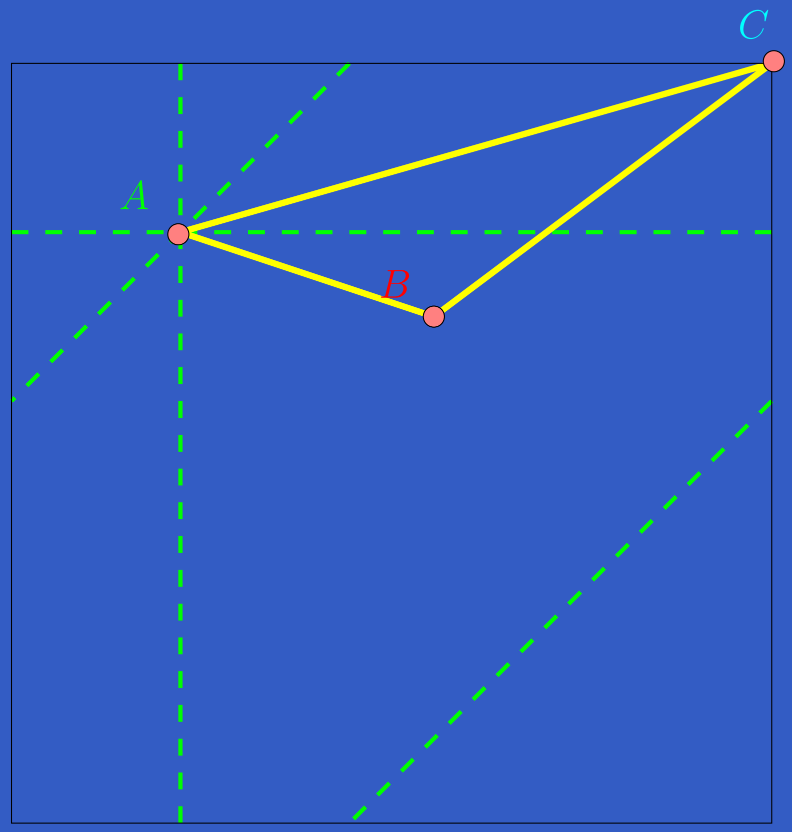

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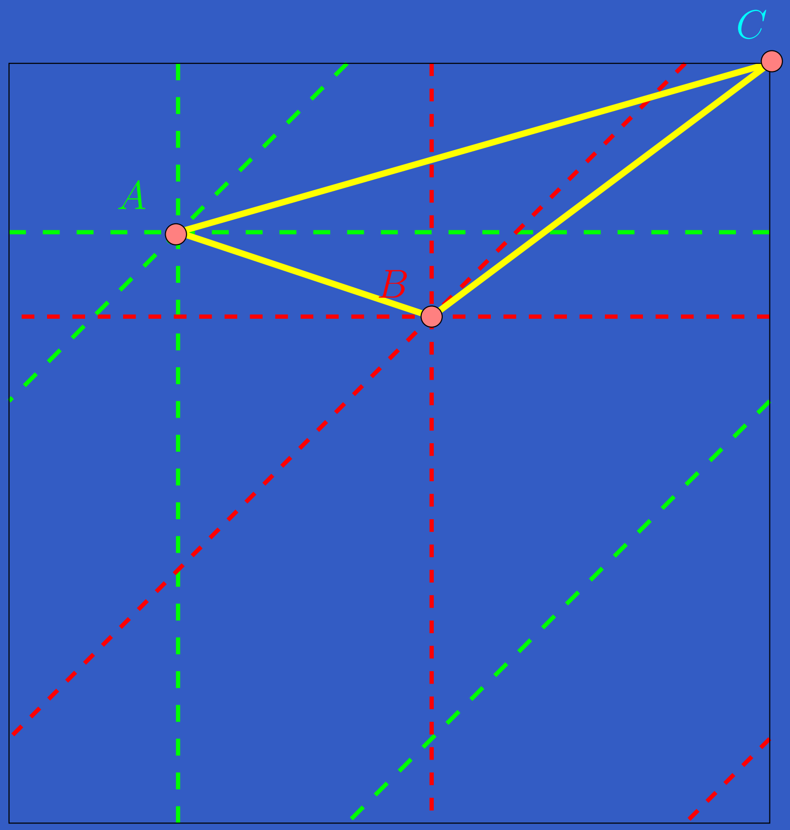

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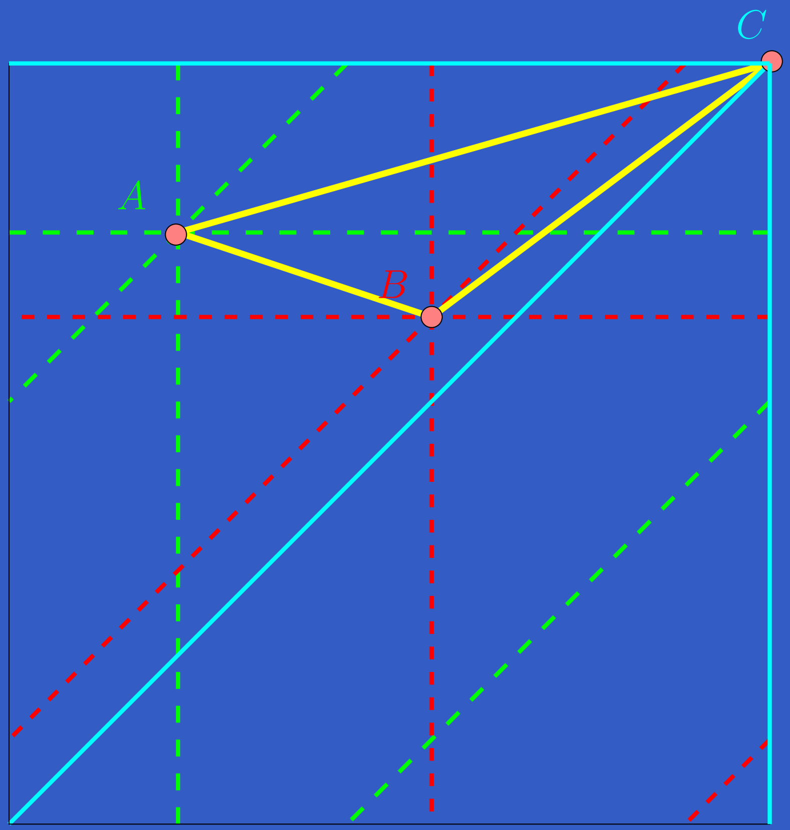

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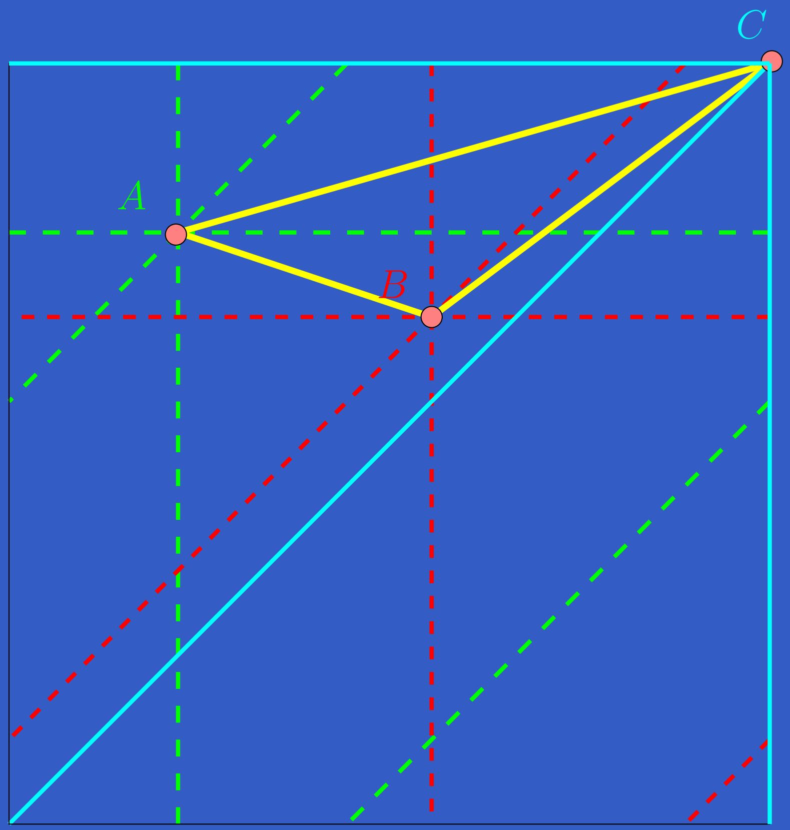

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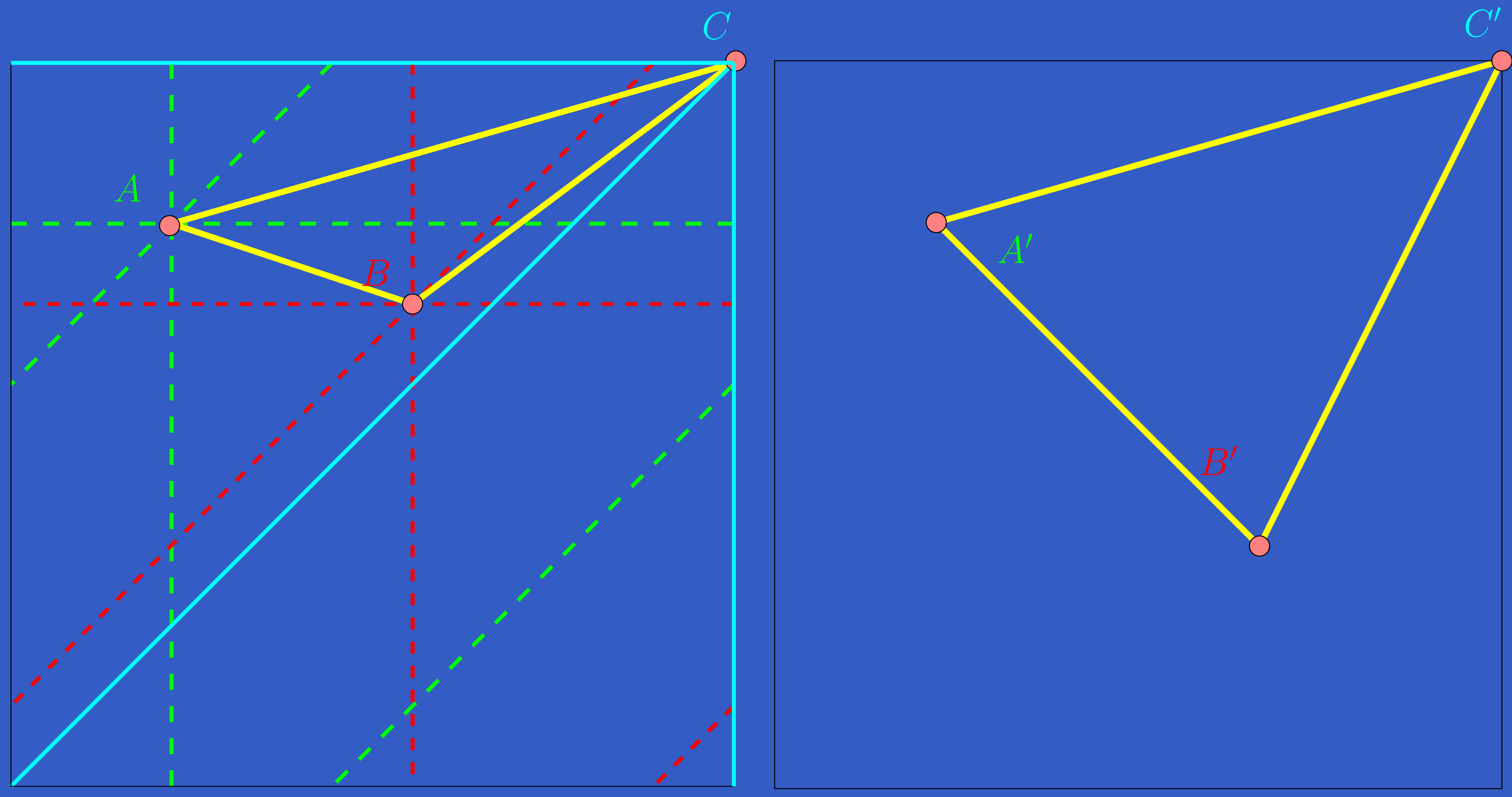

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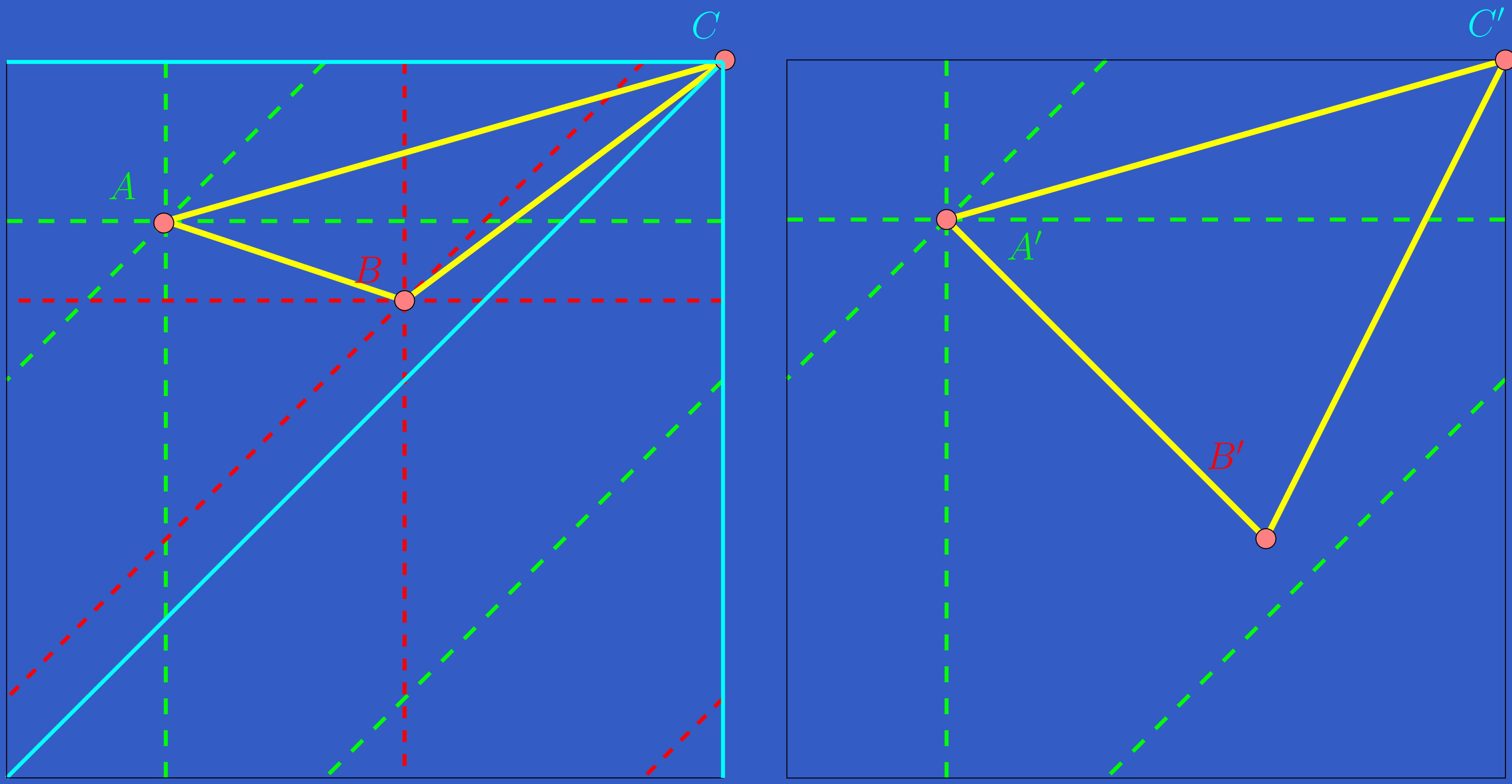

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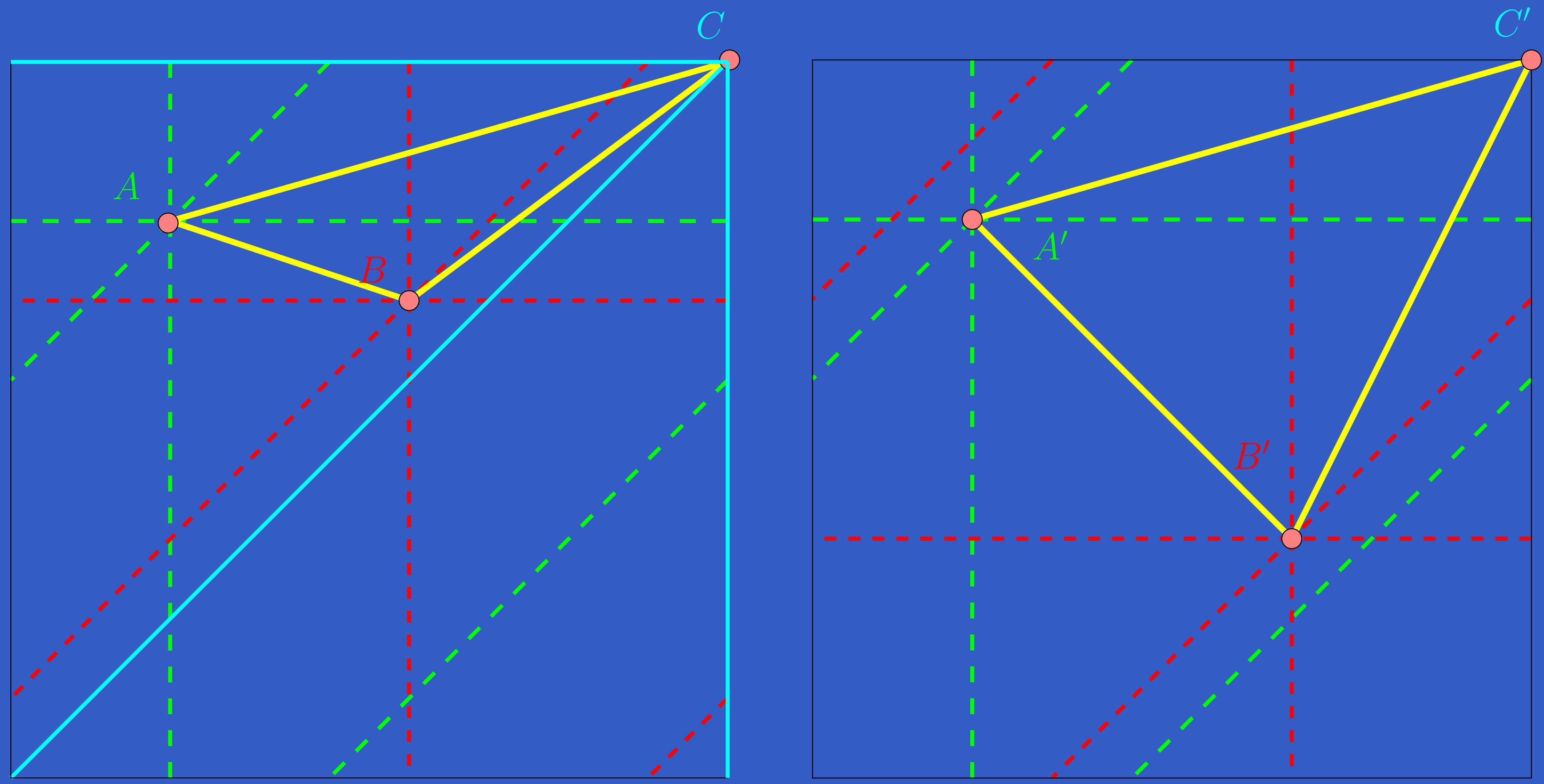

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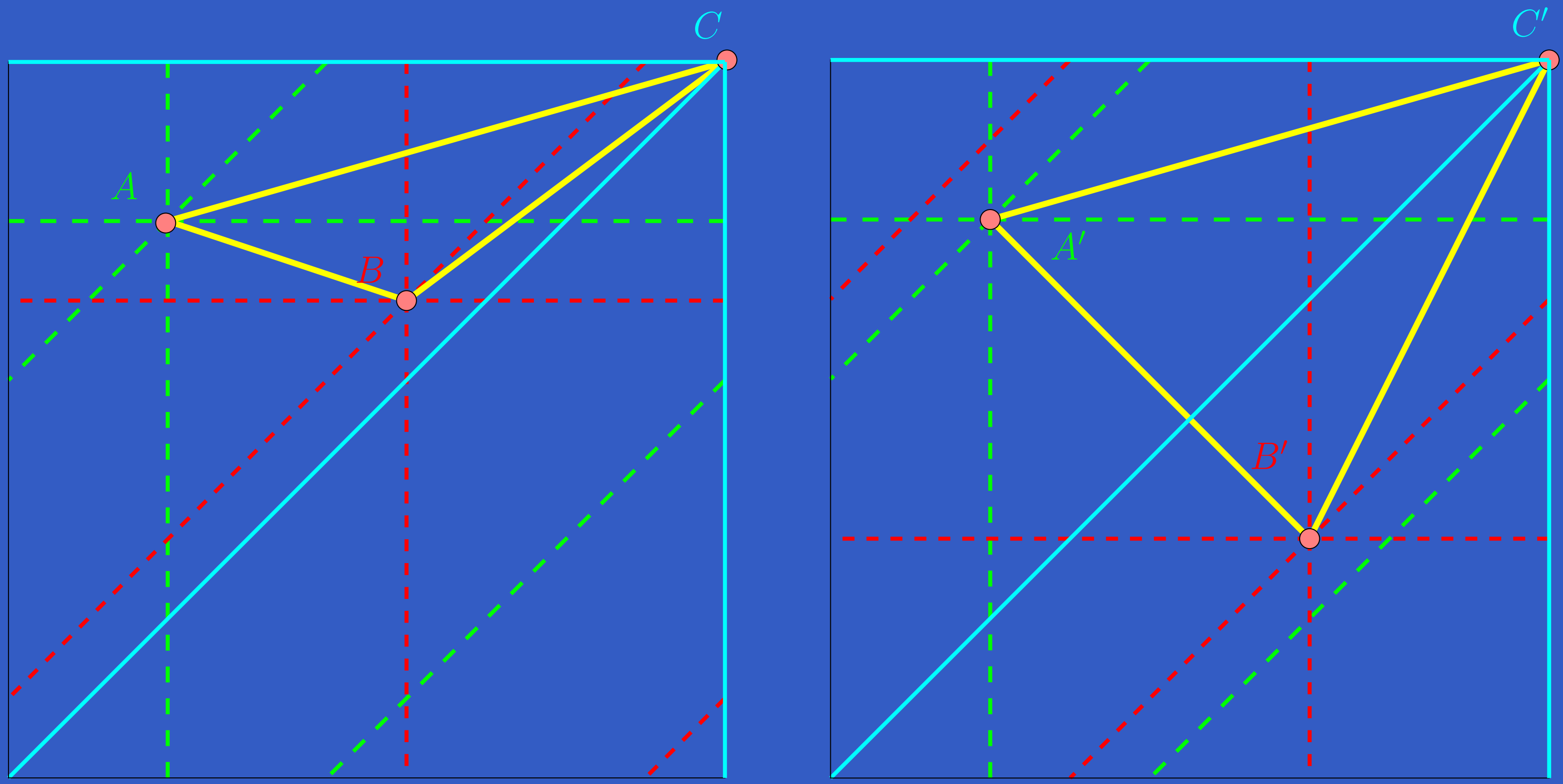

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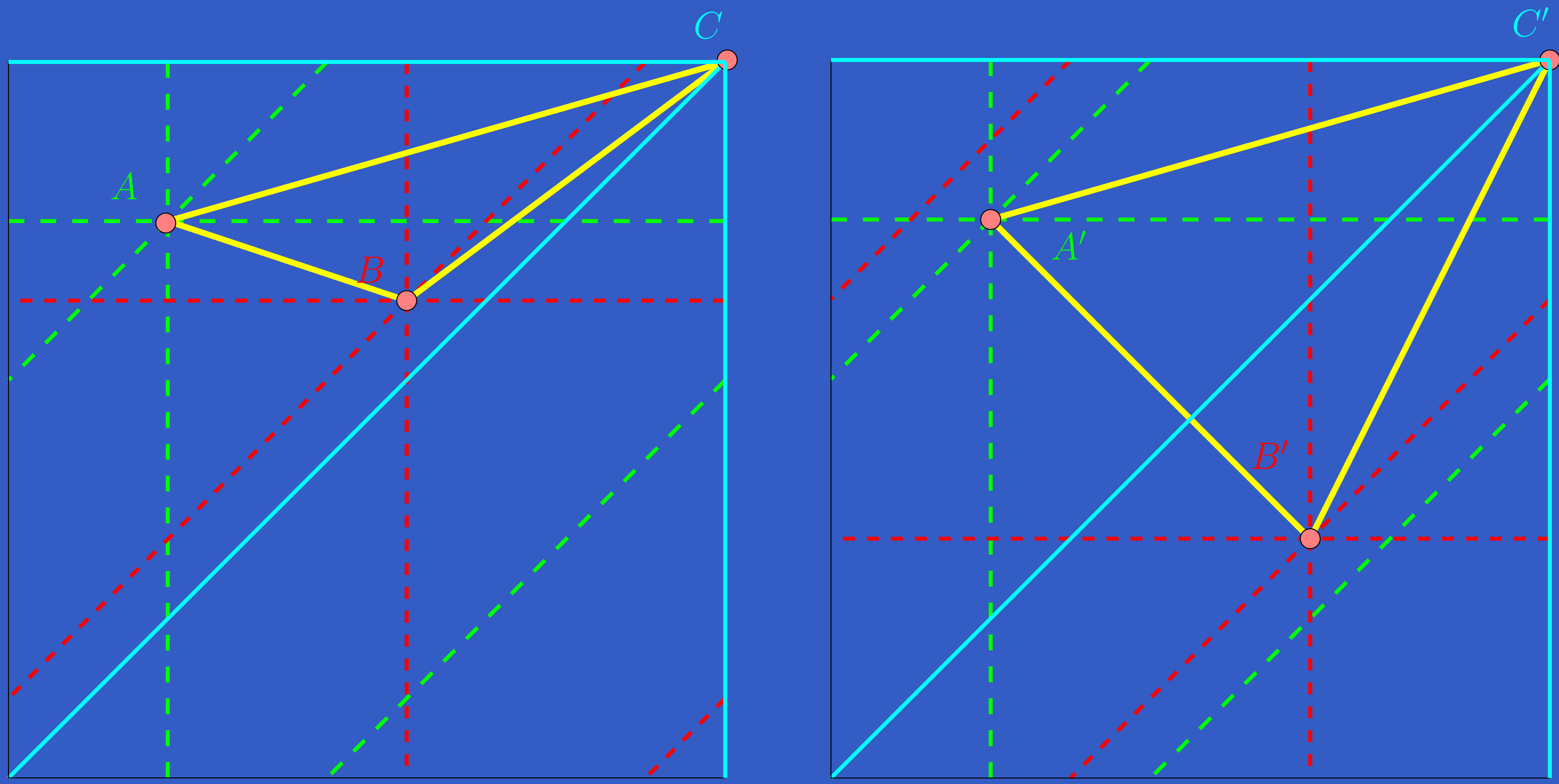

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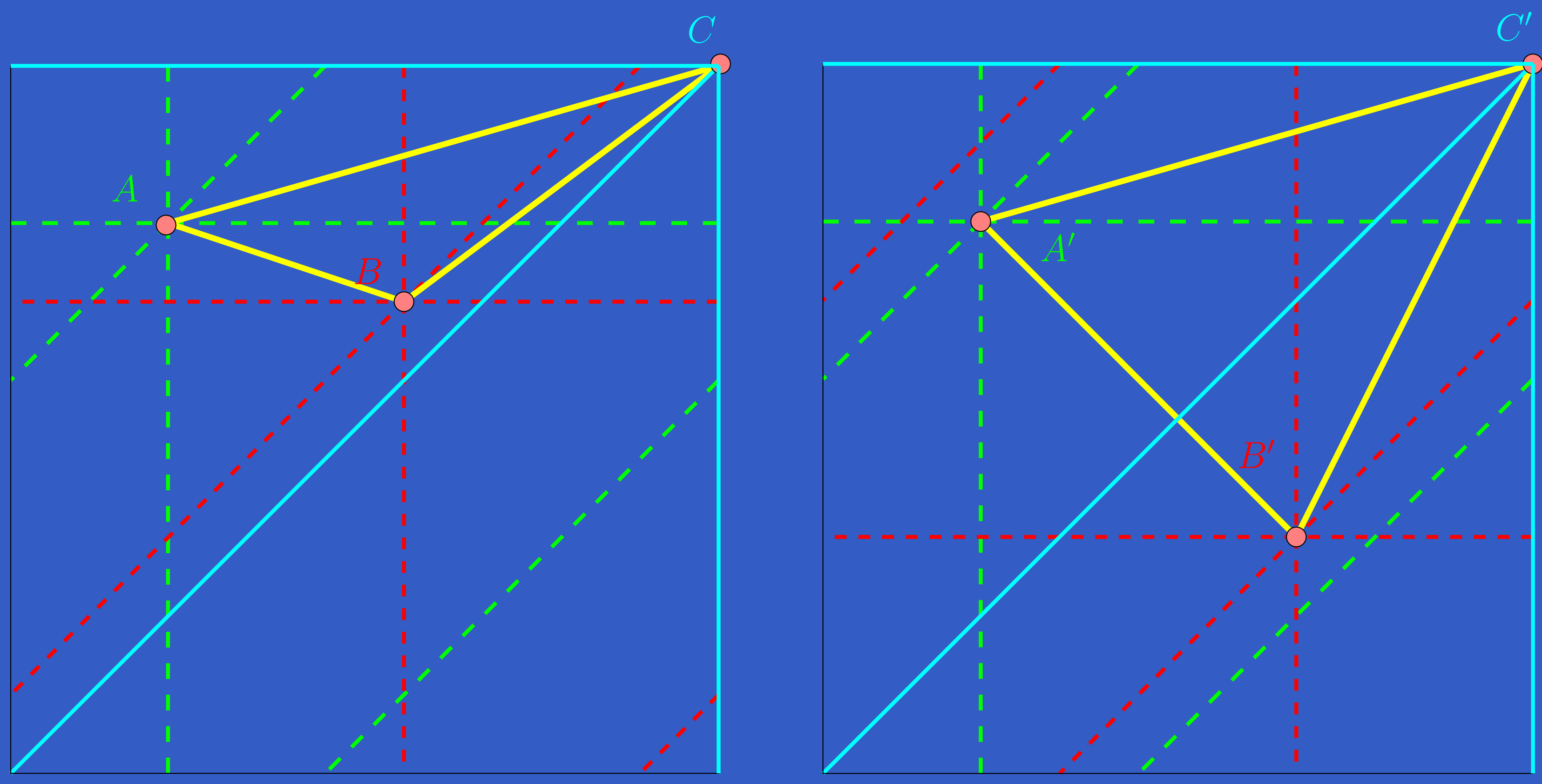

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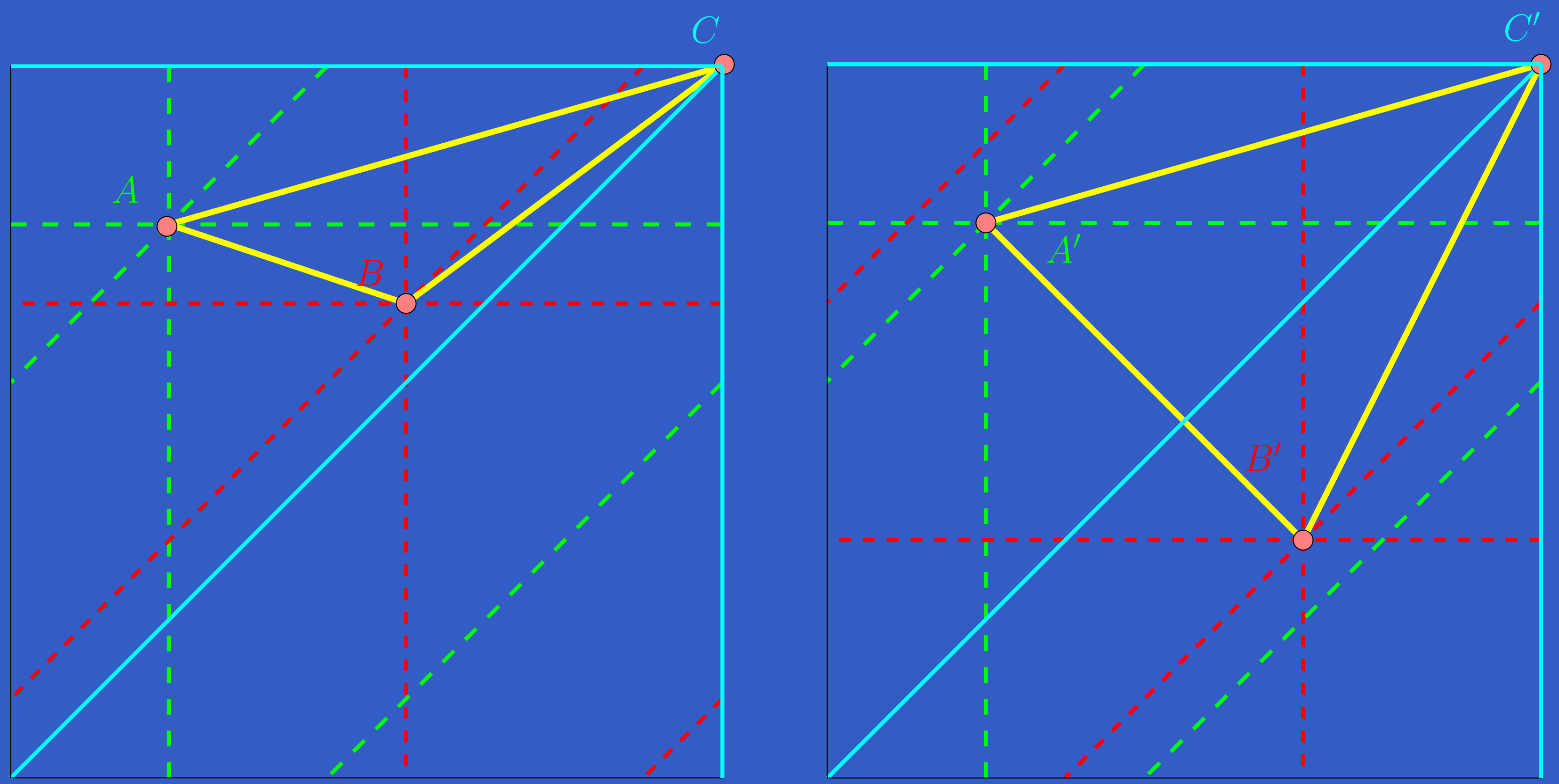

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Still ... $n_A n_B + n_4(T)$ *is a symplectic invariant*.

What next?

- Extend the theory to the A_∞-category theory of Fukaya-Oh-Ohta-Ono or to the cluster homology of Cornea-Lalonde. This would also get rid of the monotonicity assumption. This is future project planned with Cornea and Lalonde.

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- Replace QH with contact homology and structures coming from SFT. $\rightsquigarrow HF(L)$ being a module over richer algebraic objects.
- If the above works, we get a Floer homological approach to relative/real enumerative geometry. We would also get more complete picture of the relative packing problem.

HAPPY BIRTHDAY YASHA

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Till 120!