Spectral theory and frame flows on Kaehler manifolds

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Motivation

- Quantum dynamics and classical dynamics are linked via Bohr's correspondence principle
- Why at all Kähler manifolds?

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The theorem of Shnirelman, Colin de Verdiere and Zelditch

If X is a Riemannian manifold and ϕ_i and orthonormal basis with

$$\Delta\phi_i = \lambda_i\phi_i, \quad \lambda_i \nearrow \infty$$

then up to a subsequence of counting density zero

$$\langle \phi_i, A\phi_i \rangle \to \int_{T_1 X} \sigma_A(\xi) d\xi.$$

if the geodesic flow on $T_1 X$ is ergodic.

The Dirac operator

Theorem (Jakobson, AS (06))

If X is a Riemannian spin manifold with spinor bundle S and Dirac operator D and $(\phi_i)_{i\in\mathbb{Z}}$ is an orthonormal basis of eigensections with

$$D\phi_i = \lambda_i\phi_i, \quad \lambda_i \nearrow \infty$$

then up to a subsequence of counting density zero

$$\langle\phi_i, A\phi_i\rangle \to C_n \int_{T_1 X} \mathrm{tr}\left((1+\gamma_\xi)\sigma_A(\xi)\right) d\xi.$$

if the frame flow on FX is ergodic (e.g. if the curvature is not too far away from being constant).

The Laplace Beltrami operator on *p*-forms

Theorem (Jakobson, AS (06))

If X is a Riemannian manifold and Δ_p the Laplace operator on p-forms. Suppose that a complete sequence of co-closed eigenforms ϕ_i such that

$$\delta\phi_i = 0$$
$$\Delta_p \phi_i = \lambda_i \phi_i, \quad \lambda_i \nearrow \infty.$$

If the frame flow on FX is ergodic and if $p \neq \frac{n-1}{2}$ then up to a subsequence of counting density zero

$$\langle \phi_i, A\phi_i \rangle \to \omega_p(\sigma_A).$$

Kähler manifolds

What happens then if the frame flow is not ergodic, but the geodesic flow is? This is the case if the manifold is negatively curved and Kähler. A complex Riemannian manifold is called Kähler manifold if the complex structure J is covariantly constant. The functions $\langle e_i, Je_k \rangle$ are then conserved quantities under the frame flow. However,

Theorem (Brin, Gromov (1980))

In negative curvature and complex dimension 2 or in odd complex dimension the restricted frame flow is ergodic.

Properties of Kähler manifolds

The complex structure gives rise to a decomposition of the complexified tangent bundle into holomorphic and anti-homolomorphic parts. In this way one gets an additional bi-grading of the bundle of forms.

$$\Lambda X = \oplus_{p,q} \Lambda^{p,q} X$$

and this is preserved under the parallel transport. Moreover, with

$$d = \partial + \overline{\partial},$$

one has

$$\Delta = 2(\partial\partial^* + \partial^*\partial) = 2(\bar{\partial}\bar{\partial}^* + \bar{\partial}^*\bar{\partial})$$

so the bi-grading is also preserved by the Laplace operator. One gets different quantum limits.

The full symmetry

Exterior multiplication with the symplectic 2-form defines an operator L, which generates an action of $sl_2(\mathbb{C})$ on the space of differential forms commuting with the Laplace-Beltrami operator. Together with ∂ and $\overline{\partial}$ this generates a graded star Lie algebra with central element Δ

$$[L,\bar{\partial}^*] = -i\partial,\ [L^*,\partial] = i\bar{\partial}^*,\ [L^*,\bar{\partial}] = -i\partial^*,\ [L,\partial^*] = i\bar{\partial},$$
$$[L^*,L] = H,\ [H,L] = -2L,\ [H,L^*] = 2L^*,$$
$$\{\partial,\partial\} = \{\bar{\partial},\bar{\partial}\} = \{\partial^*,\partial^*\} = \{\bar{\partial}^*,\bar{\partial}^*\} = 0,$$
$$[L,\bar{\partial}] = [L,\partial] = [L^*,\bar{\partial}^*] = [L^*,\partial^*] = 0,$$
$$\{\partial,\partial^*\} = \{\bar{\partial},\bar{\partial}^*\} = \frac{1}{2}\Delta.$$

The full symmetry

On the orthogonal complement of the space of harmonic forms we therefore have a representation of the star algebra $\mathcal{U}(\mathfrak{g})$ given by

$$
\begin{gathered}
[L, \bar{a}^*] = -ia,\ [L^*, a] = i\bar{a}^*,\ [L^*, \bar{a}] = -ia^*,\ [L, a^*] = i\bar{a},\\
[L^*, L] = H,\ [H, L] = -2L,\ [H, L^*] = 2L^*,\\
\{a, a\} = \{\bar{a}, \bar{a}\} = \{a^*, a^*\} = \{\bar{a}^*, \bar{a}^*\} = 0,\\
[L, \bar{a}] = [L, a] = [L^*, \bar{a}^*] = [L^*, a^*] = 0,\\
\{a, a^*\} = \{\bar{a}, \bar{a}^*\} = 1,\\
\{\bar{a}, a^*\} = \{a, \bar{a}^*\} = 0.
\end{gathered}
$$

What is the correct gauge condition?

Complication: the $sl(2, \mathbb{C})$ action does not commute with the action of the $*$-algebra generated by ∂ and $\bar{\partial}$. For example any primitive closed form is harmonic.
Solution: Lefschetz operator only in direction transversal to the flow:

$$L_t = L - 2i\Delta^{-1}\bar{\partial}\partial$$

and then L_t, L_t^* generate an $sl(2, \mathbb{C})$ action by pseudodifferential operators commuting with ∂ and $\bar{\partial}$. This makes it possible to study the representation theory of this graded Lie algebra and guess the correct gauge condition. For example any irreducible representation of $\mathcal{U}(\mathfrak{g})$ decomposes under the action of the Lefschetz $sl_2(\mathbb{C})$ action into 4 irreducible representation.

Result 1

There is a model representation of this graded Lie algebra $\mathcal{U}(\mathfrak{g})$ on $\Lambda\mathbb{C}^m$:

$$a\alpha = z_1 \wedge \alpha,$$
$$\bar{a}\alpha = \bar{z}_1 \wedge \alpha,$$
$$L\alpha = \left(\frac{i}{2} \sum_{k=1}^{m} z_k \wedge \bar{z}_k \right) \wedge \alpha, \ldots$$

Theorem: Asymptotically, the proportions of irreps $C^\infty(\Lambda X)$ is equal to those in the model representation. Representations with high weights do not occur.

Result 2

If X is a compact Kähler manifold and ϕ_i is a complete set of co-closed primitive (p, q)-forms with

$$\Delta\phi_i = \lambda_i\phi_i, \quad \lambda_i \nearrow \infty$$

then up to a subsequence of counting density zero

$$\langle\phi_i, A\phi_i\rangle \to \omega_\infty(\sigma_A)$$

if the restricted frame flow on UX is ergodic. Here $\omega_\infty(\sigma_A)$ is a state on the matrix valued functions on T_1X that can be explicitly given.

Remarks

- We have actually determined a full ergodic decomposition of the tracial state on T_1X. But formulas are more complicated. This means we have classified all possible gauge conditions that lead to quantum ergodicity results.
- Apart from the dynamics this shows that the quantization of J leads naturally to supersymmetry.
- A similar result can be proved for the Dirac operator of any spin^c-structure. Here as well there is an additional gauge condition appearing.

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