Mixing and waves: Part II
Beyond the Ehrenfest time

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Recall: Ehrenfest time

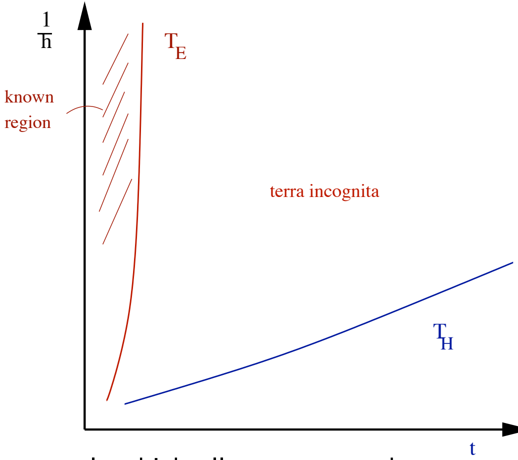
- Ehrenfest time

$$T_E \sim \frac{1}{\lambda} \ln \frac{1}{\hbar}$$

-exponential proliferation of orbits,
-small-scale oscillations

- Heisenberg time, time scale to resolve spectrum:

$$T_H \sim \frac{1}{\hbar^{d-1}}$$



Main aim in this talk is to present tools which allow to prove the accuracy of semiclassical approximations up to

$$t \sim 1/\sqrt{\hbar}$$

Poincare disc

Let $\mathbb{D} := \{z \in \mathbb{C} \; ; \; |z| < 1\}$ be the unit disk with

- metric: $\mathrm{d}s^2 = \frac{4|\mathrm{d}z|^2}{(1-|z|^2)^2}$.
- isometries: $\gamma(z) = \frac{\alpha z+\beta}{\beta^* z+\alpha^*}$ with $|\alpha|^2 - |\beta|^2 = 1$
- Laplace Beltrami operator

$$\Delta = \frac{(1-|z|^2)^2}{4}(\partial_x^2 + \partial_y^2)$$

  commutes with isometries, i.e., $(\Delta u) \circ \gamma = \Delta(u \circ \gamma)$.
- time evolution operator

$$\mathcal{U}(t) = \mathrm{e}^{\frac{\mathrm{i}\hbar}{2} t\Delta}$$

  commutes with isometries, i.e., $(\mathcal{U}(t)u) \circ \gamma = \mathcal{U}(t)(u \circ \gamma)$.

Surfaces of constant negative curvature

- Any surface of constant negative curvature is of the form

$$M = \mathbb{D}/\Gamma$$

where Γ is a discrete group of isometries, a *Fuchsian group*.

- $u : \mathbb{D} \to \mathbb{C}$ is a function on M if $u \circ \gamma = u$ for all $\gamma \in \Gamma$
- For $u : \mathbb{D} \to \mathbb{C}$ we set

$$u_\Gamma := \sum_{\gamma \in \Gamma} u \circ \gamma^{-1}$$

this is a function on M if the sum converges. Note

$$\mathcal{U}(t) u_\Gamma = \big(\mathcal{U}(t) u\big)_\Gamma$$

Exponential volume growth

- On $\mathbb{D}$ we have $\mathrm{vol}(\{d(z,0) \leq r\}) \sim \mathrm{e}^r$ so for compact M

$$\#\{\gamma \in \Gamma\,;\, d(\gamma(z), z) \leq r\} \sim \mathrm{e}^r$$

  hence u_Γ converges if for $\beta > 1$

$$|u(z)| \ll \mathrm{e}^{-\beta d(z,0)}\,.$$

- For $\alpha, \beta > 0$, $\langle d \rangle(z) = (1 + d(z,0)^2)^{1/2}$ set

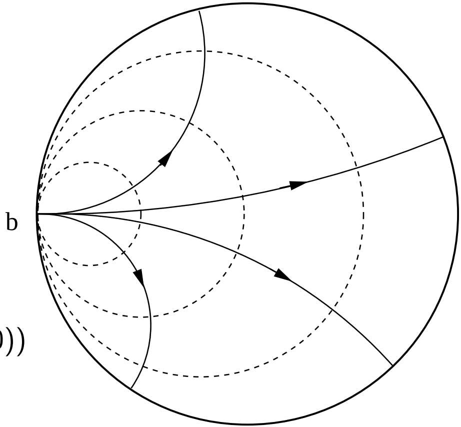
$$\|a\|_{\alpha,\beta} = \|\mathrm{e}^{\alpha\sqrt{-\Delta}}\mathrm{e}^{\beta\langle d\rangle} a\|_{L^2(\mathbb{D})}\,, \quad H_{\alpha,\beta} := \{a : \mathbb{D} \to \mathbb{C}\,;\, \|a\|_{\alpha,\beta} < \infty\}$$

  $\alpha > 0 \to a$ real analytic, $\beta > 1 \to a_\Gamma$ converges

Geodesics and horocycles

Geodesics: circles perpendicular to $\partial\mathbb{D}$

Horocycles: circles tangent to $\partial\mathbb{D}$, let $\xi(b, z)$ be the horocycle through $z \in \mathbb{D}$ touching $\partial\mathbb{D}$ at $b \in \partial\mathbb{D}$



Fix $b \in \partial\mathbb{D}$ and set

$$\varphi_b(z) = \text{signed } d(\xi(b, z), \xi(b, 0))$$

and let furthermore $\Phi_b^t(z)$ be a distance t shift along the geodesic from b through z.

Hyperbolic plane waves

The initial states we propagate are of the form u_Γ with

$$u(z) = a(z)\mathrm{e}^{\frac{\mathrm{i}}{\hbar}\varphi_b(z)}$$

- these are **Lagrangian states** with $\Lambda_b := \{(z, d\varphi_b(z)), z \in \mathbb{D}\}$ an **unstable manifold** of the geodesic flow.
- The wavefronts $\varphi_b(z) = const.$ are the horocycles associated with b.
- with $\pi : T^*\mathbb{D} \to \mathbb{D}$ let $\pi_b := \pi|_{\Lambda_b} : \Lambda_b \to \mathbb{D}$, then $\Phi_b^t = \pi_b \phi^t \pi_b^{-1}$ with geodesic flow ϕ^t.

Define the unitary operator $S_b(t)$ on $L^2(\mathbb{D})$ by

$$S_b(t)a(z) = \mathrm{e}^{-t/2}a(\Phi_b^{-t}(z))$$

it describes transport along the geodesics emanating from b.

main result

For $a \in H_{\alpha.\beta}$ set

$$u^{(0)}(t) = \mathrm{e}^{-\frac{\mathrm{i}}{\hbar}\frac{t}{2}} \big(S_b(t)a\big)\, \mathrm{e}^{\frac{\mathrm{i}}{\hbar}\varphi_b} \ .$$

- Leading order semiclassical approximation for $\mathcal{U}(t)[a\mathrm{e}^{\frac{\mathrm{i}}{\hbar}\varphi_b}]$
- since the effective support of $S_b(t)a$ grows exponentially, proliferation of overlaps between $(S_b(t)a)\circ\gamma$ and $(S_b(t)a)\circ\gamma'$ gives $\|u^{(0)}(t)_\Gamma\| \sim \mathrm{e}^{t/2}$.

Theorem

Let $M = \mathbb{D}/\Gamma$, then for $\alpha > 0, \beta > 1$ there exist constants $C > 0$, $\delta > 0$ such that for all $a \in H_{\alpha,\beta}(\mathbb{D})$ and $b \in \partial\mathbb{D}$,

$$\left\| u^{(0)}(t)_\Gamma - \mathcal{U}(t)[a\,\mathrm{e}^{\frac{\mathrm{i}}{\hbar}\varphi_b}]_\Gamma \right\|_{L^2(M)} \leq C\|a\|_{\alpha,\beta}\, t\sqrt{\hbar}$$

for

$$0 \leq t \leq \delta\frac{1}{\sqrt{\hbar}} \ .$$

Main tool: decomposition of $\mathcal{U}(t)$

Set

$$\Delta_b(t) := S_b^*(t)\Delta S_b(t)$$

and let $V_b(t)$ be the solution of

$$\mathrm{i}\partial_t V_b(t) = -\frac{\hbar}{2}\Delta_b(t)V_b(t) \quad \text{with } V_b(0) = I$$

Theorem

For $a \in L^2(\mathbb{D})$ we have

$$\mathcal{U}(t)\big(a\mathrm{e}^{\frac{\mathrm{i}}{\hbar}\varphi_b}\big) = \mathrm{e}^{-\frac{\mathrm{i}}{\hbar}\frac{t}{2}}\big(S_b(t)V_b(t)a\big)\,\mathrm{e}^{\frac{\mathrm{i}}{\hbar}\varphi_b}$$

Remarks:

- proof follows by inserting into Schrödinger equation
- interpretation:
  - $S_b(t)$ is classical transport
  - $V_b(t)$ describes dispersion

Dispersive part

In upper halfplane with $b = \mathrm{i}\infty$ we have

- $\Delta = y^2(\partial_y^2 + \partial_x^2)$
- $S_b(t)a(x,y) = \mathrm{e}^{-t/2}a(x, \mathrm{e}^t y)$
- $\Delta_b(t) = y^2\partial_y^2 + \mathrm{e}^{-2t}y^2\partial_x^2$

Note the Volterra expansion

$$V_b(t)a = a + \frac{\hbar}{2\mathrm{i}}\int_0^t \Delta_b(t_1)a\,\mathrm{d}t_1 + \cdots + \frac{\hbar^k}{(2\mathrm{i})^k}\int_0^t \cdots \int_0^{t_{k-1}} \Delta_b(t_1)\cdots\Delta_b(t_k)V_b(t_k)a\,\mathrm{d}t_1\cdots\mathrm{d}t_k$$

so if a has bounded derivatives $V_b(t)$ describes dispersion on a scale $\hbar t$ along the geodesics emanating from b.

Main idea: Turning S into V with $\mathcal{U}$

We want to estimate $\|u^{(0)}(t)_\Gamma\|_{L^2(M)}$ using

$$\mathcal{U}(t)\big(a\mathrm{e}^{\frac{\mathrm{i}}{\hbar}\varphi_b}\big) = \mathrm{e}^{-\frac{\mathrm{i}}{\hbar}\frac{t}{2}}\big(S_b(t)V_b(t)a\big)\,\mathrm{e}^{\frac{\mathrm{i}}{\hbar}\varphi_b}$$

Since V_b is unitary

$$\begin{aligned} u^{(0)}(t) &= \mathrm{e}^{-\frac{\mathrm{i}}{\hbar}\frac{t}{2}}\big(S_b(t)a\big)\,\mathrm{e}^{\frac{\mathrm{i}}{\hbar}\varphi_b} \\ &= \mathrm{e}^{-\frac{\mathrm{i}}{\hbar}\frac{t}{2}}\big(S_bV_bV_b^*a\big)\,\mathrm{e}^{\frac{\mathrm{i}}{\hbar}\varphi_b} \\ &= \mathcal{U}(t)\big([V_b^*a]\mathrm{e}^{\frac{\mathrm{i}}{\hbar}\varphi_b}\big) \end{aligned}$$

but $\mathcal{U}(t)$ commutes with the action of Γ and is unitary, so

$$\|u^{(0)}(t)_\Gamma\|_{L^2(M)} = \Big\|\big([V_b^*a]\mathrm{e}^{\frac{\mathrm{i}}{\hbar}\varphi_b}\big)_\Gamma\Big\|_{L^2(M)}$$

RHS contains no transport, but dispersion instead. Supposed to scale with $\hbar t$

use same idea for remainder

Using $u^{(0)}(t) = \mathcal{U}(t)\big([V_b^* a]\mathrm{e}^{\frac{\mathrm{i}}{\hbar}\varphi_b}\big)$ gives

$$u^{(0)}(t) - \mathcal{U}(t)(a\mathrm{e}^{\frac{\mathrm{i}}{\hbar}\varphi_b}) = \mathcal{U}(t)\big([V_b^* a]\mathrm{e}^{\frac{\mathrm{i}}{\hbar}\varphi_b}\big) - \mathcal{U}(t)(a\mathrm{e}^{\frac{\mathrm{i}}{\hbar}\varphi_b})$$
$$= \mathcal{U}(t)\big([V_b^* a - a]\mathrm{e}^{\frac{\mathrm{i}}{\hbar}\varphi_b}\big)$$

but from Schrödinger equation for V_b

$$V_b^* a - a = \frac{\hbar}{2\mathrm{i}} \int_0^t V_b^*(t')\Delta_b(t') a \, \mathrm{d}t' \ ,$$

so

$$\|u^{(0)}(t)_\Gamma - \mathcal{U}(t)(a\mathrm{e}^{\frac{\mathrm{i}}{\hbar}\varphi_b})_\Gamma\|_{L^2} \ll \hbar \int_0^t \big\|\big[\big(V_b^*(t')\Delta_b(t')a\big)\mathrm{e}^{\frac{\mathrm{i}}{\hbar}\varphi_b}\big]_\Gamma\big\|_{L^2(M)} \, \mathrm{d}t'$$

So we need estimates on $V_b^*(t)a$ which ensure convergence of the sum over Γ.

Dispersive estimates

Main question: under which conditions on a do we have

$$|V_b^*(t)a(z)| \ll \mathrm{e}^{-\beta\langle d\rangle(z)}$$

Model problem: $U(t) = \mathrm{e}^{\mathrm{i}\frac{\hbar t}{2}\Delta}$ on $\mathbb{R}$, by Paley Wiener

$$|U(t)a(x)| \ll \mathrm{e}^{-\beta|x|}$$

if $\widehat{U(t)a}(\xi)$ is analytic in $|\mathrm{Im}\,\xi| < \beta$ and $\widehat{U(t)a}(\xi \pm \mathrm{i}\beta) \in L^1_\xi$. But

$$\widehat{U(t)a}(\xi) = \mathrm{e}^{\mathrm{i}\frac{\hbar t}{2}\xi^2}\hat{a}(\xi) \ ,$$

so

$$|\hat{a}(\xi \pm \mathrm{i}\beta)| \ll \mathrm{e}^{-\beta\hbar|t||\xi|}$$

So a has to be analytic and be quickly decaying.

Dispersive conjecture

This leads to

$$\|a\|_{\alpha,\beta} = \|e^{\alpha\sqrt{-\Delta}} e^{\beta\langle d\rangle} a\|_{L^2(\mathbb{D})} ,$$

and $H_{\alpha,\beta} := \{a : \mathbb{D} \to \mathbb{C} ; \|a\|_{\alpha,\beta} < \infty\}$.

Conjecture

For $\alpha, \beta > 0$ and $\alpha' < \alpha$ there exist $C, \delta > 0$ such that

$$\|V(t)a\|_{\alpha',\beta} \leq C\|a\|_{\alpha,\beta}$$
$$\|V^*(t)a\|_{\alpha',\beta} \leq C\|a\|_{\alpha,\beta}$$

for $a \in H_{\alpha,\beta}$ and

$$t \leq \delta \frac{\alpha - \alpha'}{\beta} \frac{1}{\hbar} .$$

Remark: Not hard to prove for $\mathcal{U}(t)$ instead of $V(t)$.

Mollifying

Let $\chi \in C^\infty(\mathbb{R})$ with $\operatorname{supp} \chi \in (-2, 2)$ and $\chi(x) = 1$ for $x \in [-1, 1]$ and set with $\varepsilon > 0$ $J_\varepsilon := \chi(\varepsilon\Delta)$. Then we can define

$$\Delta^{(\varepsilon)}(t) = J_\varepsilon \Delta(t) J_\varepsilon \ ,$$

$$\mathrm{i}\partial_t V^{(\varepsilon)}(t) = -\frac{\hbar}{2}\Delta^{(\varepsilon)}(t) V^{(\varepsilon)}(t) \quad \text{with } V^{(\varepsilon)}(0) = I$$

Then we have for $t \leq c\frac{\alpha-\alpha'}{\alpha}\frac{\varepsilon}{\hbar}$

$$\|V^{(\varepsilon)}(t)a\|_{\alpha',\beta} \leq C\|a\|_{\alpha,\beta} \qquad \|V^{(\varepsilon)^*}(t)a\|_{\alpha',\beta} \leq C\|a\|_{\alpha,\beta}$$

and

$$\|[(V^{(\varepsilon)}(t)V^*(t) - 1)a]_\Gamma\|_{L^2} \ll \|a\|_{\alpha,\beta} \mathrm{e}^{-\frac{1}{4}(\alpha/\varepsilon - 2t)}$$

The optimal choice for ε is then $\varepsilon \sim \sqrt{\hbar}$ which gives

$$t \ll 1/\sqrt{\hbar}$$

Remarks

- Using Volterra series for V_b one can can include higher order terms in $\hbar$ up to exponential small remainder. This allows as well to estimate accuracy in Sobolev norms, which gives pointwise estimates.
- **Localised states**: if $a \in H_{1,\beta}$ then $a\big((z - z_0)/\hbar^\delta\big) \in H_{\alpha,\beta}$ with $\alpha = \hbar^\delta$, we then obtain the conditions

$$t \ll \frac{1}{\hbar^{(1-3\delta)/2}} \,, \qquad \text{and} \quad \delta < \frac{1}{3}$$

To treat **coherent states**, i.e., $\delta = 1/2$, one would have to approximate $V_b(t)$ using Metaplectic operators. This would give $t \ll 1/\hbar^{1/4}$ and with Conjecture 1 $t \ll 1/\hbar^{1/2}$.

More general systems?

- Construction of $S(t)$ is geometrical, works for other phase-functions and non-constant curvature.
- Decomposition of $\mathcal{U}(t)$ into $S(t)$ and $V(t)$ works, too.
- Main problem: generator of $V(t)$,

$$\Delta(t) = S(t)^* \Delta S(t) \ ,$$

has coefficients which oscillate exponentially rapid.
Need to generalise analysis of dispersion.

Operators on $\mathbb{D}$

Symbol: use hyperbolic plane waves to define symbol (Zelditch 86)

$$A\mathrm{e}^{\left(\frac{\mathrm{i}}{\hbar}\lambda+\frac{1}{2}\right)\varphi_b(z)} = a(\hbar; z, b, \lambda)\mathrm{e}^{\left(\frac{\mathrm{i}}{\hbar}\lambda+\frac{1}{2}\right)\varphi_b(z)}$$

Helgason's harmonic analysis:

$$u(z) = \frac{1}{2\pi\hbar^2}\int_{\mathbb{R}^+\times\partial\mathbb{D}} \mathrm{e}^{\left(\frac{\mathrm{i}}{\hbar}\lambda+\frac{1}{2}\right)\varphi_b(z)}\mathcal{F}u(\lambda, b)\lambda\tanh\left(\frac{2\pi\lambda}{\hbar}\right)\mathrm{d}r\mathrm{d}b$$

$$\text{where}\quad \mathcal{F}u(\lambda, b) = \int_{\mathbb{D}} \mathrm{e}^{\left(-\frac{\mathrm{i}}{\hbar}\lambda+\frac{1}{2}\right)\varphi_b(z)}u(z)\ \mathrm{d}\nu(z)$$

Using this we define

$$\mathrm{Op}[a]u(z) = \frac{1}{2\pi\hbar^2}\int_{\mathbb{R}^+\times\partial\mathbb{D}} \mathrm{e}^{\left(\frac{\mathrm{i}}{\hbar}\lambda+\frac{1}{2}\right)\varphi_b(z)}a(z, b, \lambda)$$

$$\mathcal{F}u(\lambda, b)\lambda\tanh\left(\frac{2\pi\lambda}{\hbar}\right)\mathrm{d}r\mathrm{d}b\ .$$

We have

$$\mathcal{U}(t)\mathrm{e}^{\left(\frac{\mathrm{i}}{\hbar}\lambda+\frac{1}{2}\right)\varphi_b(z)} = \mathrm{e}^{-\frac{\mathrm{i}}{\hbar}\left(\lambda^2+\frac{1}{4}\right)}\mathrm{e}^{\left(\frac{\mathrm{i}}{\hbar}\lambda+\frac{1}{2}\right)\varphi_b(z)}$$

and so the symbol of $\mathcal{U}(t)\,\mathrm{Op}[a]\mathcal{U}^*(t)$ is

$$\begin{aligned}\mathcal{U}(t)\,\mathrm{Op}[a]\mathcal{U}^*(t)\mathrm{e}^{\left(\frac{\mathrm{i}}{\hbar}\lambda+\frac{1}{2}\right)\varphi_b} &= \mathrm{e}^{\frac{\mathrm{i}}{\hbar}\left(\lambda^2+\frac{1}{4}\right)}\mathcal{U}(t)\left(a\mathrm{e}^{\left(\frac{\mathrm{i}}{\hbar}\lambda+\frac{1}{2}\right)\varphi_b}\right)\\ &= \left(\hat{S}_{b,\lambda}(t)\hat{V}_{b,\lambda}a\right)\mathrm{e}^{\left(\frac{\mathrm{i}}{\hbar}\lambda+\frac{1}{2}\right)\varphi_b}\end{aligned}$$

where

- $\hat{S}_{b,\lambda}(t) = \mathrm{e}^{-\frac{1}{2}\varphi_b}S_b(\lambda t)\mathrm{e}^{\frac{1}{2}\varphi_b}$
- $\hat{V}_{b,\lambda}$ is generated by $\hat{\Delta}_{b,\lambda} := \mathrm{e}^{-\frac{1}{2}\varphi_b}S_b^*(\lambda t)\Delta S_b(\lambda t)\mathrm{e}^{\frac{1}{2}\varphi_b}$.

Symbol of $\mathcal{U}(t)\,\mathrm{Op}[a]\mathcal{U}^*(t)$

Define the operators $\hat{\Delta}$ and $\hat{V}$ by

$$(\hat{\Delta}a)(z,b,\lambda) = \hat{\Delta}_{b,\lambda}a(z,b,\lambda) \quad , \qquad (\hat{V}a)(z,b,\lambda) = \hat{V}_{b,\lambda}a(z,b,\lambda) \ ,$$

and $\hat{S}(t)$ by $(\hat{S}(t)a)(z,b,\lambda) = \hat{S}_{b,\lambda}(t)a(z,b,\lambda)$, then

- $\hat{S}(t)a = a \circ \phi^{-t}$, where ϕ^t is geodesic flow.,
- and
$$\mathcal{U}(t)\,\mathrm{Op}[a]\mathcal{U}^*(t) = \mathrm{Op}[(\hat{V}(t)a)\circ\phi^{-t}] \ .$$

Remark To obtain Egorov Theorem we have to expand $\hat{V}(t)a$ into a Volterra series

$$\hat{V}(t)a = a + \frac{\hbar}{2\mathrm{i}}\int_0^t \hat{\Delta}(t_1)a\,\mathrm{d}t_1 + \cdots$$

Egorov's Theorem for large times

Zeditch 86: let $\hat{\gamma}(z, b, \lambda) := (\gamma(z), \gamma(b), \lambda)$ be the lift of γ, and let $T_\gamma u := u \circ \gamma^{-1}$ then

$$T_\gamma^* \operatorname{Op}[a] T_\gamma = \operatorname{Op}[a \circ \hat{\gamma}^{-1}] \ ,$$

so we can identify operators on M with operators on $\mathbb{D}$ with Γ-invariant symbols $S^{m,k}(M)$.

Theorem

Let $a \in S^{0,0}(M)$, be analytic, then there is a $\delta > 0$ such that

$$\|\mathcal{U}(t) \operatorname{Op}[a] \mathcal{U}^*(t) - \operatorname{Op}[a \circ \phi^{-t}]\| \ll t\sqrt{\hbar} \ .$$

for

$$t \leq \delta \frac{1}{\sqrt{\hbar}}$$

Summary and Conclusions

- We extended the time range where semiclassical approximations are accurate from $T_E \sim \ln 1/\hbar$ to $1/\sqrt{\hbar}$. Two main ingredients:
  1. Separation of propagation into two parts, classical propagation and dispersion. Using unitarity remainder estimates could be reduced to estimates of the dispersive part.
  2. We then used energy type inequalities to obtain estimates on the dispersive part.
- The $1/\sqrt{\hbar}$ scale is probably not optimal, we conjecture that the results hold up to $1/\hbar$. The main open problem is to obtain sharper estimates on the dispersive part.
- Using a semiclassical calculus adapted to the phase space geometry the same techniques can be used to obtain a version of Egorov's theorem valid for large times.