Mixing and Waves: Part I
The Ehrenfest time

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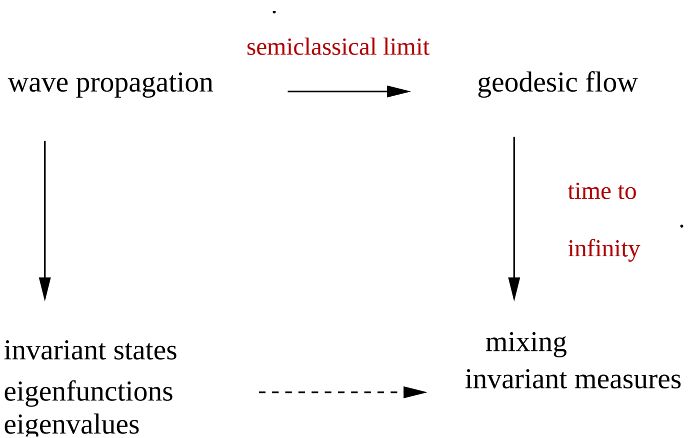
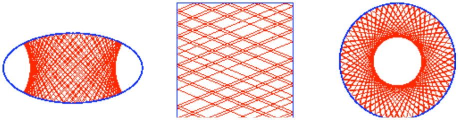
Spectrum and dynamics



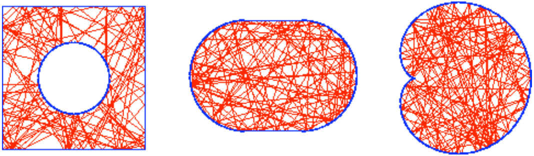
- semiclassical limit and $t \to \infty$ don't commute
- need semiclassical techniques which are uniform in t

regular and irregular motion

regular motion:



irregular motion:



- neighbouring trajectories diverge typically
- a typical trajectory fills the space with uniform density

mixing and universality

A dynamical system $(X, \phi^t, \mathrm{d}\mu)$ is *mixing* if for $a, \rho \in L^2(X)$ $(\int_X \rho \, \mathrm{d}\mu = 1)$

$$\lim_{t\to\infty} \int_X a \circ \phi^t \rho \, \mathrm{d}\mu = \int_X a \, \mathrm{d}\mu$$

Interpretation:

- a observable, ρ a state (i.e., a probability density), then $\int a \circ \phi^t \rho \, \mathrm{d}\mu$ expected value of a at time t: the system forgets where it came from - "Universality"
- Other manifestations of universality: If mixing is rapid enough a Central Limit Theorem (CLT) holds

$$\frac{1}{\sqrt{T}} \int_0^T a \circ \Phi^t \mathrm{d}t \quad \text{becomes normally distributed}$$

for $T \to \infty$ (if $\int_X a \, \mathrm{d}\mu = 0$).

Anosov flows

Definition

ϕ^t is called Anosov if for all $x \in X$ there is a splitting

$$T_x X = E^s(x) \oplus E^u(x) \oplus E^0(x)$$

such that $E^0(x)$ is spanned by the flow direction and there are constants $\lambda > 0, C$ such that

- $\|\mathrm{d}\phi^t u\| \leq C\mathrm{e}^{-\lambda t}\|u\|$ *for all $u \in E^s(x)$ and $t > 0$*
- $\|\mathrm{d}\phi^t u\| \leq C\mathrm{e}^{\lambda t}\|u\|$ *for all $u \in E^u(x)$ and $t < 0$*

Example: geodesic flow on compact manifold M.

- $X = S^*M$, $\mathrm{d}\mu$ Liouville measure on S^*M
- ϕ^t Hamiltonian flow generated by $H(x, \xi) = \frac{1}{2}|\xi|^2_{g(x)}$

Geodesic flow is Anosov if all sectional curvatures are negative.

exponential mixing

Theorem (Dolgopyat 98, Liverani 05)

Assume X is contact and $\phi^t : X \to X$ is Anosov and volume preserving, then there exists a $\gamma > 0$ such that for all $a, \rho \in C^1(X)$, $\int \rho \, \mathrm{d}\mu = 1$,

$$\int a \circ \phi^t \rho \, \mathrm{d}\mu = \int a \, \mathrm{d}\mu + O(\|a\|_{C^1} \|\rho\|_{C^1} \mathrm{e}^{-\gamma|t|})$$

Localise initial conditions: $\rho_\varepsilon(x) := \frac{1}{\varepsilon^{2d-1}} \rho_0\big(\frac{d(x,x_0)}{\varepsilon}\big)$, then $\|\rho_\varepsilon\|_{C^1} \sim 1/\varepsilon^{2d}$, so

$$\lim_{\varepsilon \to 0} \int a \circ \phi^t \rho_\varepsilon \, \mathrm{d}\mu = \begin{cases} a\big(\phi^t(x_0)\big) & \text{for} \quad t \ll \frac{1}{\lambda} \ln \frac{1}{\varepsilon} \\ \int a \, \mathrm{d}\mu & \text{for} \quad t \gg \frac{1}{\gamma} \ln \frac{1}{\varepsilon} \end{cases}$$

transition from local to global behaviour. Later $\varepsilon \sim \sqrt{\hbar}$, uncertainty relation

The quantum classical correspondence: Observables

Main idea: classify operators by their action on rapidly oscillating functions, e.g., plane waves:

- for an operator A we define its *symbol* a by

$$A\mathrm{e}^{\frac{\mathrm{i}}{\hbar}\langle x,\xi\rangle} = a(\hbar, x, \xi)\mathrm{e}^{\frac{\mathrm{i}}{\hbar}\langle x,\xi\rangle}$$

- Symbol classes: $a \in S_\rho^{m,k}$ if

$$|\partial_x^\alpha \partial_\xi^\beta a| \leq C_{\alpha,\beta}\hbar^{-k}\hbar^{-\rho(|\alpha|+|\beta|)}\langle\xi\rangle^{m-|\beta|}\ , \quad \text{where} \quad \langle\xi\rangle = (1+|\xi|^2)^{1/2}$$

- Operator classes: $A \in \Psi_\rho^{k,m}$ if $a \in S_\rho^{k,m}$
- $\hbar \to 0$ semiclassical limit, ρ governs "classicality" of symbol and A
- Example: $\hat{H} = -\hbar^2\Delta \in \Psi_0^{0,2}$, $H = |\xi|^2_{g(x)} + O(\hbar)$.

using Fourier transformation one gets

$$A\psi(x) = \frac{1}{(2\pi\hbar)^d} \iint a(\hbar, x, \xi) \mathrm{e}^{\frac{\mathrm{i}}{\hbar}\langle x-y,\xi\rangle} \psi(y)\, \mathrm{d}y \mathrm{d}\xi$$

symbol determines the operator and we will write $A = \mathrm{Op}[a]$.

- product formula: $\mathrm{Op}[a]\,\mathrm{Op}[b] = \mathrm{Op}[a\sharp b]$ where

$$a\sharp b = \mathrm{e}^{-\mathrm{i}\hbar\langle\partial_\xi,\partial_y\rangle} a(x,\xi) b(y,\eta)|_{x=y,\xi=\eta} \sim \sum_\alpha \frac{(-\mathrm{i}\hbar)^{|\alpha|}}{\alpha!} \partial_\xi^\alpha a \partial_x^\alpha b$$

  asymptotic expansion in $\hbar$ if $\rho < 1/2$: cannot localize on scales smaller that $\sqrt{\hbar}$, Heisenberg $\Delta q \Delta p \geq \hbar/2$.

- $a\sharp b - b\sharp a \sim \mathrm{i}\hbar\{a, b\} + \cdots$ Commutator $\to$ Poissonbracket
- $\|\mathrm{Op}[a]\|_{L^2} \leq C \sum_{|\alpha|\leq 2d+1} \hbar^{|\alpha|/2} |\partial^\alpha a|$.

Operators and symbols on manifolds

- use covering by local charts to define $\mathrm{Op}[a]$, different choice of local charts defines different $\mathrm{Op}'[a]$. In general $\mathrm{Op}[a] = \mathrm{Op}'[a] + O(\hbar)$ for $a \in S_0^{0,0}$.
- use global phase function: $\phi : M \times T^*M \to \mathbb{R}$, homogeneous of degree one in ξ, non-degenerate and $\mathrm{d}_\xi \phi(x, y, \xi) = O(x - y)$, and define

$$\mathrm{Op}[a]\psi(x) = \frac{1}{(2\pi\hbar)^d} \iint a(\hbar, x, \xi) \mathrm{e}^{\frac{\mathrm{i}}{\hbar}\phi(x,y,\xi)} \psi(y) \, \mathrm{d}y \mathrm{d}\xi$$

  such functions can be constructed, e.g., using exponential map. Can define symbols on T^*M using covariant derivatives. (Widom 80, Safarov 97)
- Results on calculus of operators carry over from $\mathbb{R}^n$. (On non-compact manifolds operators have to be properly supported)

Quantum time evolution

- unitary operator $\mathcal{U}(t) = \mathrm{e}^{-\frac{\mathrm{i}}{\hbar} t\hat{H}}$, solution to

$$\mathrm{i}\hbar\partial_t \mathcal{U}(t) = \hat{H}\mathcal{U}(t) \ , \quad \mathcal{U}(t=0) = I$$

- time evolution of states $\psi \to \mathcal{U}(t)\psi$
- expectation values of observables A

$$\langle \mathcal{U}(t)\psi, A\mathcal{U}(t)\psi\rangle = \langle \psi, \mathcal{U}^*(t)A\mathcal{U}(t)\psi\rangle$$

- time evolution of observables $A \to A(t) = \mathcal{U}^*(t)A\mathcal{U}(t)$ solution to

$$\mathrm{i}\hbar\frac{\mathrm{d}A(t)}{\mathrm{d}t} = [A(t), \hat{H}] \ , \quad A(t=0) = A \ .$$

The correspondence principle: Egorov's theorem

Ansatz $A(t) = \mathrm{Op}[a(t)]$, $a \in S_\rho^{0,0}$. commutator $[A(t), \hat{H}]$ has symbol

$$a(t)\sharp H - H\sharp a(t) = \mathrm{i}\hbar\{a(t), H\} + O(\hbar^{2-2\rho})$$

so in leading order $\partial_t a(t) = \{a(t), H\}$ which is solved by $a(t) = a \circ \phi^t$.

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Theorem (Bambusi, Graffi and Paul (99); Bouzouina and Robert (02))

There exists a $k > 1$ such that for all $a \in S_0^{0,0}(T^*M)$

$$\|\mathcal{U}(t)^* \mathrm{Op}[a]\mathcal{U}(t) - \mathrm{Op}[a \circ \phi^t]\|_{L^2} = O(\hbar\|a \circ \phi^t\|_{C^k}) \ .$$

Remarks:

- Correspondence principle: for $\hbar \to 0$ we find quantum $\to$ classical.
- if ϕ^t is Anosov, then $\|a \circ \phi^{-t}\|_{C^k} = O(\mathrm{e}^{k\lambda|t|})$

Ehrenfest time I

Since

$$|\partial a \circ \phi^t| \ll |\partial a| \mathrm{e}^{\lambda t}$$

we have for $a \in S_0^{0,0}$ that

$$a \circ \phi^t \in S_\rho^{0,0} \quad \text{for} \quad \mathrm{e}^{\lambda t} \ll \hbar^{-\rho}$$

Ehrenfest time

$$T \sim \frac{\rho}{\lambda} \ln \frac{1}{\hbar}$$

classical and quantum fluctuations are of the same order

pseudo-differential operator calculus breaks down

1978-1979 Berman Zaslavsky, Balasy Berry Tabor Voros: log breaking time, Ehrenfest time, limit of validity of semiclassics?

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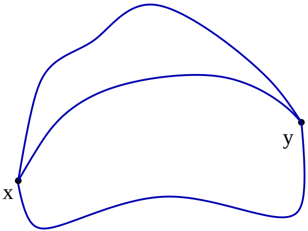
But for larger times?

- oscillations on smaler scale than Heisenberg $\rightarrow$ averaging $\rightarrow$ mixing $\rightarrow$ universality?
- Averaged Egorov, e.g., $\operatorname{Tr} B(\mathcal{U}(t)^* \operatorname{Op}[a] \mathcal{U}(t) - \operatorname{Op}[a \circ \phi^t]) = ?$

Van-Vleck formula

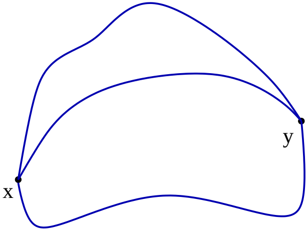
Let $\mathcal{U}(t)\psi(x) = \int K(t,x,y)\psi(y)\,\mathrm{d}y$
then

$$K(t,x,y) = \sum_{\gamma_{x,y}(t)} \left[A_\gamma(t) + O_{\gamma,t}(\hbar)\right] \mathrm{e}^{\frac{\mathrm{i}}{\hbar} S_\gamma(t)}$$



where the sum is over all geodesics $\gamma_{x,y}$ from x to y in time t (in some energy window), and S_γ is the action along γ.

Van-Vleck formula



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where the sum is over all geodesics $\gamma_{x,y}$ from x to y in time t (in some energy window), and S_γ is the action along γ.

- If ϕ^t is Anosov then $A_\gamma(t) \sim t\mathrm{e}^{-\lambda_\gamma t/2}$, $O_{\gamma,t}(\hbar) \ll \hbar \mathrm{e}^{-\lambda_\gamma t/2}$ ($1/\sqrt{}$ of unstable Jacobian), so the remainder is, roughly, bounded by

$$\sum_{\gamma_{x,y}(t)} \hbar t \mathrm{e}^{-\lambda_\gamma t/2} \sim \hbar \mathrm{e}^{Pt}$$

with pressure $P = P(-\mathcal{H}/2)$ ($\mathcal{H}$-SRB potential, unstable Jacobian).

Ehrenfest time II

- remainder term small for

$$t \ll T \sim \frac{1}{P} \ln \frac{1}{\hbar}$$

- What for larger times?
  - note that

$$\sum_{\gamma_{x,y}(t)} |A_\gamma|^2 < \infty$$

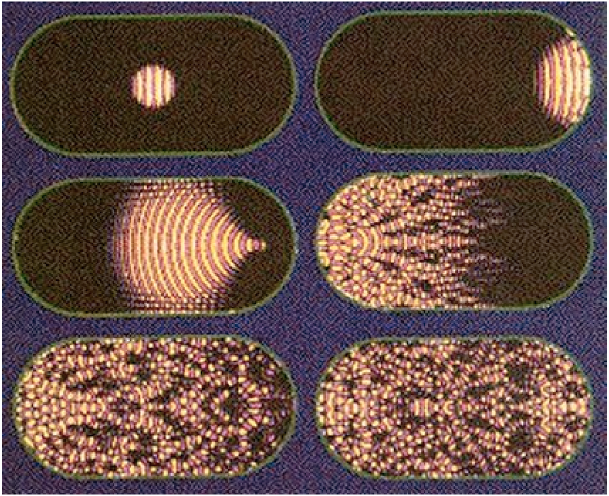
  - so CLT for long orbits might lead to CLT for time evolution

$$K(t,x,y) \sim \sum_{\gamma_{x,y}(t)} A_\gamma(t) \mathrm{e}^{\frac{\mathrm{i}}{\hbar} S_\gamma(t)}$$

1979 Balasz Berry - random wave conjecture: equidistribution and universal fluctuations.

Tomsovic Heller 1991

Numerical experiments: Semiclassics remains accurate for $t \ll 1/\sqrt{\hbar}$. waves become equidistributed, fluctuations satisfy CLT



Remarks: Ehrenfest time

- Can control accuracy of semiclassical approximations up to Ehrenfest time scales

$$t \ll T_E \sim \ln \frac{1}{\hbar}$$

two sources of breakdown:

  - positive Liapunov exponents, inducing rapid oscillations on scales shorter than $\sqrt{\hbar}$, standart semicassical techniques break down
  - exponential proliferation of orbits, one would have to incorporate the cancellation from rapidly oscillating phase-factors to go beyond

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- Beyond Ehrenfest time: expect universality from exponential mixing and CLT
  - effective averaging from uncertainty principle
  - generic long orbits become dense and behave universal.

Related problems: equidistribution

Theorem (RS 05)

Let M is compact and of neg. curvature, take $\psi = f\mathrm{e}^{\frac{\mathrm{i}}{\hbar}\varphi}$ with $\operatorname{supp} f$ *small, and assume*

$$\Lambda := \{(x, \mathrm{d}\varphi(x)) \ , \ \ x \in \operatorname{supp} f\} \subset S^*M \ ,$$

and Λ is transversal to the stable foliation (i.e. for all $x \in \Lambda$, $T_x\Lambda \cap E^s(x) = \{0\}$). Then there exists constants $\Gamma, \gamma > 0$ such that for all $a \in S_0^{0,0}$

$$\langle \mathcal{U}(t)\psi, \mathrm{Op}[a]\mathcal{U}(t)\psi\rangle = \|\psi\|_{L^2} \int_{S^*M} a \ \mathrm{d}\mu + O(\hbar \mathrm{e}^{\Gamma|t|}) + O(\mathrm{e}^{-\gamma|t|})$$

Problems:

- larger times
- coherent states

Localized states

Theorem

Let $\delta > 0$, $a_0 \in C^\infty(M)$ and set

$$a(\hbar, x) = \frac{1}{\hbar^{\frac{d\delta}{2}}} a_0\Big(\frac{x - x_0}{\hbar^\delta}\Big) \qquad \psi_0(x) = a(\hbar, x)\mathrm{e}^{\frac{\mathrm{i}}{\hbar}\varphi(x)} \ .$$

Assume ϕ^t is Anosov, $\Lambda_\varphi \subset S^*M$, and Λ_φ is transversal to the stable foliation. Then there exists constants $\Gamma, \lambda \geq \gamma > 0$ such that for all $f \in C_0^\infty(T^*M)$ and if $\|\psi\|_{L^2} = 1$

$$\lim_{\hbar \to 0} \langle \mathcal{U}(t)\psi, \mathrm{Op}[f]\mathcal{U}(t)\psi\rangle = \begin{cases} f\left(\phi^{-t}(x_0, \mathrm{d}\varphi(x_0))\right) & \text{if} \quad t << \frac{\delta}{\lambda} \ln \frac{1}{\hbar} \\ \int_{S^*M} f \, \mathrm{d}\mu & \text{if} \quad \frac{\delta}{\gamma} \ln \frac{1}{\hbar} << t << \frac{1}{\ } \end{cases}$$

Related problems: rate of quantum ergodicity

$$-\Delta\psi_n = \lambda_n^2 \psi_n$$

- $N(\lambda) := |\{\lambda_n \leq \lambda\}|$
- on manifolds of negative curvature (Zelditch 94)

$$\frac{1}{N(\lambda)} \sum_{\lambda_n \leq \lambda} \left| \langle \psi_n, \mathrm{Op}[a]\psi_n \rangle - \int_{S^*M} a \mathrm{d}\mu \right|^2 \ll 1/\ln\lambda$$

main tool: avaraging $\mathrm{Op}[a] \to \frac{1}{T}\int_0^T \mathcal{U}^*(t)\,\mathrm{Op}[a]\mathcal{U}(t)\,\mathrm{d}t$, RHS is $1/T$,

- Conjecture: RHS$= O(\lambda^{-1})$ (on surfaces)

Related problems: Weyls law

- for surfaces

$$N(\lambda) = c\lambda^2 + R(\lambda)$$

  - upper bound in negative curvature: Berard (77) $R(\lambda) = O(\lambda/\ln\lambda)$
  - lower bound: Jacobson, Polterovich, Toth (07) $R(\lambda) = \Omega((\log\lambda)^{P(-\mathcal{H}/2)/h-\varepsilon})$
  - Conjecture: Randol (81): $R(\lambda) = O(\lambda^{1/2})$
- Estimates on L^∞ norms:
  - Berard (77): $\|\psi_n\|_{L^\infty} = O(\lambda/\ln\lambda)$
  - Conjecture (Iwaniec Sarnak 95): $\|\psi_n\|_{L^\infty} = O_\varepsilon(\lambda^\varepsilon)$ for all $\varepsilon > 0$.

Summary: time scales

- Ehrenfest time

$$T_E \sim \frac{1}{\lambda} \ln \frac{1}{\hbar}$$

-exponential proliferation of orbits,
-small-scale oscillations

- Heisenberg time, time scale to resolve spectrum:

$$T_H \sim \frac{1}{\hbar^{d-1}}$$