Spectral Asymptotics and Dynamics
Workshop "Spectrum and Dynamics" (CRM, Montreal)

Victor Ivrii

Department of Mathematics, University of Toronto

Wednesday, April 9, 2008

Table of Contents

I am going to show that

1. Eigenvalue asymptotics is intimately related to the corresponding dynamics;

I am going to show that

1. Eigenvalue asymptotics is intimately related to the corresponding dynamics;
2. There is quantum dynamics and classical dynamics and both of them are intimately related;

I am going to show that

1. Eigenvalue asymptotics is intimately related to the corresponding dynamics;
2. There is quantum dynamics and classical dynamics and both of them are intimately related;
3. Quantum dynamics is richer than the classical one;

I am going to show that

1. Eigenvalue asymptotics is intimately related to the corresponding dynamics;
2. There is quantum dynamics and classical dynamics and both of them are intimately related;
3. Quantum dynamics is richer than the classical one;
4. Standard notions of the classical dynamics are not that obvious in many situations.

You are kidding, right?

My first claim is that If we know nothing about classical dynamics, we cannot say anything about eigenvalue asymptotics.

You are kidding, right?

My first claim is that If we know nothing about classical dynamics, we cannot say anything about eigenvalue asymptotics.
This claim seems to be extreme and in complete contradiction with the facts:

- Hermann Weyl paper where asymptotical distribution of eigenvalues of Dirichlet Laplacian

$$N(\lambda) = \mathrm{Weyl} + o(\lambda^{(d-1)/2}) \qquad \text{as } \lambda \to +\infty \tag{1}$$

  did not mention anything about classical dynamics; here $N(\lambda)$ is the number of the eigenvalues of the (positive) Laplacian not exceeding λ and

$$\mathrm{Weyl} = (2\pi)^{-d}\omega_d \lambda^{d/2} \mathrm{vol} X, \tag{2}$$

  d is the dimension and ω_d is a volume of the unit ball in $\mathbb{R}^d$;

- My own results: for Schrödinger operator

$$A = h^2\Delta + V(x) \tag{3}$$

asymptotics

$$N_h(\lambda) = \mathrm{Weyl} + O(h^{1-d}) \qquad \text{as } h \to +0 \tag{4}$$

with

$$\mathrm{Weyl} = (2\pi)^{-d} h^{-d} \iint_{a(x,\xi)<\lambda} dx\, d\xi \tag{5}$$

and

$$a(x,\xi) = |\xi|^2 + V(x) \tag{6}$$

also does not mention any dynamics;

- My own results: for Schrödinger operator

$$A = h^2\Delta + V(x) \tag{3}$$

asymptotics

$$N_h(\lambda) = \mathrm{Weyl} + O(h^{1-d}) \qquad \text{as } h \to +0 \tag{4}$$

with

$$\mathrm{Weyl} = (2\pi)^{-d} h^{-d} \iint_{a(x,\xi)<\lambda} dx\, d\xi \tag{5}$$

and

$$a(x,\xi) = |\xi|^2 + V(x) \tag{6}$$

also does not mention any dynamics;
- Similar more general results for matrix operators under microhyperbolicity condition;

- My own results: for Schrödinger operator

$$A = h^2\Delta + V(x) \tag{3}$$

asymptotics

$$N_h(\lambda) = \mathrm{Weyl} + O(h^{1-d}) \qquad \text{as } h \to +0 \tag{4}$$

with

$$\mathrm{Weyl} = (2\pi)^{-d} h^{-d} \iint_{a(x,\xi)<\lambda} dx\, d\xi \tag{5}$$

and

$$a(x,\xi) = |\xi|^2 + V(x) \tag{6}$$

also does not mention any dynamics;

- Similar more general results for matrix operators under microhyperbolicity condition;
- Only asymptotics with the second term and the remainder estimates $o(\lambda^{(d-1)/2})$ and $O(h^{1-d})$ require conditions to dynamics.

Nope, I am serious!

No, I am not kidding. All these results do not mention any dynamics, but it is still here. Classical dynamics associated with the Laplacian contains no stationary points;

Nope, I am serious!

No, I am not kidding. All these results do not mention any dynamics, but it is still here. Classical dynamics associated with the Laplacian contains no stationary points; so does dynamics associated with matrix operators under microhyperbolicity condition;

Nope, I am serious!

No, I am not kidding. All these results do not mention any dynamics, but it is still here. Classical dynamics associated with the Laplacian contains no stationary points; so does dynamics associated with matrix operators under microhyperbolicity condition; in the case of the Schrödinger operator (3) there could be stationary points as $V(x)=\lambda$, $\nabla V(x)=0$ but the measure (there is a natural measure $d\mu_\lambda = dxd\xi : da$ on the energy surface $\{(x,\xi) : a(x,\xi)=\lambda\}$) of points with $|\nabla a(x,\xi)|\le\varepsilon$ is $O(\varepsilon^{d-1})$.

Nope, I am serious!

No, I am not kidding. All these results do not mention any dynamics, but it is still here. Classical dynamics associated with the Laplacian contains no stationary points; so does dynamics associated with matrix operators under microhyperbolicity condition; in the case of the Schrödinger operator (3) there could be stationary points as $V(x)=\lambda$, $\nabla V(x)=0$ but the measure (there is a natural measure $d\mu_\lambda = dxd\xi : da$ on the energy surface $\{(x,\xi) : a(x,\xi)=\lambda\}$) of points with $|\nabla a(x,\xi)|\le \varepsilon$ is $O(\varepsilon^{d-1})$.

All these are quite strong conditions to local dynamics.

Nope, I am serious!

No, I am not kidding. All these results do not mention any dynamics, but it is still here. Classical dynamics associated with the Laplacian contains no stationary points; so does dynamics associated with matrix operators under microhyperbolicity condition; in the case of the Schrödinger operator (3) there could be stationary points as $V(x)=\lambda$, $\nabla V(x)=0$ but the measure (there is a natural measure $d\mu_\lambda = dxd\xi : da$ on the energy surface $\{(x,\xi) : a(x,\xi)=\lambda\}$) of points with $|\nabla a(x,\xi)|\le\varepsilon$ is $O(\varepsilon^{d-1})$.

All these are quite strong conditions to local dynamics.

On the contrary, consider $a(hD)$ on $\mathbb{R}^d/\mathbb{Z}^d$. Assuming that $\operatorname{mes}\{\xi : a(\xi)=\lambda_0\}>0$ we see that even remainder estimate $o(h^{-d})$ is not possible for infinitesimal perturbations of such operator.

Global dynamics

But let us talk about more sharp asymptotics (with the smaller remainder estimates).

Global dynamics

But let us talk about more sharp asymptotics (with the smaller remainder estimates). These asymptotics require conditions of the global nature. Here our worst enemy are (or seem to be) periodic trajectories.

Global dynamics

But let us talk about more sharp asymptotics (with the smaller remainder estimates). These asymptotics require conditions of the global nature. Here our worst enemy are (or seem to be) periodic trajectories. Reason: many periodic trajectories indicate either highly degenerate eigenvalues or at least tight clusters of eigenvalues and the eigenvalue counting function $N_h(\lambda)$ grows too fast when λ cross "bad" values.

Examples (with the standard metrics):

1. Laplace(-Beltrami) on the sphere $\mathbb{S}^d$ (or on semi-sphere), periodic trajectories are large circles;

Examples (with the standard metrics):

1. Laplace(-Beltrami) on the sphere $\mathbb{S}^d$ (or on semi-sphere), periodic trajectories are large circles;
2. Schrödinger with the potential $|x|^2$ on $\mathbb{R}^d$, periodic trajectories are ellipses with the center at 0;

Examples (with the standard metrics):

1. Laplace(-Beltrami) on the sphere $\mathbb{S}^d$ (or on semi-sphere), periodic trajectories are large circles;
2. Schrödinger with the potential $|x|^2$ on $\mathbb{R}^d$, periodic trajectories are ellipses with the center at 0;
3. Schrödinger with the potential $-|x|^{-1}$ on $\mathbb{R}^d$, periodic trajectories are Kepler ellipses ($\lambda < 0$);

Examples (with the standard metrics):

1. Laplace(-Beltrami) on the sphere $\mathbb{S}^d$ (or on semi-sphere), periodic trajectories are large circles;
2. Schrödinger with the potential $|x|^2$ on $\mathbb{R}^d$, periodic trajectories are ellipses with the center at 0;
3. Schrödinger with the potential $-|x|^{-1}$ on $\mathbb{R}^d$, periodic trajectories are Kepler ellipses ($\lambda < 0$);
4. Magnetic Schrödinger in $\mathbb{R}^2$ $A = (-ih\nabla(x) - F(x))^2 + V(x)$ with the linear vector potential $F(x)$ and $V(x) = 0$; then A has pure point (of infinite multiplicity) spectrum $\{(2n+1)bh, n = 0, 1, \dots\}$ where $b = |\nabla \times F|$ is a scalar magnetic intensity; periodic trajectories are circles of radius $\sqrt{\lambda/b}$.

To avoid the bad effect of periodic trajectories we assume

Non-periodicity condition

The measure of periodic trajectories is 0.

To avoid the bad effect of periodic trajectories we assume

Non-periodicity condition

The measure of periodic trajectories is 0.

Then

"Theorem"

Under non-periodicity condition asymptotics

$$N_h(\lambda) = \mathrm{Weyl} + \kappa h^{1-d} + o(h^{1-d}) \qquad \text{as } h \to +0 \tag{7}$$

and similar asymptotics with respect to λ hold.

We did not specified very precisely what we are studying; however it works for scalar operators on closed manifolds and dynamics is the dynamics of the principal symbol (under some pretty weak nondegeneracy condition).

We did not specified very precisely what we are studying; however it works for scalar operators on closed manifolds and dynamics is the dynamics of the principal symbol (under some pretty weak nondegeneracy condition). Similarly we can study asymptotics with respect to $\lambda \to +\infty$ for such operators on closed manifolds and for Schrödinger operator with $V \to +\infty$ as $|x| \to \infty$ on $\mathbb{R}^d$ (and its ilk)

We did not specified very precisely what we are studying; however it works for scalar operators on closed manifolds and dynamics is the dynamics of the principal symbol (under some pretty weak nondegeneracy condition). Similarly we can study asymptotics with respect to $\lambda \to +\infty$ for such operators on closed manifolds and for Schrödinger operator with $V \to +\infty$ as $|x| \to \infty$ on $\mathbb{R}^d$ (and its ilk) and asymptotics as $\lambda \to -0$ for Schrödinger operator with $V \to 0$ as $|x| \to \infty$ on $\mathbb{R}^d$ (and its ilk).

We did not specified very precisely what we are studying; however it works for scalar operators on closed manifolds and dynamics is the dynamics of the principal symbol (under some pretty weak nondegeneracy condition). Similarly we can study asymptotics with respect to $\lambda \to +\infty$ for such operators on closed manifolds and for Schrödinger operator with $V \to +\infty$ as $|x| \to \infty$ on $\mathbb{R}^d$ (and its ilk) and asymptotics as $\lambda \to -0$ for Schrödinger operator with $V \to 0$ as $|x| \to \infty$ on $\mathbb{R}^d$ (and its ilk).

For matrix operators the similar result would hold if the eigenvalues $f_j(x,\xi)$ of the principal symbol have constant multiplicities and we consider the dynamics generated by each of symbol $f_j(x,\xi)$.

Let's play billiard

What about manifolds with boundary? Let us consider just Laplacian or Schrödinger.

Let's play billiard

What about manifolds with boundary? Let us consider just Laplacian or Schrödinger. Then instead of Hamiltonian trajectories we need to consider corresponding billiards which are composed from the pieces of Hamiltonian trajectories reflecting according to

Geometric optics law

Reflection angle = Incidence angle

Let's play billiard

What about manifolds with boundary? Let us consider just Laplacian or Schrödinger. Then instead of Hamiltonian trajectories we need to consider corresponding billiards which are composed from the pieces of Hamiltonian trajectories reflecting according to

Geometric optics law

Reflection angle = Incidence angle

Of course, there is a caveat: there could be points (called dead-end points) such that billiards passing through them began behave badly.

Let's play billiard

What about manifolds with boundary? Let us consider just Laplacian or Schrödinger. Then instead of Hamiltonian trajectories we need to consider corresponding billiards which are composed from the pieces of Hamiltonian trajectories reflecting according to

Geometric optics law

Reflection angle = Incidence angle

Of course, there is a caveat: there could be points (called dead-end points) such that billiards passing through them began behave badly. Fortunately the measure of such dead-end points is 0 (and this is true not only for manifolds with smooth boundaries but also having vertices, edges or conical points – we have powerful methods of analysis allowing to derive sharp asymptotics in these cases as well).

Let's play billiard

What about manifolds with boundary? Let us consider just Laplacian or Schrödinger. Then instead of Hamiltonian trajectories we need to consider corresponding billiards which are composed from the pieces of Hamiltonian trajectories reflecting according to

Geometric optics law

Reflection angle = Incidence angle

Of course, there is a caveat: there could be points (called dead-end points) such that billiards passing through them began behave badly. Fortunately the measure of such dead-end points is 0 (and this is true not only for manifolds with smooth boundaries but also having vertices, edges or conical points – we have powerful methods of analysis allowing to derive sharp asymptotics in these cases as well). Unfortunately it is only first taste of troubles to come.

In 1979 I proved asymptotics with the second term for Laplacian on smooth manifolds under Non-periodicity condition (to billiards) and made

In 1979 I proved asymptotics with the second term for Laplacian on smooth manifolds under Non-periodicity condition (to billiards) and made

Conjecture

Non-periodicity condition is fulfilled for ordinary billiards in Euclidean domains.

In 1979 I proved asymptotics with the second term for Laplacian on smooth manifolds under Non-periodicity condition (to billiards) and made

Conjecture

Non-periodicity condition is fulfilled for ordinary billiards in Euclidean domains.

Billiard people (the best of them) promised to prove it in a couple of days.

In 1979 I proved asymptotics with the second term for Laplacian on smooth manifolds under Non-periodicity condition (to billiards) and made

Conjecture

Non-periodicity condition is fulfilled for ordinary billiards in Euclidean domains.

Billiard people (the best of them) promised to prove it in a couple of days. Unfortunately in this case "couple" $> 10,000$.

In 1979 I proved asymptotics with the second term for Laplacian on smooth manifolds under Non-periodicity condition (to billiards) and made

Conjecture

Non-periodicity condition is fulfilled for ordinary billiards in Euclidean domains.

Billiard people (the best of them) promised to prove it in a couple of days. Unfortunately in this case "couple" $> 10,000$.

Fortunately in some cases (f.e. strictly convex domains with analytic boundaries) we can prove this conjecture.

Sharper than sharp

Can we get better than $o(h^{1-d})$ etc?

Sharper than sharp

Can we get better than $o(h^{1-d})$ etc? Yes, but we need to assume

Growth condition

With the exception of the set of measure $\le \varepsilon$ through each point passes the billiard of the length t such that all angles are greater than ε^L and $|D\Phi_t(z)| \le \varepsilon^{-L}$ as $|t| \le T$;

Sharper than sharp

Can we get better than $o(h^{1-d})$ etc? Yes, but we need to assume

Growth condition

With the exception of the set of measure $\le \varepsilon$ through each point passes the billiard of the length t such that all angles are greater than ε^L and $|D\Phi_t(z)| \le \varepsilon^{-L}$ as $|t| \le T$;

and

Strong non-periodicity condition

Further, with the exception of the set of measure $\le \varepsilon$ $\mathrm{dist}(z, \Phi_t(z)) \ge \varepsilon^L$ as $T_0 \le |t| \le T$ where T_0 is a small constant.

Here Φ_t denotes billiard flow.

Basically, there are two cases:

- More generic case when one can take T arbitrarily large and $\varepsilon = e^{-CT}$. Example: manifolds of negative sectional curvature (P. Bérard, 1977). Then asymptotics with the remainder estimate $O(\lambda^{(d-1)/2}/\log\lambda)$ holds.

Basically, there are two cases:

- More generic case when one can take T arbitrarily large and $\varepsilon = e^{-CT}$. Example: manifolds of negative sectional curvature (P. Bérard, 1977). Then asymptotics with the remainder estimate $O(\lambda^{(d-1)/2}/\log\lambda)$ holds.
- Some completely integrable cases: Balls, interior of ellipses or domains between two confocal ellipses, but also polyhedra when one can take T arbitrarily large and $\varepsilon = T^{-L}$. Then asymptotics with the remainder estimate $O(\lambda^{(d-1-\delta)/2})$ holds; $\delta > 0$ is unspecified small exponent.

Periodic case

Assume that all trajectories are periodic. Then period depends only on energy level λ: $T = T(\lambda)$ (but there could be subperiodic points with periods $T(\lambda)/m$, $m = 2, 3, \dots$).

Periodic case

Assume that all trajectories are periodic. Then period depends only on energy level λ: $T = T(\lambda)$ (but there could be subperiodic points with periods $T(\lambda)/m$, $m = 2, 3, \dots$).

Examples

1. If all geodesics are periodic then $T(\lambda) = \text{const}\lambda^{-1/2}$;

Periodic case

Assume that all trajectories are periodic. Then period depends only on energy level λ: $T = T(\lambda)$ (but there could be subperiodic points with periods $T(\lambda)/m$, $m = 2, 3, \dots$).

Examples

1. If all geodesics are periodic then $T(\lambda) = \text{const}\lambda^{-1/2}$;
2. For harmonic oscillator and for 2D magnetic Schrödinger (with constant b and $V = 0$ $T(\lambda) = \text{const}$;

Periodic case

Assume that all trajectories are periodic. Then period depends only on energy level λ: $T = T(\lambda)$ (but there could be subperiodic points with periods $T(\lambda)/m$, $m = 2, 3, \dots$).

Examples

1. If all geodesics are periodic then $T(\lambda) = \text{const}\lambda^{-1/2}$;
2. For harmonic oscillator and for 2D magnetic Schrödinger (with constant b and $V = 0$ $T(\lambda) = \text{const}$;
3. For Schrödinger with Coulomb potential $T(\lambda) = \text{const}(-\lambda)^{-3/2}$ (Kepler's law).

Periodic case

Assume that all trajectories are periodic. Then period depends only on energy level λ: $T = T(\lambda)$ (but there could be subperiodic points with periods $T(\lambda)/m$, $m = 2, 3, \dots$).

Examples

1. If all geodesics are periodic then $T(\lambda) = \text{const}\lambda^{-1/2}$;
2. For harmonic oscillator and for 2D magnetic Schrödinger (with constant b and $V = 0$ $T(\lambda) = \text{const}$;
3. For Schrödinger with Coulomb potential $T(\lambda) = \text{const}(-\lambda)^{-3/2}$ (Kepler's law).

Then replacing Hamiltonian a by $f(a)$ with $f(\lambda) = \int^{\lambda} T(\lambda)\, d\lambda$ we change the speed but not the trajectories; after this change $T = 1$ and $\Phi_1 = I$.

Classical and quantum

So, let us replace operator A by operator $f(A)$ (which could be problematic if A has singularities).

Classical and quantum

So, let us replace operator A by operator $f(A)$ (which could be problematic if A has singularities). Since $e^{ih^{-1}tA}$ is a Fourier integral operator corresponding to symplectomorphism Φ_t we conclude that $e^{ih^{-1}A}$ corresponds to I and therefore is a h-pseudo-differential operator:

$$e^{ih^{-1}A} = e^{i\varepsilon h^{-1}B} \tag{8}$$

where at this moment $\varepsilon \le h$ and one can select B commuting with A.

Classical and quantum

So, let us replace operator A by operator $f(A)$ (which could be problematic if A has singularities). Since $e^{ih^{-1}tA}$ is a Fourier integral operator corresponding to symplectomorphism Φ_t we conclude that $e^{ih^{-1}A}$ corresponds to I and therefore is a h-pseudo-differential operator:

$$e^{ih^{-1}A} = e^{i\varepsilon h^{-1}B} \tag{8}$$

where at this moment $\varepsilon \le h$ and one can select B commuting with A. So, quantum dynamics $e^{ih^{-1}tA}$ is not necessarily periodic but quantum dynamics $e^{ih^{-1}tA_0}$ is where $A_0 = A - \varepsilon B$:

$$e^{ih^{-1}A_0} = I. \tag{9}$$

Classical and quantum

So, let us replace operator A by operator $f(A)$ (which could be problematic if A has singularities). Since $e^{ih^{-1}tA}$ is a Fourier integral operator corresponding to symplectomorphism Φ_t we conclude that $e^{ih^{-1}A}$ corresponds to I and therefore is a h-pseudo-differential operator:

$$e^{ih^{-1}A} = e^{i\varepsilon h^{-1}B} \tag{8}$$

where at this moment $\varepsilon \le h$ and one can select B commuting with A. So, quantum dynamics $e^{ih^{-1}tA}$ is not necessarily periodic but quantum dynamics $e^{ih^{-1}tA_0}$ is where $A_0 = A - \varepsilon B$:

$$e^{ih^{-1}A_0} = I. \tag{9}$$

We can consider A as perturbation of A_0.

More generally: consider symbol a_0 satisfying $e^{H_{a_0}} = I$ where H_a is the Hamiltonian field of a.

More generally: consider symbol a_0 satisfying $e^{H_{a_0}} = I$ where H_a is the Hamiltonian field of a. Consider $a = a_0 + \eta a_1$ with $\eta \ll 1$; then (8) holds with $\varepsilon \le \eta$.

More generally: consider symbol a_0 satisfying $e^{H_{a_0}} = I$ where H_a is the Hamiltonian field of a. Consider $a = a_0 + \eta a_1$ with $\eta \ll 1$; then (8) holds with $\varepsilon \le \eta$. For generic perturbations $\varepsilon \asymp \eta$ and b is an average of a_1 along closed trajectories but sometimes can be made smaller.

More generally: consider symbol a_0 satisfying $e^{H_{a_0}} = I$ where H_a is the Hamiltonian field of a. Consider $a = a_0 + \eta a_1$ with $\eta \ll 1$; then (8) holds with $\varepsilon \le \eta$. For generic perturbations $\varepsilon \asymp \eta$ and b is an average of a_1 along closed trajectories but sometimes can be made smaller.

Spectrum of A_0 snaps to points $2\pi h\mathbb{Z}$.

More generally: consider symbol a_0 satisfying $e^{H_{a_0}} = I$ where H_a is the Hamiltonian field of a. Consider $a = a_0 + \eta a_1$ with $\eta \ll 1$; then (8) holds with $\varepsilon \le \eta$. For generic perturbations $\varepsilon \asymp \eta$ and b is an average of a_1 along closed trajectories but sometimes can be made smaller.

Spectrum of A_0 snaps to points $2\pi h\mathbb{Z}$. Spectrum of A will be contained in $C\varepsilon$-vicinities of such points (thus we get eigenvalue clusters as $C\varepsilon < \frac{1}{2}$).

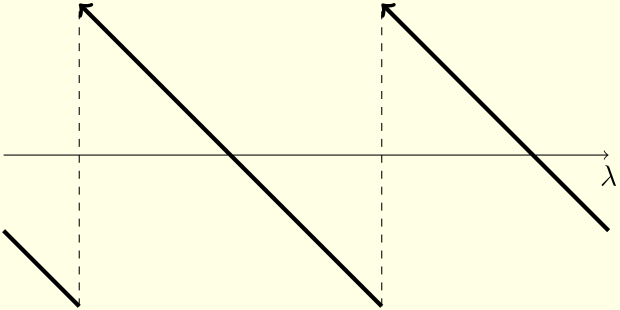


Figure: Graph of $N(\lambda) -$ Weyl for A_0

More generally: consider symbol a_0 satisfying $e^{H_{a_0}} = I$ where H_a is the Hamiltonian field of a. Consider $a = a_0 + \eta a_1$ with $\eta \ll 1$; then (8) holds with $\varepsilon \le \eta$. For generic perturbations $\varepsilon \asymp \eta$ and b is an average of a_1 along closed trajectories but sometimes can be made smaller.

Spectrum of A_0 snaps to points $2\pi h\mathbb{Z}$. Spectrum of A will be contained in $C\varepsilon$-vicinities of such points (thus we get eigenvalue clusters as $C\varepsilon < \frac{1}{2}$).

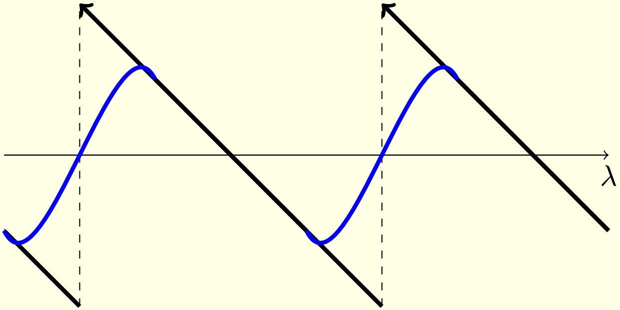


Figure: Graph of $N(\lambda) -$ Weyl for

A_0 and A; blue shows clusters

More generally: consider symbol a_0 satisfying $e^{H_{a_0}} = I$ where H_a is the Hamiltonian field of a. Consider $a = a_0 + \eta a_1$ with $\eta \ll 1$; then (8) holds with $\varepsilon \le \eta$. For generic perturbations $\varepsilon \asymp \eta$ and b is an average of a_1 along closed trajectories but sometimes can be made smaller.

Spectrum of A_0 snaps to points $2\pi h\mathbb{Z}$. Spectrum of A will be contained in $C\varepsilon$-vicinities of such points (thus we get eigenvalue clusters as $C\varepsilon < \frac{1}{2}$).

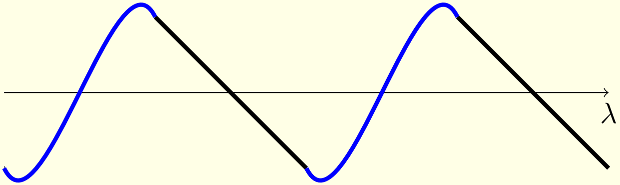


Figure: Graph of $N(\lambda) - $ Weyl for A; blue shows clusters

While compensate “the sharp saw” for A_0 is hopeless, we are able to to compensate “the dull saw” for A.

While compensate "the sharp saw" for A_0 is hopeless, we are able to to compensate "the dull saw" for A.
So, consider dynamics

$$e^{ih^{-1}tA} = e^{ih^{-1}(n+t')A} \tag{10}$$

with $n = \lfloor t \rfloor$, $t' = t - n$,

While compensate "the sharp saw" for A_0 is hopeless, we are able to to compensate "the dull saw" for A.
So, consider dynamics

$$e^{ih^{-1}tA} = e^{ih^{-1}(n+t')A} = e^{ih^{-1}nA} \cdot e^{ih^{-1}t'A} \tag{10}$$

with $n = \lfloor t \rfloor$, $t' = t - n$,

While compensate "the sharp saw" for A_0 is hopeless, we are able to to compensate "the dull saw" for A.
So, consider dynamics

$$e^{ih^{-1}tA} = e^{ih^{-1}(n+t')A} = e^{ih^{-1}nA} \cdot e^{ih^{-1}t'A} = e^{ih^{-1}\varepsilon nB} \cdot e^{ih^{-1}t'A} \tag{10}$$

with $n = \lfloor t \rfloor$, $t' = t - n$, using (8),

While compensate “the sharp saw” for A_0 is hopeless, we are able to to compensate “the dull saw” for A.
So, consider dynamics

$$e^{ih^{-1}tA} = e^{ih^{-1}(n+t')A} = e^{ih^{-1}nA}\cdot e^{ih^{-1}t'A} = e^{ih^{-1}\varepsilon nB}\cdot e^{ih^{-1}t'A} = e^{ih^{-1}t''B}\cdot e^{ih^{-1}t'A} \tag{10}$$

with $n=\lfloor t\rfloor$, $t'=t-n$, using (8), $t''=\varepsilon n$.

While compensate "the sharp saw" for A_0 is hopeless, we are able to to compensate "the dull saw" for A.
So, consider dynamics

$$e^{ih^{-1}tA} = e^{ih^{-1}(n+t')A} = e^{ih^{-1}nA}\cdot e^{ih^{-1}t'A} = e^{ih^{-1}\varepsilon nB}\cdot e^{ih^{-1}t'A} = e^{ih^{-1}t''B}\cdot e^{ih^{-1}t'A} \qquad (10)$$

with $n=\lfloor t\rfloor$, $t'=t-n$, using (8), $t''=\varepsilon n$.
Right-hand expression is a Fourier integral operator as $|t''|\le C$ i.e. $|t|\le c\varepsilon^{-1}$.

While compensate "the sharp saw" for A_0 is hopeless, we are able to to compensate "the dull saw" for A.
So, consider dynamics

$$e^{ih^{-1}tA} = e^{ih^{-1}(n+t')A} = e^{ih^{-1}nA} \cdot e^{ih^{-1}t'A} = e^{ih^{-1}\varepsilon nB} \cdot e^{ih^{-1}t'A} = e^{ih^{-1}t''B} \cdot e^{ih^{-1}t'A} \qquad (10)$$

with $n = \lfloor t \rfloor$, $t' = t - n$, using (8), $t'' = \varepsilon n$.
Right-hand expression is a Fourier integral operator as $|t''| \le C$ i.e. $|t| \le c\varepsilon^{-1}$.
So, long-time dynamics for A is replaced by dynamics with the slow time t'' and the normal time t'. Both these dynamics commute, so do classical dynamics $e^{t'H_a}$ and $e^{t''H_b}$.

While compensate "the sharp saw" for A_0 is hopeless, we are able to to compensate "the dull saw" for A.
So, consider dynamics

$$e^{ih^{-1}tA} = e^{ih^{-1}(n+t')A} = e^{ih^{-1}nA} \cdot e^{ih^{-1}t'A} = e^{ih^{-1}\varepsilon nB} \cdot e^{ih^{-1}t'A} = e^{ih^{-1}t''B} \cdot e^{ih^{-1}t'A} \tag{10}$$

with $n = \lfloor t \rfloor$, $t' = t - n$, using (8), $t'' = \varepsilon n$.
Right-hand expression is a Fourier integral operator as $|t''| \le C$ i.e. $|t| \le c\varepsilon^{-1}$.
So, long-time dynamics for A is replaced by dynamics with the slow time t'' and the normal time t'. Both these dynamics commute, so do classical dynamics $e^{t'H_a}$ and $e^{t''H_b}$.
In other words: there is normal periodic movement and the slow (with the speed ε) drift of periodic trajectories.

This drift breaks classical periodicity instantly provided f.e.

Drift condition

H_a and H_b are not parallel (as $a = \lambda$ and $b = \mu$).

This drift breaks classical periodicity instantly provided f.e.

Drift condition

H_a and H_b are not parallel (as $a = \lambda$ and $b = \mu$).

Quantum periodicity is more delicate: we must observe that it is broken.

This drift breaks classical periodicity instantly provided f.e.

Drift condition

H_a and H_b are not parallel (as $a = \lambda$ and $b = \mu$).

Quantum periodicity is more delicate: we must observe that it is broken. Under drift condition for time t the drift is $\asymp \varepsilon t$ and it is observable if

Microlocal uncertainty principle

$\mathrm{osc}(x) \times \mathrm{osc}(\xi) \geq h^{1-\delta}$

is fulfilled with arbitrarily small exponent $\delta > 0$.

This drift breaks classical periodicity instantly provided f.e.

Drift condition

H_a and H_b are not parallel (as $a = \lambda$ and $b = \mu$).

Quantum periodicity is more delicate: we must observe that it is broken. Under drift condition for time t the drift is $\asymp \varepsilon t$ and it is observable if

Microlocal uncertainty principle

$\mathrm{osc}(x) \times \mathrm{osc}(\xi) \geq h^{1-\delta}$

is fulfilled with arbitrarily small exponent $\delta > 0$.
We can weaken it to logarithmic uncertainty principle but not to the standard one.

As $h^{1-\delta} \le \varepsilon \ll 1$ the microlocal uncertainty principle is fulfilled after first turn (thus instantly)

As $h^{1-\delta} \le \varepsilon \ll 1$ the microlocal uncertainty principle is fulfilled after first turn (thus instantly) and our knowledge of $\mathrm{Tr} e^{ih^{-1}tA}$ for $|t| \le T_0$ (small constant) instantly translates into knowledge of $\mathrm{Tr} e^{ih^{-1}tA}$ for $|t| \le T_0\varepsilon^{-1}$ and we get a standard asymptotics with the remainder estimate $O(T^{-1}h^{1-d}) = O(\varepsilon h^{1-d})$

As $h^{1-\delta} \le \varepsilon \ll 1$ the microlocal uncertainty principle is fulfilled after first turn (thus instantly) and our knowledge of $\operatorname{Tr} e^{ih^{-1}tA}$ for $|t| \le T_0$ (small constant) instantly translates into knowledge of $\operatorname{Tr} e^{ih^{-1}tA}$ for $|t| \le T_0\varepsilon^{-1}$ and we get a standard asymptotics with the remainder estimate $O(T^{-1}h^{1-d}) = O(\varepsilon h^{1-d})$ because in this case

$$F_{t\to h^{-1}\tau}\bar{\chi}_T(t)\operatorname{Tr} e^{ih^{-1}tA} \equiv F_{t\to h^{-1}\tau}\bar{\chi}_{T_0}(t)\operatorname{Tr} e^{ih^{-1}tA} \mod O(h^\infty) \qquad (11)$$

As $h^{1-\delta} \le \varepsilon \ll 1$ the microlocal uncertainty principle is fulfilled after first turn (thus instantly) and our knowledge of $\operatorname{Tr} e^{ih^{-1}tA}$ for $|t| \le T_0$ (small constant) instantly translates into knowledge of $\operatorname{Tr} e^{ih^{-1}tA}$ for $|t| \le T_0\varepsilon^{-1}$ and we get a standard asymptotics with the remainder estimate $O(T^{-1}h^{1-d}) = O(\varepsilon h^{1-d})$ because in this case

$$F_{t\to h^{-1}\tau}\bar{\chi}_T(t)\operatorname{Tr} e^{ih^{-1}tA} \equiv F_{t\to h^{-1}\tau}\bar{\chi}_{T_0}(t)\operatorname{Tr} e^{ih^{-1}tA} \mod O(h^\infty) \qquad (11)$$

and the absolute value (of both) does not exceed Ch^{1-d}. Here $\bar{\chi}_T(t) = \bar{\chi}(t/T)$ where $\bar{\chi} \in C_0^\infty([-1,1])$, equals 1 on $[-\frac{1}{2}, \frac{1}{2}]$.

Case $h^L \le \varepsilon \le h^{1-\delta}$ is much more subtle.

Case $h^L \le \varepsilon \le h^{1-\delta}$ is much more subtle. We can observe the drift only for $t \in [T', T]$ with $T' = \varepsilon^{-1} h^{1-\delta}$; then

$$F_{t\to h^{-1}\tau} \bar{\chi}_T(t) \operatorname{Tr} e^{ih^{-1}tA} \equiv F_{t\to h^{-1}\tau} \bar{\chi}_{T'}(t) \operatorname{Tr} e^{ih^{-1}tA} \mod O(h^\infty) \quad (12)$$

Case $h^L \le \varepsilon \le h^{1-\delta}$ is much more subtle. We can observe the drift only for $t \in [T', T]$ with $T' = \varepsilon^{-1} h^{1-\delta}$; then

$$
\begin{aligned}
F_{t\to h^{-1}\tau}\bar\chi_T(t)\mathrm{Tr} e^{ih^{-1}tA} &\equiv F_{t\to h^{-1}\tau}\bar\chi_{T'}(t)\mathrm{Tr} e^{ih^{-1}tA}\\
&\equiv \sum_{n:|n|\le T'} F_{t\to h^{-1}\tau}\bar\chi_{T_0}(t-n)\mathrm{Tr} e^{ih^{-1}tA} \mod O(h^\infty)
\end{aligned}
\tag{12}
$$

and one can prove that

$$
|F_{t\to h^{-1}\tau}\bar\chi_{T_0}(t-n)\mathrm{Tr} e^{ih^{-1}tA}| \le Ch^{1-d}(|n|\varepsilon/h+1)^{-L}; \tag{13}
$$

and the absolute value of (12) does not exceed $C\bigl(\varepsilon^{-1}h^{2-d}+h^{1-d}\bigr)$.

Case $h^L \le \varepsilon \le h^{1-\delta}$ is much more subtle. We can observe the drift only for $t \in [T', T]$ with $T' = \varepsilon^{-1} h^{1-\delta}$; then

$$\begin{aligned} F_{t\to h^{-1}\tau}\bar{\chi}_T(t)\mathrm{Tr} e^{ih^{-1}tA} &\equiv F_{t\to h^{-1}\tau}\bar{\chi}_{T'}(t)\mathrm{Tr} e^{ih^{-1}tA} \\ &\equiv \sum_{n:|n|\le T'} F_{t\to h^{-1}\tau}\bar{\chi}_{T_0}(t-n)\mathrm{Tr} e^{ih^{-1}tA} \mod O(h^\infty) \end{aligned} \tag{12}$$

and one can prove that

$$|F_{t\to h^{-1}\tau}\bar{\chi}_{T_0}(t-n)\mathrm{Tr} e^{ih^{-1}tA}| \le Ch^{1-d}(|n|\varepsilon/h+1)^{-L}; \tag{13}$$

and the absolute value of (12) does not exceed $C\bigl(\varepsilon^{-1}h^{2-d} + h^{1-d}\bigr)$. Then we can recover asymptotics with the remainder estimate

$$C\bigl(\varepsilon^{-1}h^{2-d} + h^{1-d}\bigr)/T = C(\varepsilon h^{1-d} + h^{2-d}). \tag{14}$$

However wingings with $1 \le |n| \le T'$ contribute to the main part of the asymptotics

$$\mathcal{N} = h^{-1} \int^{\lambda} F_{t\to h^{-1}\tau} \bar{\chi}_T(t) \operatorname{Tr} e^{ih^{-1}tA}\, d\tau; \tag{15}$$

namely they generate an extra term

$$\mathcal{N}' = h^{-1} \int^{\lambda} \sum_{n:|n|\le T'} F_{t\to h^{-1}\tau} \bar{\chi}_{T_0}(t-n) \operatorname{Tr} e^{ih^{-1}tA}\, d\tau; \tag{16}$$

which after series of transformations modulo remainder estimate could be rewritten as $\mathcal{Q}(\lambda, \varepsilon^{-1}\lambda) h^{1-d}$ with $\mathcal{Q}(\lambda, \varepsilon^{-1}\lambda)$ depending on the normal and fast λ;

However wingings with $1 \le |n| \le T'$ contribute to the main part of the asymptotics

$$\mathcal{N} = h^{-1}\int^{\lambda} F_{t\to h^{-1}\tau}\bar{\chi}_T(t)\operatorname{Tr} e^{ih^{-1}tA}\,d\tau; \tag{15}$$

namely they generate an extra term

$$\mathcal{N}' = h^{-1}\int^{\lambda} \sum_{n:|n|\le T'} F_{t\to h^{-1}\tau}\bar{\chi}_{T_0}(t-n)\operatorname{Tr} e^{ih^{-1}tA}\,d\tau; \tag{16}$$

which after series of transformations modulo remainder estimate could be rewritten as $\mathcal{Q}(\lambda,\varepsilon^{-1}\lambda)h^{1-d}$ with $\mathcal{Q}(\lambda,\varepsilon^{-1}\lambda)$ depending on the normal and fast λ; this leads to asymptotics

$$N_h(\lambda) = \mathrm{Weyl} + (\kappa + \mathcal{Q})h^{1-d} + O(\varepsilon h^{1-d} + h^{2-d}) \qquad \text{as } h \to +0. \tag{17}$$

As $\varepsilon \le h$ and there are clusters we get distribution of eigenvalues

$$N_{n,h}(\lambda) = (2\pi)^{1-d} h^{1-d} \int_{a=2\pi hn,\, \varepsilon b<\lambda-2\pi nh} dxd\xi + O(h^{2-d})$$

$$\text{as } h \to +0, |2\pi nh - \lambda| \le ch \quad (18)$$

inside of clusters.

Remark

1. We can weaken Drift-condition (but we cannot skip it).

As $\varepsilon \le h$ and there are clusters we get distribution of eigenvalues

$$N_{n,h}(\lambda) = (2\pi)^{1-d} h^{1-d} \int_{a=2\pi hn,\, \varepsilon b<\lambda-2\pi nh} dxd\xi + O(h^{2-d})$$
$$\text{as } h \to +0, |2\pi nh - \lambda| \le ch \quad (18)$$

inside of clusters.

Remark

1. We can weaken Drift-condition (but we cannot skip it).
2. Yu. Safarov got similar results but with remainder estimate $o(h^{1-d})$ as the measure of periodic trajectories is positive but not all the trajectories are periodic.

As $\varepsilon \le h$ and there are clusters we get distribution of eigenvalues

$$N_{n,h}(\lambda) = (2\pi)^{1-d} h^{1-d} \int_{a=2\pi hn,\, \varepsilon b<\lambda-2\pi nh} dx d\xi + O(h^{2-d}) \qquad \text{as } h\to +0, |2\pi nh-\lambda| \le ch \quad (18)$$

inside of clusters.

Remark

1. We can weaken Drift-condition (but we cannot skip it).
2. Yu. Safarov got similar results but with remainder estimate $o(h^{1-d})$ as the measure of periodic trajectories is positive but not all the trajectories are periodic.
3. Similar results with remainder estimate $o(h^{1-d})$ got recently for perturbed Schrödinger with Coulomb potential A. Uribe and C. Villegas-Blas;

As $\varepsilon \le h$ and there are clusters we get distribution of eigenvalues

$$N_{n,h}(\lambda) = (2\pi)^{1-d} h^{1-d} \int_{a=2\pi hn,\, \varepsilon b<\lambda-2\pi nh} dx d\xi + O(h^{2-d}) \qquad \text{as } h \to +0, |2\pi nh - \lambda| \le ch \tag{18}$$

inside of clusters.

Remark

1. We can weaken Drift-condition (but we cannot skip it).
2. Yu. Safarov got similar results but with remainder estimate $o(h^{1-d})$ as the measure of periodic trajectories is positive but not all the trajectories are periodic.
3. Similar results with remainder estimate $o(h^{1-d})$ got recently for perturbed Schrödinger with Coulomb potential A. Uribe and C. Villegas-Blas;
4. And I got remainder estimate $O(\varepsilon h^{1-d} + h^{2-d} + h^{\frac{5}{6}(1-d)})$ (work in progress; singularity at 0 is very nasty here).

Curiosier and curiosier

If we want to consider more general situations, we need to understand why periodic trajectories come into this problem.

Curiosier and curiosier

If we want to consider more general situations, we need to understand why periodic trajectories come into this problem. Let us consider $\mathrm{Tr} e^{ih^{-1}tA}Q$ where Q is h-pseudo-differential operator with symbol q. Then

$$\mathrm{WF}(\mathrm{Tr} e^{ih^{-1}tA}Q) \subset \{(t,\tau) : \exists (x,\xi) \in \mathrm{supp} q, (x,\xi,x,-\xi) \in \mathrm{WF}(u)\} \quad (19)$$

where WF means wave front set and $u(x,y,t)$ is the Schwartz kernel of $e^{ih^{-1}tA}$

Curiosier and curiosier

If we want to consider more general situations, we need to understand why periodic trajectories come into this problem. Let us consider $\mathrm{Tr} e^{ih^{-1}tA}Q$ where Q is h-pseudo-differential operator with symbol q. Then

$$\mathrm{WF}(\mathrm{Tr} e^{ih^{-1}tA}Q) \subset \{(t,\tau) : \exists (x,\xi) \in \mathrm{supp} q, (x,\xi,x,-\xi) \in \mathrm{WF}(u)\} \tag{19}$$

where WF means wave front set and $u(x,y,t)$ is the Schwartz kernel of $e^{ih^{-1}tA}$ while also

$$\mathrm{Tr}(e^{ih^{-1}tA}Q) = \int e^{ih^{-1}t\tau}\, d_\tau \mathrm{Tr}(E(\tau)Q) \tag{20}$$

$E(\tau)$ is the spectral projector of A.

When we consider a scalar operator on closed manifold

$$\mathrm{WF}(u) \subset \{(x, \xi, y, -\eta, t, \tau) : a(x, \xi) = \tau, (x, \xi) = \Phi_t(y, \eta))\} \tag{21}$$

and $\Phi_t = e^{-tH_a}$ is defined by

$$\frac{dz}{dt} = -H_a(z). \tag{22}$$

When we consider a scalar operator on closed manifold

$$\mathrm{WF}(u) \subset \{(x,\xi,y,-\eta,t,\tau) : a(x,\xi) = \tau, (x,\xi) = \Phi_t(y,\eta))\} \tag{21}$$

and $\Phi_t = e^{-tH_a}$ is defined by

$$\frac{dz}{dt} = -H_a(z). \tag{22}$$

For system however (22) becomes

$$\frac{dz}{dt} \in K(z) \qquad \text{a.e.} \tag{23}$$

where $K(z)$ is a set.

When we consider a scalar operator on closed manifold

$$\mathrm{WF}(u) \subset \{(x,\xi,y,-\eta,t,\tau) : a(x,\xi)=\tau, (x,\xi)=\Phi_t(y,\eta))\} \tag{21}$$

and $\Phi_t = e^{-tH_a}$ is defined by

$$\frac{dz}{dt} = -H_a(z). \tag{22}$$

For system however (22) becomes

$$\frac{dz}{dt} \in K(z) \qquad \text{a.e.} \tag{23}$$

where $K(z)$ is a set. This is a differential inclusion (fortunately heuristically in the majority of points $K(z)$ consists of one point.

When we consider a scalar operator on closed manifold

$$\mathrm{WF}(u) \subset \{(x,\xi,y,-\eta,t,\tau) : a(x,\xi) = \tau, (x,\xi) = \Phi_t(y,\eta))\} \tag{21}$$

and $\Phi_t = e^{-tH_a}$ is defined by

$$\frac{dz}{dt} = -H_a(z). \tag{22}$$

For system however (22) becomes

$$\frac{dz}{dt} \in K(z) \qquad \text{a.e.} \tag{23}$$

where $K(z)$ is a set. This is a differential inclusion (fortunately heuristically in the majority of points $K(z)$ consists of one point. This leads to the branching (probably without implications for problem we discuss).

Under conditions to trajectories we can get remainder estimate $o(h^{1-d})$ for differential operators with coefficients with first derivatives continuous with continuity modulus $|\log t|^{-1}$. Methods of analysis are immensely powerful. However H_a is not Lipschitz function and solutions of (22) are not necessarily unique.

Under conditions to trajectories we can get remainder estimate $o(h^{1-d})$ for differential operators with coefficients with first derivatives continuous with continuity modulus $|\log t|^{-1}$. Methods of analysis are immensely powerful. However H_a is not Lipschitz function and solutions of (22) are not necessarily unique. So we get another branching phenomena.

Branching at reflection

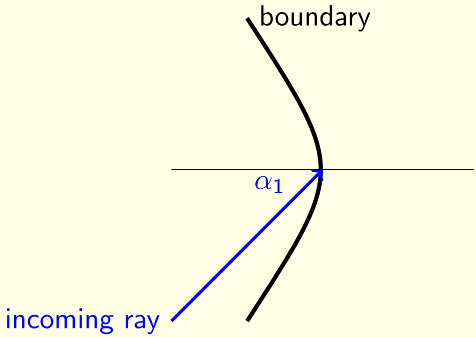
The last branching phenomena (and we will discuss it) happens when incoming trajectory generates more than one outgoing trajectory.

Branching at reflection

The last branching phenomena (and we will discuss it) happens when incoming trajectory generates more than one outgoing trajectory.

Example

Two wave equations in two domains or manifolds having a common boundary:

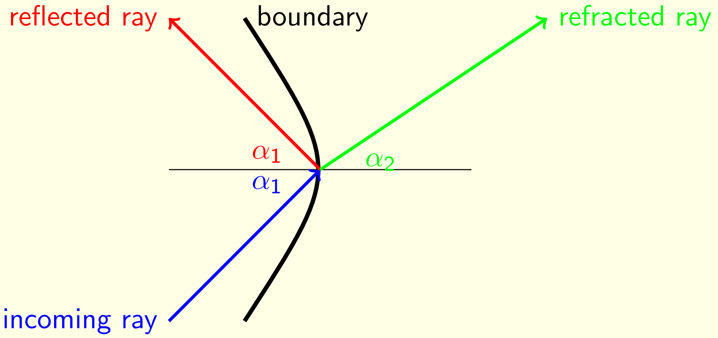


Branching at reflection

The last branching phenomena (and we will discuss it) happens when incoming trajectory generates more than one outgoing trajectory.

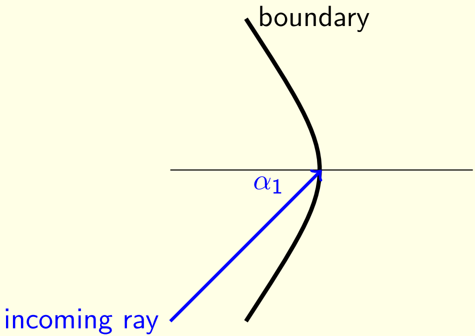
Example

Two wave equations in two domains or manifolds having a common boundary:



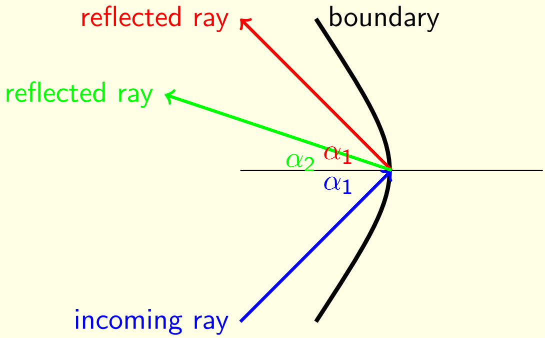
Example

Isotropic elasticity has two kinds of waves: compression waves and shear waves with two different speeds ($c_1 > c_2$)



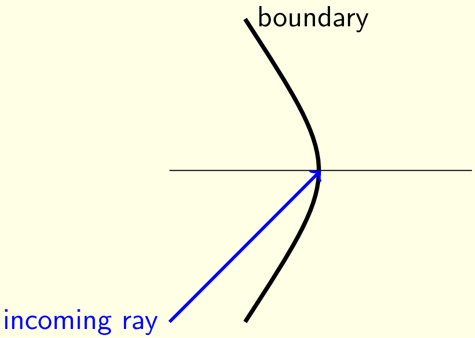
Example

Isotropic elasticity has two kinds of waves: compression waves and shear waves with two different speeds ($c_1 > c_2$)



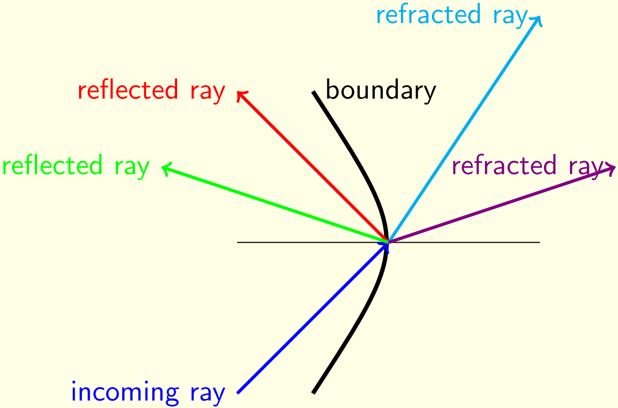
Example

Isotropic elasticity system in two domains with a common boundary



Example

Isotropic elasticity system in two domains with a common boundary



All these reflections/refractions are according to

Snell's law

$$\frac{\sin\alpha_i}{c_i}=\frac{\sin\alpha_o}{c_o}. \tag{24}$$

All these reflections/refractions are according to

Snell's law

$$\frac{\sin\alpha_i}{c_i} = \frac{\sin\alpha_o}{c_o}. \tag{24}$$

Actually if $c_i < c_o$ and $\sin\alpha_i$ is close to 1 we get $\sin\alpha_o > 1$ and there is no corresponding wave.

All these reflections/refractions are according to

Snell's law

$$\frac{\sin\alpha_i}{c_i} = \frac{\sin\alpha_o}{c_o}. \tag{24}$$

Actually if $c_i < c_o$ and $\sin\alpha_i$ is close to 1 we get $\sin\alpha_o > 1$ and there is no corresponding wave. Actually is: complex phase wave fast decaying from the boundary; no geometrical implications.

All these reflections/refractions are according to

Snell's law

$$\frac{\sin\alpha_i}{c_i} = \frac{\sin\alpha_o}{c_o}. \tag{24}$$

Actually if $c_i < c_o$ and $\sin\alpha_i$ is close to 1 we get $\sin\alpha_o > 1$ and there is no corresponding wave. Actually is: complex phase wave fast decaying from the boundary; no geometrical implications.
Things which happen when two waves are tangent to the boundary are beyond our knowledge.

All these reflections/refractions are according to

Snell's law

$$\frac{\sin\alpha_i}{c_i} = \frac{\sin\alpha_o}{c_o}. \tag{24}$$

Actually if $c_i < c_o$ and $\sin\alpha_i$ is close to 1 we get $\sin\alpha_o > 1$ and there is no corresponding wave. Actually is: complex phase wave fast decaying from the boundary; no geometrical implications.
Things which happen when two waves are tangent to the boundary are beyond our knowledge.
So, in these examples we have multivalued classical flow Φ_t. This implies a question

What the hell means periodic point

What the hell means periodic point

First definition

z is periodic if $z \in \Phi_t(z)$ for some $t \neq 0$.

What the hell means periodic point

First definition

z is periodic if $z \in \Phi_t(z)$ for some $t \neq 0$.

With this definition everything is very simple: if non-periodicity condition is fulfilled then "normal" asymptotics with the second term holds.

What the hell means periodic point

First definition

z is periodic if $z \in \Phi_t(z)$ for some $t \neq 0$.

With this definition everything is very simple: if non-periodicity condition is fulfilled then "normal" asymptotics with the second term holds.

Unfortunately we know very few cases when this condition is fulfilled: f.e. disk (or ball) and $c_j = \text{const}$: then reflection angle equals to the next incidence angle and along each branching trajectory each type of waves has a fixed angle.

Weirder than weird

But there is another case: assume that there are many periodic points but due to branching sizeable part of high-frequency energy does not come back:

Weirder than weird

But there is another case: assume that there are many periodic points but due to branching sizeable part of high-frequency energy does not come back:

Example

Consider wave equation on the upper half-sphere $\mathbb{S}^2_+$ and in the disk $\mathbb{D}$ which are glued together by their boundaries.

Weirder than weird

But there is another case: assume that there are many periodic points but due to branching sizeable part of high-frequency energy does not come back:

Example

Consider wave equation on the upper half-sphere $\mathbb{S}^2_+$ and in the disk $\mathbb{D}$ which are glued together by their boundaries. Trajectories in $\mathbb{S}^2_+$ are half-circles and periodic.

Weirder than weird

But there is another case: assume that there are many periodic points but due to branching sizeable part of high-frequency energy does not come back:

Example

Consider wave equation on the upper half-sphere $\mathbb{S}^2_+$ and in the disk $\mathbb{D}$ which are glued together by their boundaries. Trajectories in $\mathbb{S}^2_+$ are half-circles and periodic. Assume that the speed in $\mathbb{S}^2_+$ is larger than in $\mathbb{D}$; then there will be no complete internal reflection from this side and under generic boundary conditions providing energy conservation a certain portion of high-frequency energy would go from $\mathbb{S}^2_+$ to $\mathbb{D}$ and for generic angle it does not come to the point where it originated.

In the example above

$$N_h(\lambda) = \mathrm{Weyl} + (\kappa + \mathcal{Q})h^{1-d} + O(h^{1+\delta-d}) \qquad \text{as } h \to +0 \tag{25}$$

for unspecified $\delta > 0$.

In the example above

$$N_h(\lambda) = \mathsf{Weyl} + (\kappa + \mathcal{Q})h^{1-d} + O(h^{1+\delta-d}) \qquad \text{as } h \to +0 \tag{25}$$

for unspecified $\delta > 0$. However now there are no clusters but only inhomogeneity in distribution of eigenvalues.

What to do?

Tell me the wisest and the greatest, what to do to make these dynamics people to prove what we need?

What to do?

Tell me the wisest and the greatest, what to do to make these dynamics people to prove what we need?

Put them in jail and don't feed them well until they do whatever they must.

What to do?

Tell me the wisest and the greatest, what to do to make these dynamics people to prove what we need?

Put them in jail and don't feed them well until they do whatever they must. And you need to beat them too.

What to do?

Tell me the wisest and the greatest, what to do to make these dynamics people to prove what we need?

Put them in jail and don't feed them well until they do whatever they must. And you need to beat them too.

Unfortunately it is not possible and we can only beg:

Pretty-pretty-pretty-please!!!