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Singularities of two-plane fields, Legendrian curves, and the friendly Monster

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Abstract

Overview

There are families of two-plane fields in k-dimensions, called Goursat distributions which have interesting singularities. In the first lecture we will show how to construct, a universal space, "the Monster", which contains all Goursat singularities. In the second lecture we relate points of the Monster to singularities of Legendrian curves and in this way are able to solve classification problems for Goursat distributions. The main tool is Cartan's method of prolongation. Prolongation can be used to resolve plane curve singularities. In the final lecture we present work in progress relating resolution by prolongation to the classical method of resolution, resolution by blow-up. As we proceed through the lectures, we will point out open problems which occur, some of which would be appropriate thesis problems. All the work reported on here is joint with Mikhail Zhitomirskii of Technion (Haifa).

Lecture 1. Distributions, prolongation, and the universal Goursat distribution.

By a 'distribution' we mean a sub-bundle of the tangent bundle of a manifold. We begin with a taxonomy of distributions and their symmetry groups, and then focus on a particular family of rank 2 distributions called Goursat distributions. The first three Goursat distributions are a surface with its tangent bundle, a contact three-manifold, and an Engel 4-manifold. After that, Piotr Mormul discovered that the number of distinct Goursat germs living in a k-dimensional manifold, grows rapidly, like the odd Fibonacci numbers, up to dimension 10, at which point continuua of distinct Goursat germs appear.

We can construct a Goursat distribution on a manifold of dimension $k+1$ out of one on a manifold of dimension k by using Cartan's method of prolongation, as adapted by Robert Bryant and Lucas Hsu. In fact, as we will show, *all* Goursat distribution germs arise this way, starting from a surface M_0. In this way, we are able to construct a tower of manifolds with Goursat distributions $\ldots \to M_{n+1} \to M_n \to M_{n-1} \to \ldots \to M_0$, $dim(M_n) = n+2$, which we call the 'Monster'. The Monster has the universal property that any Goursat distribution germ in dimension n is realized somewhere within M_{n-2}. A good chunk of the lecture will thus be devoted to the prolongation method, and how it applies to distributions, to symmetries of distributions, and to integral curves of distributions.

Lecture 2. Touching the Monster with singular plane curves.

Symmetries of the Monster all arise out of contact transformations (acting on M_1). This fact suggests that anything we want to know about the Monster (and hence about Goursat distributions) might be somehow encoded by singularites of Legendrian curve germs. We describe how to implement this encoding and how to use it to solve essentially any classification o problems concerning Goursat distributions. In so doing, we will have to recall the (Newton-)Puiseux expansion, and the Puiseux characteristic of a plane curve singularity, following C.T.C. Wall's book 'Singularities of Plane Curves'.

Lecture 3. Prolongation = blow-up?

Plane curve singularities are traditionally "resolved" (made non-singular) by the method of blow-up. Prolongation provides an alternative method of resolution. Are the two resolutions the same? Probably not in general, since blow-up involves choices along the way, while prolongation does not. However, for "unibranched" singularities the two methods do yield isomorphic resolutions, according to results of Vidya Swaminathan (graduate student; UCSC). We will describe the two methods and some details of the isomorphism. At the end we will present several other open problems.

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Our first paper together: R.Montgomery, M.Zhitomirskii, *Geometric Approach to Goursat Flags.* Annales de L'Institut Henri Poincare. Analyse non Lineaire **18** (2001), no. 4, 459-493.

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