Hyperdeterminants as integrable 3D difference equations

(joint project with Sergey Tsarev, Krasnoyarsk State Pedagogical University)

Thomas Wolf

Brock University, Ontario

Ste-Adèle, June 26, 2008

Plan:

- The simplest (hyper)determinants: 2×2 and $2 \times 2 \times 2$.

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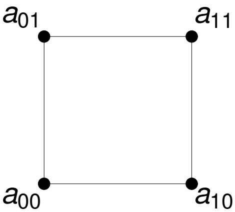
- Is $2 \times 2 \times 2 \times 2$ hyperdeterminant integrable??

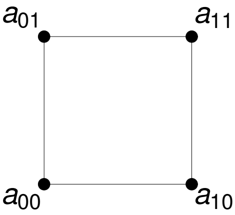
- 2×2



$$det = a_{00}a_{11} - a_{01}a_{10}$$

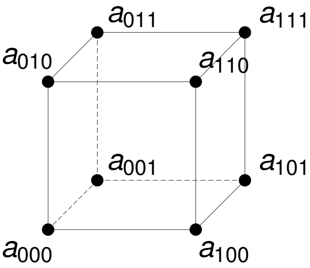
2×2 and $2 \times 2 \times 2$ (hyper)determinants

- 2×2



$$det = a_{00}a_{11} - a_{01}a_{10}$$

- $2 \times 2 \times 2$



$$\begin{aligned}
hyperdet = \\
& a_{111}^2 a_{000}^2 + a_{100}^2 a_{011}^2 + a_{101}^2 a_{010}^2 + a_{110}^2 a_{001}^2 - \\
& 2a_{111}a_{110}a_{001}a_{000} - 2a_{111}a_{101}a_{010}a_{000} - \\
& 2a_{111}a_{100}a_{011}a_{000} - 2a_{110}a_{101}a_{010}a_{001} - \\
& 2a_{110}a_{100}a_{011}a_{001} - 2a_{101}a_{100}a_{011}a_{010} + \\
& 4a_{111}a_{100}a_{010}a_{001} + 4a_{110}a_{101}a_{011}a_{000}
\end{aligned}$$

Modern applications of hyperdeterminants

- quantum information ("Multipartite Entanglement and Hyperdeterminants", *A. Miyake, M. Wadati*, Quant. Info. Comp. 2 (Special), 540-555 (2002),
- biomathematics ("Estimating vaccine coverage by using computer algebra", *R.Altmann, K. Altmann*, Mathematical Medicine and Biology 2000 17(2):137-146,
- numerical analysis ("Tensor rank and the ill-posedness of the best low-rank approximation problem", *V. de Silva and L.-H. Lim*, `http://arxiv.org/abs/math/0607647`)
- data analysis ("Kruskal's condition for uniqueness in Candecomp/Parafac..." *A. Stegeman, J. M. F. Ten Berge* Computational Statistics & Data Analysis 50(1): 210-220 (2006)),
- a few other fields, in particular *discrete integrable systems*.

The polynomial invariants of four qubits

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(Dated: December 14, 2002)

We describe explicitly the algebra of polynomial functions on the Hilbert space of four qubit states which are invariant under the SLOCC group $SL(2,\mathbb{C})^4$. From this description, we obtain a closed formula for the hyperdeterminant in terms of low degree invariants.

PACS numbers: O3.67.Hk, 03.65.Ud, 03.65.Fd

I. INTRODUCTION

Various classifications of states with up to four qubits have been recently proposed, with the aim of understanding the different ways in which multipartite systems can be entangled [1, 2, 3, 4, 5]. However, one cannot expect that such classifications will be worked out for an arbitrary number k of qubits, and there is a need for a coarser classification scheme which would be computable for general k. In [6], Klyachko proposed to assimilate entanglement with the notion of *semi-stability* of geometric invariant theory. In this context, a semi-stable state is one which can be separated from 0 by a polynomial invariant of $SL(2,\mathbb{C})^k$, the point in the geometric approach being that explicit knowledge of the invariants is in principle not necessary to check this property.

In this paper, we construct a complete set of algebraic invariants of 4-qubit states. This allows us to identify under the SLOCC group $SL(2,\mathbb{C})^4$. This amounts to the construction of a moduli space for four qubit states. Our strategy is to find first the Hilbert series of the algebra of invariants $\mathcal{J}$. Next, we construct by classical methods four invariants of the required degrees. The knowledge of the Hilbert series reduces then the proof of algebraic independence and completeness to simple verifications. The values of the invariants on the orbits of [4] are tabulated in the Appendix.

II. THE HILBERT SERIES

Let $\mathcal{J}_d$ be the space of G-invariant homogeneous polynomial functions of degree d in the variables A_{ijkl}. Using some elementary representation theory, it is not difficult to show that $\mathcal{J}_d$ is zero for d odd, and that for $d = 2m$ even, the dimension of $\mathcal{J}_d$ is equal to the multiplicity of

Strings, black holes, and quantum information

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We find multiple relations between extremal black holes in string theory and 2- and 3-qubit systems in quantum information theory. We show that the entropy of the axion-dilaton extremal black hole is related to the concurrence of a 2-qubit state, whereas the entropy of the STU black holes, Bogomol'nyi-Prasad-Sommerfield (BPS) as well as non-BPS, is related to the 3-tangle of a 3-qubit state. We relate the 3-qubit states with the string theory states with some number of D-branes. We identify a set of large black holes with the maximally entangled Greenberger, Horne, Zeilinger (GHZ) class of states and small black holes with separable, bipartite, and W states. We sort out the relation between 3-qubit states, twistors, octonions, and black holes. We give a simple expression for the entropy and the area of stretched horizon of small black holes in terms of a norm and 2-tangles of a 3-qubit system. Finally, we show that the most general expression for the black hole and black ring entropy in $N = 8$ supergravity/M theory, which is given by the famous quartic Cartan $E_{7(7)}$ invariant, can be reduced to Cayley's hyperdeterminant describing the 3-tangle of a 3-qubit state.

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I. INTRODUCTION

During the last 15 years there was a significant progress in two different fields of knowledge: a description of black holes in string theory and the theory of quantum information and quantum computing. At the first glance these two subjects may seem quite distant from each other. However, there are some general themes, such as entanglement, information, and entropy, which repeatedly appear both in the theory of black holes and in the theory of quantum information.

Studies of stringy black holes began with a discovery of a broad class of new extremal black hole solutions [1], investigation of their supersymmetry [2], a discovery of the black hole attractor mechanism [3], and the microscopic calculation of black hole entropy [4]. Investigation of stringy black holes resulted in a better understanding of the information loss paradox in the theory of black holes, revealed nonperturbative symmetries between different versions of string theory, and stimulated what is now called "the second string theory revolution" [5–7]. For reviews on stringy black holes see [8]. On the other hand, there were many exciting developments in the theory of quantum computation, quantum cryptography, quantum cloning, quantum teleportation, classification of entangled states, and investigation of a measure of entanglement in the context of the quantum information theory; for a review see e.g. [9]. It would be quite useful to find some links between these different sets of results.

One of the first steps in this direction was made in a recent paper by Michael Duff [10]. He discovered that a complicated expression for the entropy of the so-called extremal STU black holes[1] obtained in [12] can be represented in a very compact way as Cayley's hyperdeterminant [13], which appears in the theory of quantum information in the calculation of the measure of entanglement of the 3-qubit system (3-tangle) [14,15]. The STU black holes represent a broad class of classical solutions of the effective supergravity derived from string theory in [16].

As emphasized in [10], the intriguing relation between STU extremal black holes and 3-qubit systems in quantum information theory may be coincidental. It may be explained, e.g., by the fact that both theories have the same underlying symmetry. At the level of classical supergravity the symmetry of extremal STU black holes is $[SL(2, \mathbb{R})]^3$. This symmetry may be broken down to $[SL(2, \mathbb{Z})]^3$ by quantum corrections or by the requirement that the electric and magnetic charges have to be quantized. In string theory a consistent microscopic description of the extremal black holes requires $[SL(2, \mathbb{Z})]^3$ symmetry. In ABC system the symmetry is $[SL(2, \mathbb{C})]^3$.

But even if the relation between the STU black holes and the 3-qubit system boils down to their underlying symmetry, this fact by itself can be quite useful. It may allow us to obtain new classes of black hole solutions and provide their interpretation based on the general formalism of quantum information. It may also provide us with an extremely

[1]The explicit construction of BPS black holes with four charges and a finite area of the horizon within $D = 4$ $N = 4$ toroidally compactified string theory was obtained in [11]. This solution has an embedding as a generating solution in the STU model.

Classification of multipartite entangled states by multidimensional determinants

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We find that multidimensional determinants "hyperdeterminants," related to entanglement measures (the so-called concurrence, or 3-tangle for two or three qubits, respectively), are derived from a duality between entangled states and separable states. By means of the hyperdeterminant and its singularities, the single copy of multipartite pure entangled states is classified into an onion structure of every closed subset, similar to that by the local rank in the bipartite case. This reveals how inequivalent multipartite entangled classes are partially ordered under local actions. In particular, the generic entangled class of the maximal dimension, distinguished as the nonzero hyperdeterminant, does not include the maximally entangled states in Bell's inequalities in general (e.g., in the $n \geqslant 4$ qubits), contrary to the widely known bipartite or three-qubit cases. It suggests that not only are they never locally interconvertible with the majority of multipartite entangled states, but they would have no grounds for the canonical n-partite entangled states. Our classification is also useful for the mixed states.

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I. INTRODUCTION

Entanglement is the quantum correlation exhibiting nonlocal (nonseparable) properties. It is supposed to be never strengthened, on average, by local operations and classical communication (LOCC). In particular, entanglement in multiparties is of fundamental interest in quantum many-body theory [1], and makes quantum information processing (QIP), e.g., distillation protocol, more efficient than relying on entanglement only in two parties [2]. Here, we classify and characterize the multipartite entanglement that has yet to be understood, compared with the bipartite one.

For the single copy of bipartite pure states on $\mathcal{H}(\mathbb{C}^{k+1}) \otimes \mathcal{H}(\mathbb{C}^{k+1})$, we are interested in whether a state $|\Psi\rangle$ can convert to another state $|\Phi\rangle$ by LOCC. It is convenient to consider the Schmidt decomposition,

$$|\Psi\rangle = \sum_{i_1,i_2=0}^{k} a_{i_1,i_2}|i_1\rangle\otimes|i_2\rangle = \sum_{j=0}^{k} \lambda_j |e_j\rangle\otimes|e'_j\rangle, \qquad (1)$$

where the computational basis $|i_j\rangle$ is transformed to a local biorthonormal basis $|e_j\rangle,|e'_j\rangle$, and the Schmidt coefficients λ_j can be taken as $\lambda_j \geqslant 0$. We call the number of nonzero λ_j the (Schmidt) local rank r. Then the LOCC convertibility is given by a majorization rule between the coefficients λ_j of $|\Psi\rangle$ and those of $|\Phi\rangle$ [3]. This suggests that the structure of entangled states consists of partially ordered, continuous classes labeled by a set of λ_j. In particular, $|\Psi\rangle$ and $|\Phi\rangle$ belong to the same class under the LOCC classification if and only if all continuous λ_j coincide.

Suppose we are concerned with a coarse-grained classification by the so-called stochastic LOCC (SLOCC) [4,5], where we identify $|\Psi\rangle$ and $|\Phi\rangle$ that are interconvertible back and forth with (maybe different) nonvanishing probabilities. This is because $|\Psi\rangle$ and $|\Phi\rangle$ are supposed to perform the same tasks in QIP although their probabilities of success differ. Later, we find that this SLOCC classification is still fine grained to classify the multipartite entanglement. Mathematically, two states belong to the same class under SLOCC if and only if they are converted by an *invertible* local operation having a nonzero determinant [5]. Thus the SLOCC classification is equivalent to the classification of orbits of the natural action: direct product of general linear groups $\mathrm{GL}_{k+1}(\mathbb{C}) \times \mathrm{GL}_{k+1}(\mathbb{C})$ [6]. The local rank r in Eq. (1) [7], equivalently the rank of a_{i_1,i_2}, is found to be preserved under SLOCC. A set S_j of states of the local rank $\leqslant j$ is a *closed* subvariety under SLOCC, and S_{j-1} is the singular locus of S_j. This is how the local rank leads to an "onion" structure (mathematically the stratification):

$$S_{k+1} \supset S_k \supset \cdots \supset S_1 \supset S_0 = \varnothing, \qquad (2)$$

and $S_j - S_{j-1}(j=1,\ldots,k+1)$ give $k+1$ classes of entangled states. Since the local rank can decrease by *noninvertible* local operations, i.e., general LOCC [4,5,8], these classes are totally ordered such that, in particular, the outermost generic set $S_{k+1}-S_k$ is the class of maximally entangled states and the innermost set $S_1 (=S_1 - S_0)$ is that of separable states.

For the single copy of multipartite pure states,

$$|\Psi\rangle = \sum_{i_1,\ldots,i_n=0}^{k_1,\ldots,k_n} a_{i_1,\ldots,i_n}|i_1\rangle\otimes\cdots\otimes|i_n\rangle, \qquad (3)$$

there are difficulties in extending the Schmidt decomposition for a multiorthonormal basis [9]. Moreover, an attempt to use the tensor rank of $a_{i_1,\ldots,i_n}$ [10] falls down since S_j, defined by it, is not always closed [11,12]. For three qubits, Dür *et al.* showed that SLOCC classifies the whole states M into *finite* classes, and in particular there exist two inequivalent, Greenberger-Horne-Zeilinger (GHZ) and W, classes of the tripartite entanglement [5]. They also pointed out that this case is exceptional since the action $\mathrm{GL}_{k_1+1}(\mathbb{C}) \times \cdots \times \mathrm{GL}_{k_n+1}(\mathbb{C})$ has *infinitely many* orbits in general (e.g., for $n \geqslant 4$).

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Green–Schwarz, Nambu–Goto actions, and Cayley's hyperdeterminant

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Abstract

It has been recently shown that Nambu–Goto action can be re-expressed in terms of Cayley's hyperdeterminant with the manifest $SL(2,\mathbb{R}) \times SL(2,\mathbb{R}) \times SL(2,\mathbb{R})$ symmetry. In the present Letter, we show that the same feature is shared by Green–Schwarz σ-model for $N=2$ superstring whose target space–time is $D=2+2$. When its zweibein field is eliminated from the action, it contains the Nambu–Goto action which is nothing but the square root of Cayley's hyperdeterminant of the pull-back in superspace $\sqrt{\mathcal{D}\mathrm{et}(\Pi_{i\alpha\dot\alpha})}$ manifestly invariant under $SL(2,\mathbb{R}) \times SL(2,\mathbb{R}) \times SL(2,\mathbb{R})$. The target space–time $D=2+2$ can accommodate self-dual supersymmetric Yang–Mills theory. Our action has also fermionic κ-symmetry, satisfying the criterion for its light-cone equivalence to Neveu–Schwarz–Ramond formulation for $N=2$ superstring.

PACS: 11.25.-w; 11.30.Pb; 11.30.Fs; 02.30.Ik

Keywords: Cayley's hyperdeterminant; Green–Schwarz and Nambu–Goto actions; 2 + 2 dimensions; Self-dual supersymmetric Yang–Mills; $N = (1, 1)$ space–time supersymmetry; $N = 2$ superstring

1. Introduction

Cayley's hyperdeterminant [1], initially an object of mathematical curiosity, has found its way in many applications to physics [2]. For instance, it has been used in the discussions of quantum information theory [3,4], and the entropy of the STU black hole [5,6] in four-dimensional string theory [7].

More recently, it has been shown [8] that Nambu–Goto (NG) action [9,10] with the $D=2+2$ target space–time possesses the manifest global $SL(2,\mathbb{R}) \times SL(2,\mathbb{R}) \times SL(2,\mathbb{R}) \equiv [SL(2,\mathbb{R})]^3$ symmetry. In particular, the square root of the determinant of an inner product of pull-backs can be rewritten exactly as a Cayley's hyperdeterminant [1] realizing the manifest $[SL(2,\mathbb{R})]^3$ symmetry.

It is to be noted that the space–time dimensions $D=2+2$ pointed out in [8] are nothing but the consistent target space–time of $N=2$[1] NSR superstring [13–19]. However, the NSR formulation [16,17] has a drawback for rewriting it purely in terms of a determinant, due to the presence of fermionic superpartners on the 2D world-sheet. On the other hand, it is well known that a GS formulation [12] without explicit world-sheet supersymmetry is classically equivalent to a NSR formulation [11] on the light-cone, when the former has fermionic κ-symmetry [15,20]. From this viewpoint, a GS σ-model formulation in [14] of $N=2$ superstring [16–18] seems more advantageous, despite the temporary sacrifice of world-sheet supersymmetry. However, even the GS formulation [14] itself has an obstruction, because obviously the kinetic term in the

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[1] The $N=2$ here implies the number of world-sheet supersymmetries in the Neveu–Schwarz–Ramond (NSR) formulation [11]. Its corresponding Green–Schwarz (GS) formulation [12–14] might be also called '$N=2$' GS superstring in the present Letter. Needless to say, the number of world-sheet supersymmetries should *not* be confused with that of space–time supersymmetries, such as $N=1$ for type I superstring, or $N=2$ for type IIA or IIB superstring [15].

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KRUSKAL'S POLYNOMIAL FOR $2 \times 2 \times 2$ ARRAYS AND A GENERALIZATION TO $2 \times n \times n$ ARRAYS

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A remarkable difference between the concept of rank for matrices and that for three-way arrays has to do with the occurrence of non-maximal rank. The set of $n \times n$ matrices that have a rank less than n has zero volume. Kruskal pointed out that a $2 \times 2 \times 2$ array has rank three or less, and that the subsets of those $2 \times 2 \times 2$ arrays for which the rank is two or three both have positive volume. These subsets can be distinguished by the roots of a certain polynomial. The present paper generalizes Kruskal's results to $2 \times n \times n$ arrays. Incidentally, it is shown that two $n \times n$ matrices can be diagonalized simultaneously with positive probability.

Key words: rank, three-way arrays, PARAFAC, CANDECOMP, simultaneous diagonalization.

Kruskal (1989, p. 10) has drawn attention to the remarkable fact that the subset of those $2 \times 2 \times 2$ arrays for which the rank is less than the maximum possible rank, has positive volume. Specifically, a $2 \times 2 \times 2$ array cannot have a rank greater than three,

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Estimating vaccine coverage by using computer algebra

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The approach of N Gay for estimating the coverage of a multivalent vaccine from antibody prevalence data in certain age cohorts is complemented by using computer aided elimination theory of variables. Hereby, Gay's usage of numerical approximation can be replaced by exact formulae which are surprisingly nice, too.

Keywords: multivalent vaccine; coverage rate; antibody prevalence; hyperdeterminants.

1. Introduction

(1.1) Monitoring of vaccine preventable diseases is an important public health issue. But in some European countries like Germany there is a lack of surveillance systems, and vaccine coverage is not monitored systematically. Sometimes antibody

I. M. Gelfand
M. M. Kapranov
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Discriminants, Resultants, and Multidimensional Determinants

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On discrete 3-dimensional equations associated with the local Yang-Baxter relation

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December 1995

Abstract

The local Yang-Baxter equation (YBE), introduced by Maillet and Nijhoff, is a proper generalization to 3 dimensions of the zero curvature relation. Recently, Korepanov has constructed an infinite set of integrable 3-dimensional lattice models, and has related them to solutions to the local YBE. The simplest Korepanov's model is related to the star-triangle relation in the Ising model. In this paper the corresponding discrete equation is derived. In the continuous limit it leads to a differential 3d equation, which is symmetric with respect to all permutations of the three coordinates. A similar analysis of the star-triangle transformation in electric networks leads to the discrete bilinear equation of Miwa, associated with the BKP hierarchy.

Some related operator solutions to the tetrahedron equation are also constructed.

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the star-triangle transformation formulae (3.3) together with (3.5) and (3.6) imply the following equation on the function $f(n)$:

$$2\sum_{j=1}^{4}(f(n+e_j)f(n-e_j))^2+4\prod_{j=1}^{4}f(n+e_j)+4\prod_{j=1}^{4}f(n-e_j)$$
$$=\left(\sum_{j=1}^{4}f(n+e_j)f(n-e_j)\right)^2, \quad e_4=-e_1-e_2-e_3. \tag{3.7}$$

To prove the theorem one has to exclude the dependent variables by calculating the corresponding resultants, which in intermediate steps are cumbersome. In these calculations the Maple computer system has been used.

Consider now a particular continuous limit of the equation (3.7). Let ϵ be the lattice spacing. Introduce the field

$$\phi(\epsilon n)=3\log(f(n)), \quad n\in M. \tag{3.8}$$

Theorem 3 *In the limit $\epsilon\to 0$ with $x=(x_1,x_2,x_3)=\epsilon n\in R^3$ fixed, the equation (3.7) for the function (3.8) is reduced to the following differential equation:*

$$6\Delta_2\phi\sum_{j=1}^{4}(\partial_j^2\phi)^2=2(\Delta_3\phi)^2+(\Delta_2\phi)^3+8\sum_{j=1}^{4}(\partial_j^2\phi)^3, \tag{3.9}$$

where $\phi=\phi(x)$, $x=(x_1,x_2,x_3)$,

$$\Delta_a=\sum_{j=1}^{4}\partial_j^a, \quad a=2,3,$$

$$\partial_j=\partial/\partial x_j, \quad j=1,2,3; \quad \partial_4=-\partial_1-\partial_2-\partial_3.$$

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Lattice Geometry of the Discrete Darboux, KP, BKP and CKP Equations. Menelaus' and Carnot's Theorems

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Abstract

Möbius invariant versions of the discrete Darboux, KP, BKP and CKP equations are derived by imposing elementary geometric constraints on an (irregular) lattice in a three-dimensional Euclidean space. Each case is represented by a fundamental theorem of plane geometry. In particular, classical theorems due to Menelaus and Carnot are employed. An interpretation of the discrete CKP equation as a permutability theorem is also provided.

1 Introduction

so that, in terms of the potential τ, the system $(6.8)_1$ becomes

$$\begin{aligned}(\tau\tau_{123} - \tau_1\tau_{23} - \tau_2\tau_{31} - \tau_3\tau_{12})^2 \\ = 4(\tau_1\tau_2\tau_{23}\tau_{31} + \tau_2\tau_3\tau_{31}\tau_{12} + \tau_3\tau_1\tau_{12}\tau_{23} - \tau_1\tau_2\tau_3\tau_{123} - \tau\tau_{12}\tau_{23}\tau_{31}).\end{aligned} \tag{6.11}$$

This quartic integrable lattice equation may also be obtained in a different manner. Thus, it is observed that the discrete KP equation may be identified with the superposition principle of solutions of the KP hierarchy generated by the classical Darboux transformation (see, e.g., [22]). In a similar manner, the generic binary Darboux transformation [23] associated with the KP hierarchy gives rise to the discrete Darboux system. The binary Darboux transformation may be specialized in such a way that it induces a Bäcklund transformation for the BKP hierarchy. The corresponding permutability theorem is precisely of the form of the discrete BKP equation [24]. The KP hierarchy of C type is generated via the compatibility of an infinite hierarchy of linear differential equations of

The classical heritage: A.Cayley *et al.*

A. Cayley. “On the theory of linear transformations”, Cambridge Math. J., 1845, v. IV, p. 193-209.

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I.M.Gelfand, M.M.Kapranov, A.V.Zelevinsky. “Discriminants, resultants, and multidimensional determinants”, Birkhauser, 1994.

The definition of hyperdeterminants and its variations

(A.Cayley, 1845)
Let $A = (a_{i_1 i_2 \cdots i_r})$ be a hypermatrix with $0 \leq i_s \leq n_s$. The polylinear form

$$U = \sum_{i_1 \cdots i_r} a_{i_1 \cdots i_r} x^{(1)}_{i_1} \cdots x^{(r)}_{i_r}$$

defines a hypersurface $U = 0$ in $\mathbb{C}P^{n_1} \times \ldots \times \mathbb{C}P^{n_r}$.

It is *singular* $\iff$ $\left\{ \frac{\partial U}{\partial x^{(k)}_{i_s}} = 0 \right\}$ $\iff$ $hyperdet(A) = 0$

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- it is an irreducible polynomial in the variables $a_{i_1 i_2 \cdots i_r}$;
- it has "the same" symmetry properties as the usual determinant, in particular it is $SL(\mathbb{C}, n_1) \times \cdots \times SL(\mathbb{C}, n_r)$-invariant.

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NOTE: $(n_1 + 1) \times ... \times (n_r + 1)$ - hyperdet !!

The variations of the definition of hyperdeterminants

A.Cayley, 1845

A review of other definitions:
N.P. Sokolov, Prostranstvennye matricy i ih prilozhenija, 1960, (299 p.) in Russian.

The next step: $2 \times 2 \times 2 \times 2$

Reminder:

2×2: 2 terms, $deg = 2$

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$2 \times 2 \times 2 \times 2$ computed for the first time in:
Debbie Grier, Peter Huggins, Bernd Sturmfels, and Josephine Yu, The Hyperdeterminant and Triangulations of the 4-Cube, `www.arXiv.org:math.CO/0602149`

$2 \times 2 \times 2 \times 2$: The *intrigue*

A.Cayley: an expression for the $2 \times 2 \times 2 \times 2$ - hyperdet with around 340 terms...

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J.Weyman, A.Zelevinski. Singularities of hyperdeterminants, Ann. Inst. Fourier, 1996, v. 46, p. 591-644:
Schläfli's method works only for a few first steps, including computation of the $2 \times 2 \times 2 \times 2$ - hyperdeterminant.

FORM computations: how far can we reach now?

October 2007 (Tsarev & Wolf): an “easy” computation of the $2 \times 2 \times 2 \times 2$ - hyperdeterminant using FORM.

The computation required 8 hours on a 3Mhz processor and some 800 Mb of temporary disk storage.

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Testing $SL(\mathbb{C}, n_1) \times \cdots \times SL(\mathbb{C}, n_r)$-invariance required some 10 hours and 200 Gb of temporary disk storage (number of terms in intermediate expresions: $\sim 800,000,000$ terms.

```
            det4 2864941 Terms left      =    2864940
                         Bytes used      =  422423784

Time =   29657.90 sec    Generated terms =    2883583
            det4 2883584 Terms left      =    2883583
                         Bytes used      =  424704220

Time =   29658.14 sec    Generated terms =    2894276
            det4 2894276 Terms left      =    2894276
                         Bytes used      =  425939052

Time =   29659.61 sec
            det4         Terms active    =    2894276
                         Bytes used      =  427167534

Time =   29674.63 sec    Generated terms =    2894276
            det4         Terms in output =    2894276
                         Bytes used      =  425929418

   det4 =
      256*aa(1,1,1,1)^9*aa(1,1,2,2)*aa(1,2,1,2)*aa(1,2,2,1)*aa(2,1,1,2)*aa(2,1
      ,2,1)*aa(2,2,1,1)*aa(2,2,2,2)^9 - 256*aa(1,1,1,1)^9*aa(1,1,2,2)*aa(1,2,1
      ,2)*aa(1,2,2,1)*aa(2,1,1,2)*aa(2,1,2,1)*aa(2,2,1,2)*aa(2,2,2,1)*aa(2,2,2
      ,2)^8 - 256*aa(1,1,1,1)^9*aa(1,1,2,2)*aa(1,2,1,2)*aa(1,2,2,1)*aa(2,1,1,2
      )*aa(2,1,2,2)*aa(2,2,1,1)*aa(2,2,2,1)*aa(2,2,2,2)^8 + 256*aa(1,1,1,1)^9*
      aa(1,1,2,2)*aa(1,2,1,2)*aa(1,2,2,1)*aa(2,1,1,2)*aa(2,1,2,2)*aa(2,2,1,2)*
      aa(2,2,2,1)^2*aa(2,2,2,2)^7 - 256*aa(1,1,1,1)^9*aa(1,1,2,2)*aa(1,2,1,2)*
      aa(1,2,2,1)*aa(2,1,2,1)*aa(2,1,2,2)*aa(2,2,1,1)*aa(2,2,1,2)*aa(2,2,2,2)^
      8 + 256*aa(1,1,1,1)^9*aa(1,1,2,2)*aa(1,2,1,2)*aa(1,2,2,1)*aa(2,1,2,1)*
      aa(2,1,2,2)*aa(2,2,1,2)^2*aa(2,2,2,1)*aa(2,2,2,2)^7 + 256*aa(1,1,1,1)^9*
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      )*aa(2,2,2,1)*aa(2,2,2,2)^7 - 256*aa(1,1,1,1)^9*aa(1,1,2,2)*aa(1,2,1,2)*
```

```
Time =   31290.03 sec    Generated terms = 1281987895
            tst1       1 Terms left      =  835386178
                         Bytes used     =171824731780

Time =   31290.37 sec    Generated terms = 1282006278
            tst1       1 Terms left      =  835400556
                         Bytes used     =171827842210

Time =   31290.73 sec    Generated terms = 1282024661
            tst1       1 Terms left      =  835415471
                         Bytes used     =171831034024

Time =   31291.07 sec    Generated terms = 1282043044
            tst1       1 Terms left      =  835431117
                         Bytes used     =171834256902

Time =   31291.31 sec    Generated terms = 1282056228
            tst1       1 Terms left      =  835442033
                         Bytes used     =171836439360

Time =   31291.82 sec
            tst1         Terms active    =  835364033
                         Bytes used     =171819119808

Time =   34056.96 sec    Generated terms = 1282056228
            tst1         Terms in output =    2894276
                         Bytes used      =  421572450

    skip;
    L rez1=tst1-d4sch;

    .end
```

Principal Minor Assignment Problem

Olga Holtz, Bernd Sturmfels,
Hyperdeterminantal relations among symmetric principal minors,
Journal of Algebra, v. 316 (2007), p. 634–648,
also `arXiv:math.RA/0604374v2`

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This result $\Longleftrightarrow$ integrability of the $2 \times 2 \times 2$ - hyperdeterminant!

Hyperdeterminantal relations among symmetric principal minors

Olga Holtz

Department of Mathematics

University of California-Berkeley

`holtz@math.berkeley.edu`

joint work with Bernd Sturmfels

Goal and problem

Given a matrix A, let A_I denote its principal minor with rows and columns indexed by I.

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$$\textbf{Example :} \quad A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix}$$

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$$A_{1,3} = \det \begin{bmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{bmatrix}$$

Goal and problem

The principal minors of an $n \times n$ matrix form a vector

$$A_* := (A_I)_{I \subseteq [n]}$$

of length 2^n ($A_\emptyset := 1$).

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We give an answer in the symmetric case.

Why should we care?

This problem is related to

- **matrix theory**: inverse eigenvalue problems, detection of P-matrices and GKK-matrices;

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- **matrix theory**: inverse eigenvalue problems, detection of P-matrices and GKK-matrices;
- **combinatorics**: counting spanning trees of a graph;
- **probability & math. physics**: determinantal processes (quantum mechanincs of fermions, eigenvalues of random matrices, random spanning trees and non-intersecting paths).

Hyperdeterminantal relations

$2\times 2\times 2$-hyperdeterminant of A_* must vanish:

$$\begin{aligned}
&A_\emptyset^2 A_{123}^2 + A_1^2 A_{23}^2 + A_2^2 A_{13}^2 + A_3^2 A_{12}^2 \\
&+ 4\cdot A_\emptyset A_{12} A_{13} A_{23} + 4\cdot A_1 A_2 A_3 A_{123} \\
&- 2\cdot A_\emptyset A_1 A_{23} A_{123} - 2\cdot A_\emptyset A_2 A_{13} A_{123} \\
&- 2\cdot A_\emptyset A_3 A_{12} A_{123} - 2\cdot A_1 A_2 A_{13} A_{23} \\
&- 2\cdot A_1 A_3 A_{12} A_{23} \; - 2\cdot A_2 A_3 A_{12} A_{13} \; = \; 0.
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&-\; 2\cdot A_\emptyset A_1 A_{23} A_{123} \;-\; 2\cdot A_\emptyset A_2 A_{13} A_{123} \\
&-\; 2\cdot A_\emptyset A_3 A_{12} A_{123} \;-\; 2\cdot A_1 A_2 A_{13} A_{23} \\
&-\; 2\cdot A_1 A_3 A_{12} A_{23} \;-\; 2\cdot A_2 A_3 A_{12} A_{13} \;=\; 0.
\end{aligned}$$

This comes from

$$\begin{aligned}
&(A_{123} - A_{12}A_3 - A_{13}A_2 - A_{23}A_1 + 2A_1A_2A_3)^2 \\
&=\; 4\cdot(A_1A_2 - A_{12})(A_2A_3 - A_{23})(A_1A_3 - A_{13}).
\end{aligned}$$

Generalization

Moreover, the $2{\times}2{\times}2$ hyperdeterminantal relations hold after the substitution

$$\begin{aligned} A_{\emptyset} &\mapsto A_{I}, & A_{1} &\mapsto A_{I\cup\{j_1\}} \\ A_{2} &\mapsto A_{I\cup\{j_2\}}, & A_{3} &\mapsto A_{I\cup\{j_3\}}, \\ A_{12} &\mapsto A_{I\cup\{j_1,j_2\}}, & A_{13} &\mapsto A_{I\cup\{j_1,j_3\}}, \\ A_{23} &\mapsto A_{I\cup\{j_2,j_3\}}, & A_{123} &\mapsto A_{I\cup\{j_1,j_2,j_3\}} \end{aligned}$$

for any subset $I \subset [n]$ and any $j_1, j_2, j_3 \in [n]\backslash I$.

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We call all of these hyperdeterminantal relations of format $2\times2\times2$.

Schur complements

The Schur complement of a principal submatrix H in a matrix A is the matrix $A/H := E - FH^{-1}G$

$$\text{where} \quad A =: \begin{bmatrix} E & F \\ G & H \end{bmatrix}.$$

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Schur's identity

$$(A/H)_\alpha = \frac{A_{I\cup\alpha}}{A_I},$$

holds, assuming H is indexed by I.

Consequences of Schur

Any identity among principal minors can be generalized using Schur complements:

- replace each minor A_α by $A_{\alpha\cup I}/A_I$,
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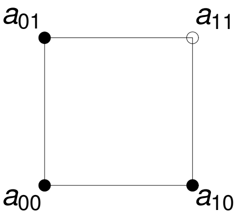
Theorem. The principal minors of a symmetric matrix satisfy hyperdeterminantal relations of format $2{\times}2{\times}2$.

Big hyperdeterminants

Theorem. Let A be a symmetric $n \times n$ matrix. Then the tensor A_* of all principal minors of A is a common zero of all the hyperdeterminants of formats up to $\underbrace{2 \times 2 \times \cdots \times 2}_{n\ terms}$.

Hyperdeterminants as discrete integrable systems: 2×2

- 2×2



$$a_{00}a_{11} - a_{01}a_{10} = 0$$

Hyperdeterminants as discrete integrable systems: 2×2

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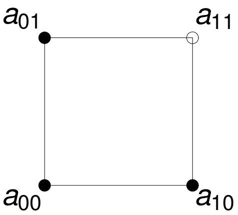

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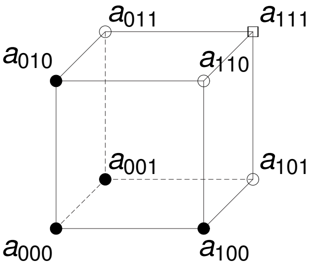
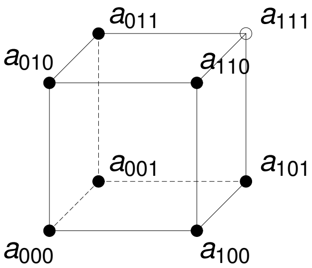
- Integrability as consistency



Hyperdeterminants as discrete integrable systems: $2 \times 2 \times 2$

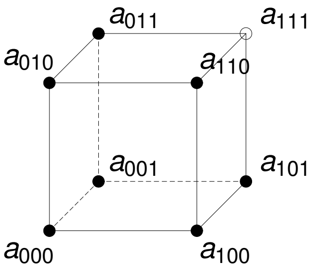


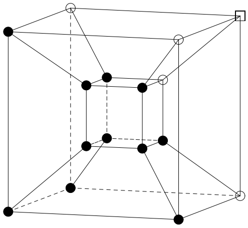
$$A_1 = a_{100}, A_{12} = a_{110}, \ldots$$

$$hyperdet = 0$$

(gives 2 possible values for a_{111})

Hyperdeterminants as discrete integrable systems: $2 \times 2 \times 2$





$A_1 = a_{100}, A_{12} = a_{110}, \ldots$
hyperdet $= 0$
(gives 2 possible values for a_{111})

Theorem of Holtz/Sturm: All eight hyperdeterminantal face relations are satisfied (for a_{ijk} being principle minors of a symmetric 4*x*4 matrix).

Hyperdeterminants as discrete integrable systems: $2 \times 2 \times 2$

Interpretations of integrability independently established by

1) R.Kashaev, 1995
2) W.Schief, 2003
3) (O.Holtz & B.Sturmfels, 2006)

Hyperdeterminants as discrete integrable systems: $2 \times 2 \times 2 \times 2$: ??

Conjecture: Any $2 \times 2 \times \ldots \times 2$ - hyperdeterminant defines an integrable discrete system.

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Test FAILED.
$2 \times 2 \times 2 \times 2$ - hyperdeterminant is not integrable.

$$0 = f01111^{9} - 87168272*f01111^{8} + 2015905513569008*f01111^{7}$$

$$- 84191163955834505565 44*f01111^{6}$$

$$- 2554368215255092620222 0279040*f01111^{5}$$

$$+ 291022627380624489760364242139078656*f01111^{4}$$

$$- 93729565318634306245059508144779490790 1952*f01111^{3}$$

$$+ 15002331239972456464734826021579258770014 17228288*f01111^{2}$$

$$- 1222312782880211859343964138643502633422676219637530624*f01111$$

$$+ 4056847131964305458737912854486373301686067476710061663 84640$$