Symmetries and reductions of integrable discrete equations on quad-graphs

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Outline

- Integrable equations on quad-graphs
  Their key properties
- Applications of symmetry methods
- Geometrical (Lie-point) symmetries and Yang-Baxter maps
- Generalized symmetries and reductions to discrete Painlevé equations

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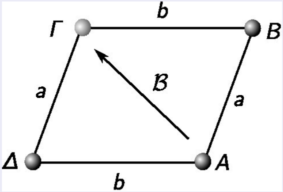
Integrable equations on quad-graphs

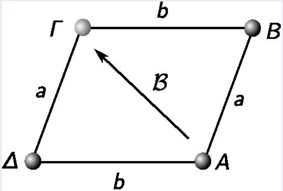


- $Q(f_A, f_B, f_\Gamma, f_\Delta; a, b) = 0$
- Affine linearity
- Symmetries of the square
- $f_\Gamma = \varphi(f_A, f_B, f_\Delta; a, b)$.

$$\mathcal{B}_{(a,b)} : \mathbb{X} \times \mathbb{X} \times \mathbb{X} \to \mathbb{X} \times \mathbb{X} \times \mathbb{X}$$

$$\mathcal{B}_{(a,b)}(f_\Delta, f_A, f_B) = (f_\Delta, f_\Gamma, f_B) ,$$

Integrable equations on quad-graphs



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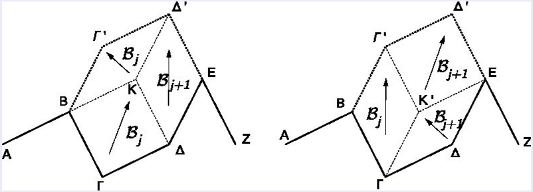
Integrability and the braid relation



$$\mathcal{B}_j : \mathbb{X}^n \to \mathbb{X}^n \qquad \mathcal{B}_j = \mathrm{Id}_{\mathbb{X}} \times \cdots \times \mathcal{B}_{(a_{j-1},a_j)} \times \cdots \times \mathrm{Id}_{\mathbb{X}}\,,$$

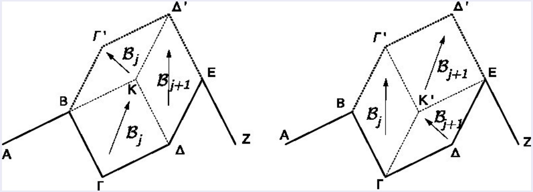
where $\mathcal{B}$ acts on the $j-1$, j and the $j+1$ factors of the map.

- Key properties of maps associated to integrable discrete equations on quad-graphs are the relations

$$\mathcal{B}_j^2 = \mathrm{Id}\,, \quad (\mathcal{B}_j\,\mathcal{B}_{j+1})^3 = \mathrm{Id}\,, \quad \mathcal{B}_j\,\mathcal{B}_i = \mathcal{B}_i\mathcal{B}_j \quad |i-j|>1$$

- $$\mathcal{B}_j\,\mathcal{B}_{j+1}\,\mathcal{B}_j = \mathcal{B}_{j+1}\,\mathcal{B}_j\,\mathcal{B}_{j+1}$$

Integrability and the braid relation



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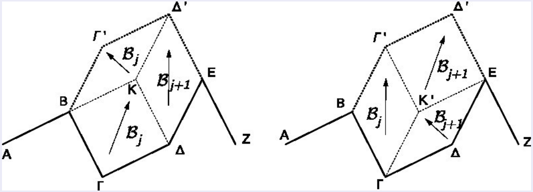

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Lie point symmetries

We consider one-parameter group of transformations

$$G\,:\,(x,y,z)\mapsto(X(x;\varepsilon)\,,Y(y;\varepsilon)\,,Z(z;\varepsilon))\ ,$$

If G commutes with $\mathcal{B}_{(a,b)}$ i.e. $G\,\mathcal{B}_{(a,b)}=\mathcal{B}_{(a,b)}\,G$, it is a symmetry of the map $\mathcal{B}_{(a,b)}$ and consequently of the equation $Q=0$

The map associated with the discrete KdV equation

$$\mathcal{B}_{(a,b)}(x,y,z)=(x,y+\frac{a-b}{x-z},z)\,,$$

- The group of Lie-point symmetries of dKdV equation

$$\begin{aligned}
G_1:(x,y,z) &\mapsto (x+\varepsilon_1\,,y+\varepsilon_1\,,z+\varepsilon_1)\,, \\
G_2:(x,y,z) &\mapsto (x-\varepsilon_2\,,y+\varepsilon_2\,,z-\varepsilon_2)\,, \qquad SO(1,1) \\
G_3:(x,y,z) &\mapsto (x\,e^{-\varepsilon_3}\,,y\,e^{\varepsilon_3}\,,z\,e^{-\varepsilon_3})\,,
\end{aligned}$$

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Projection of the map $\mathcal{B}_{(a,b)}$ to $\mathbb{X}^3/G$

$$\mathcal{B}_{(a,b)}(x,y,z) = (x, y + \frac{a-b}{x-z}, z)$$

$$G_1 : (x,y,z) \mapsto (x+\varepsilon_1\,, y+\varepsilon_1\,, z+\varepsilon_1)\ ,$$

The action of G_1 on $\mathbb{X}^3$ is regular with one-dimensional orbits. Local coordinates on the two-dimensional quotient space $\mathbb{X}^3/G_1$:

$$u = y - x\,, \quad v = z - y\,.$$

Projection of the map to $\mathbb{X}^3/G_1$ gives $B_{(a,b)} : \mathbb{X}\times\mathbb{X} \to \mathbb{X}\times\mathbb{X}$

$$B_{(a,b)}(u,v) = \left(u - \frac{a-b}{u+v}, v + \frac{a-b}{u+v}\right)\ ,$$

Braid equation

$$(\mathrm{Id}_{\mathbb{X}}\times B)(B\times\mathrm{Id}_{\mathbb{X}})(\mathrm{Id}_{\mathbb{X}}\times B) = (B\times\mathrm{Id}_{\mathbb{X}})(\mathrm{Id}_{\mathbb{X}}\times B)(B\times\mathrm{Id}_{\mathbb{X}})\,.$$

Yang-Baxter maps

$$R_{(a,b)} = \sigma \, B_{(a,b)}$$

where σ is the twist map

$$R_{(a,b)}(u,v) = \left(v + \frac{a-b}{u+v}, u - \frac{a-b}{u+v} \right) ,$$

is a YB map known as the Adler map

- Generalizations:
  The method can be applied to multi-field equations by using the multi-parameter groups of symmetry transformations and the multi-dimensional braid relations

  V.Papageorgiou, AT and A.Veselov (2006),
  V.Papageorgiou and AT (2007).

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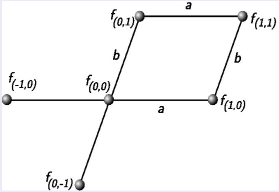
Generalized symmetries of quadrilateral equations



- $Q(f_{(0,0)}, f_{(1,0)}, f_{(0,1)}, f_{(1,1)}; a, b) = 0$
- Affine linearity, D_4 symmetry

Infinitesimal generators of Lie-point, three-point and five-point generalized symmetries

$$
\begin{aligned}
\mathbf{v} &= R(n, m, f_{(0,0)})\,\partial_{f_{(0,0)}} \\
\mathbf{v} &= R(n, m, f_{(0,0)}, f_{(1,0)}, f_{(-1,0)}; a, b)\,\partial_{f_{(0,0)}} \\
\mathbf{v} &= R(n, m, f_{(0,0)}, f_{(1,0)}, f_{(-1,0)}, f_{(0,1)}, f_{(0,-1)}; a, b)\,\partial_{f_{(0,0)}}
\end{aligned}
$$

Three point generalized symmetries

Infinitesimal invariance condition:

$$\mathbf{v}\, Q = 0 \qquad \text{mod } Q = 0$$

Biquadratic polynomials

$$h(f_{(0,0)}, f_{(1,0)}; a, b) = Q\, Q_{,f_{(0,1)} f_{(1,1)}} - Q_{,f_{(0,1)}} Q_{,f_{(1,1)}}$$

$$G(f_{(0,0)}, f_{(1,1)}; a, b) = Q\, Q_{,f_{(1,0)} f_{(0,1)}} - Q_{,f_{(1,0)}} Q_{,f_{(0,1)}}$$

Theorem: Every two-dimensional lattice equation $Q = 0$, where the function Q is affine linear and D_4-symmetric, admits a three-point generalized symmetry with generator

$$\mathbf{v}_n = \left(\frac{h(f_{(0,0)}, f_{(1,0)}; a, b)}{f_{(1,0)} - f_{(-1,0)}} - \frac{1}{2} h_{,f_{(1,0)}}(f_{(0,0)}, f_{(1,0)}; a, b) \right) \partial_{f_{(0,0)}}\,.$$

Three point generalized symmetries

In the generic case, where the matrix

$$\mathcal{G} = \left(\begin{array}{ccc} h(x,y) & G(x,z) & G(x,w) \\ h_{,x}(x,y) & G_{,x}(x,z) & G_{,x}(x,w) \\ h_{,xx}(x,y) & G_{,xx}(x,z) & G_{,xx}(x,w) \end{array} \right) \Bigg|_{x=0}$$

has rank 3, every three-point generalized symmetry generator necessarily has the form

$$\mathbf{V}_n = A(n;a,b)\,\mathbf{v}_n + \frac{1}{2}\phi(n,m,f_{(0,0)};a,b)\partial_{f_{(0,0)}}$$

$$(A(n;a,b) - A(n+1;a,b))\ h(f_{(0,0)},f_{(1,0)})^2\,\partial_{f_{(1,0)}}\left(\frac{G(f_{(1,0)},f_{(0,1)})}{h(f_{(0,0)},f_{(1,0)})}\right)$$

$$+G(f_{(1,0)},f_{(0,1)})\phi(n,m,f_{(0,0)};a,b) + h(f_{(0,0)},f_{(0,1)})\phi(n+1,m,f_{(1,0)};a,b)$$

$$+h(f_{(0,0)},f_{(1,0)})\phi(n,m+1,f_{(0,1)};a,b) = Q^2_{,f_{(1,1)}}\phi(n+1,m+1,f_{(1,1)};a,b)\,.$$

Five point generalized symmetries - the generic case

Theorem: If $\operatorname{rank} \mathcal{G} = 3$, Q is affine linear and D_4 symmetric then the characteristic R of a five-point symmetry generator of $Q = 0$ has the form

$$\begin{aligned} R &= A(n; a, b) P(f_{(0,0)}, f_{(1,0)}, f_{(-1,0)}; a, b) \\ &+ B(m; a, b) P(f_{(0,0)}, f_{(0,1)}, f_{(0,-1)}; b, a) + \frac{\psi(n, m, f_{(0,0)}; a, b)}{2} \end{aligned}$$

where

$$P(u, x, y; a, b) = \frac{h(u, x; a, b)}{x - y} - \frac{1}{2} h_{,x}(u, x; a, b)\,,$$

The determining equation

$$(A(n;a,b) - A(n+1;a,b))\ h(f_{(0,0)}, f_{(1,0)})^2\, \partial_{f_{(1,0)}} \left(\frac{G(f_{(1,0)}, f_{(0,1)})}{h(f_{(0,0)}, f_{(1,0)})} \right)$$

$$+\ (B(m;a,b) - B(m+1;a,b))\ h(f_{(0,0)}, f_{(0,1)})^2\, \partial_{f_{(0,1)}} \left(\frac{G(f_{(1,0)}, f_{(0,1)})}{h(f_{(0,0)}, f_{(0,1)})} \right)$$

$$+G(f_{(1,0)}, f_{(0,1)})\psi(n,m,f_{(0,0)};\alpha,\beta) + h(f_{(0,0)}, f_{(0,1)})\psi(n+1,m,f_{(1,0)};a,b)$$

$$+h(f_{(0,0)}, f_{(1,0)})\psi(n,m+1,f_{(0,1)};\alpha,\beta) = Q^2_{,f_{(1,1)}}\psi(n+1,m+1,f_{(1,1)};a,b)\,,$$

- Exhaustive list of the corresponding Lie point, three- and five-point generalized symmetry generators of the Adler-Bobenko-Suris classified integrable discrete equations

AT, D. Tsoubelis and P. Xenitidis (2007)

The determining equation

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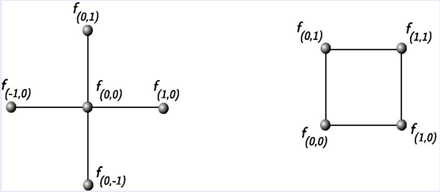
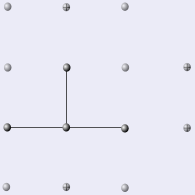
Invariant solutions

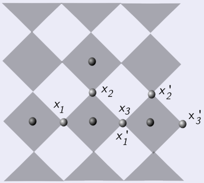


A function $f : \mathbb{Z}^2 \mapsto \mathbb{C}$ is called an invariant solution of the lattice equation $Q = 0$ under the symmetry $\mathbf{v} = R[f]\partial_{f_{(0,0)}}$, if it satisfies the lattice equation $Q = 0$ and the compatible constraint $\mathbf{v}(f) = 0$.

$$R[f] = 0 \qquad\qquad Q[f] = 0$$

Symmetry reductions and Cauchy problems





$$\mathbf{L} = R_L(n, m, f_{(0,0)})\, \partial_{f_{(0,0)}}$$

$$\mathbf{v}_{\mathcal{C}} \;=\; \mathbf{v}_n \;+\; \lambda\, \mathbf{v}_m$$

$$\mathcal{C}[f] := R(f_{(0,0)}, f_{(1,0)}, f_{(-1,0)}; a, b) \;+\; \lambda\, R(f_{(0,0)}, f_{(0,1)}, f_{(0,-1)}; b, a) = 0,$$

where

$$R(u, x, y; \alpha, \beta) = \frac{h(u, x; a, b)}{x - y} - \frac{1}{2} h_{,x}(u, x; a, b)\,.$$

$$[\mathbf{L}, \mathbf{v}_{\mathcal{C}}\,] = 0, \quad \text{mod} \quad \mathcal{C}[f] = 0$$

$$P : (x_1, x_2, x_3) \rightarrow (x_1', x_2', x_3')$$

Symmetry reduction of discrete KdV

$$\textbf{dKdV} \qquad \left(f_{(0,0)} - f_{(1,1)}\right)\left(f_{(1,0)} - f_{(0,1)}\right) - a + b = 0$$

Point symmetries :

$$\mathbf{L}_1 = \partial_{u_{(0,0)}},\ \mathbf{L}_2 = (-1)^{n+m}\partial_{u_{(0,0)}},\ \mathbf{L}_3 = (-1)^{n+m}u_{(0,0)}\partial_{u_{(0,0)}}$$

Three-point generalized symmetries :

$$\mathbf{v}_1 = \frac{1}{f_{(1,0)} - f_{(-1,0)}}\partial_{f_{(0,0)}},\ \ \mathbf{v}_2 = n\,\mathbf{v}_1 + \frac{f_{(0,0)}}{2(a-b)}\,\partial_{f_{(0,0)}},$$

$$\mathbf{v}_3 = \frac{1}{f_{(0,1)} - f_{(0,-1)}}\partial_{f_{(0,0)}},\ \ \mathbf{v}_4 = m\,\mathbf{v}_3 - \frac{f_{(0,0)}}{2(a-b)}\,\partial_{f_{(0,0)}}.$$

$$f \mapsto (-1)^{n+m} f, \qquad \mathbf{L}_2 \mapsto \mathbf{L}_1$$

$$\mathbf{v}_{\mathcal{C}} = \sum_{i=1}^{3}\lambda_i\,\mathbf{L}_i + \sum_{i=1}^{4}\mu_i\,\mathbf{v}_i$$

Symmetry reduction of discrete KdV using L_1

$$v_{(0,0)} = f_{(1,0)} - f_{(0,0)},\; w_{(0,0)} = f_{(1,1)} - f_{(1,0)},$$
$$v_{(0,1)} = f_{(1,1)} - f_{(0,1)},\, w_{(-1,0)} = f_{(0,1)} - f_{(0,0)}$$

$$[\mathbf{L}_1, \mathbf{v}_{\mathcal{C}}] = \lambda_3\, \mathbf{L}_2 + \frac{1}{2}\, \frac{\mu_2 - \mu_4}{a - b}\, \mathbf{L}_1$$

$$\lambda_3 = 0\,, \quad \mu_2 = \mu_4\,.$$

$$\mathcal{C}[g] := \lambda_1 + \lambda_2\, (-1)^{n+m} + \frac{\mu_2\, n + \mu_1}{g_1 + g_2} + \frac{\mu_2\, m + \mu_3}{g_3 + g_4} = 0\,.$$

where

$$(g_1, g_2, g_3, g_4) = \left(v_{(0,0)}, v_{(-1,0)}, w_{(-1,0)}, w_{(-1,-1)}\right).$$

The asymmetric, alternate discrete Painlevé II equation

$$
\begin{aligned}
x_1' &= x_3\,,\\
x_2' &= -x_3 + \frac{r}{x_3 - x_2}\,,\\
x_3' &= -x_3 + \frac{r\,c(n+1,\mu_1)}{\left(\frac{c(n,\mu_1)}{x_1+x_3} + \lambda(n+m)\right)(x_2-x_3)^2 + c(m,\mu_3)\,(x_2-x_3) - r\lambda(n+m+1)}\,,
\end{aligned}
$$

where $c(n,\mu) = \mu_2\, n + \mu$, $\lambda(n) = \lambda_1 + \lambda_2\,(-1)^n$ and $r = a - b$.

The above system can be decoupled for the variable $y(n) := x_2' + x_3 = x_2(n+1,m) + x_3(n,m)$

$$
\frac{r\,c(n+1,\mu_1)}{y(n+1)\,y(n)+r} + \frac{r\,c(n,\mu_1)}{y(n)\,y(n-1)+r} =
c(n+1,\mu_1) + c(m,\mu_3) + y(n)\,\lambda(n+m+1) - \frac{r}{y(n)}\,\lambda(n+m)
$$

Future work

- Complete lists of group invariant solutions of the integrable discrete equations in the Adler-Bobenko-Suris classification - new versions of discrete Painlevé equations
- Symmetries of integrable discrete equations without the D_4 symmetry - mappings arising from the corresponding symmetry reductions
- Generalizations to higher dimensions (in both dependent and independent variables)