Semi-Classical Orthogonal Polynomials and Related Difference Equations

Paul Spicer (University of Sydney)

- work done in collaboration with F. Nijhoff and N. Joshi

Outline

1. Motivation
2. What are semi-classical orthogonal polynomials?
3. The Laguerre method
   - A differential system for semi-classical orthogonal polynomials
4. A semi-discrete Lax equation
   - The compatibility between the differential system and recurrence relation
5. Some examples of semi-classical weights
   - Hermite, Laguerre and Jacobi weights

1 Motivation

The Laguerre method has been used in recent years, particularly by Magnus who found some continuous Painlevé equations, and Forrester and Witte who derived a discrete Painlevé fifth after the reduction of a coupled system. Our interest was what other difference equations could be found from different semi-classical weights.

1. E. Laguerre, *Sur la réduction en fractions continues d'une fraction qui satisfait à une équation différentielle linéaire du premier ordre dont les coefficients sont rationnels* J. Math. Pures Appl. (4) **1** (1885) 135–165 = pp. 685–711 in *Oeuvres* Vol. II, Chelsea, New York 1972.

2. A.P. Magnus, *Painlevé-type differential equations for the recurrence coefficients of semi-classical polynomials*, (J. Comput. Appl. Math. **57** (1995) 215–237. `math.CA/9307218`

3. P.J. Forrester, N.S. Witte *Bi-orthogonal polynomials on the Unit Circle, Regular Semi-Classical Weights and Integrable Systems*, `math.CA/0412394`.

4. P.J. Forrester, N.S. Witte *Discrete Painlevé Equations for a class of* P_{VI} τ*-functions given as* $U(N)$ *averages*, `math.PH/0412065`.

2 What are semi-classical orthogonal polynomials?

We define a monic orthogonal polynomial sequence $\{P_n(x)\}_{n=0}^{\infty}$ with respect to a weight function $w(x)$ on the real line $\mathbb{R}$ as

$$\int_{\mathbb{R}} P_n(x)P_m(x)w(x)dx = \delta_{nm}h_n \quad \text{where } h_n \neq 0 \ .$$

Then they always satisfy the three point recurrence relation

$$P_{n+1} = (x - S_n)P_n - R_nP_{n-1} \ ,$$

where R_n and S_n are explicitly defined and with the initial conditions $P_0 = 1, P_{-1} = 0$.

Now classical orthogonal polynomials have a weight function $w(x)$ which satisfies the Pearson equation

$$\frac{d}{dx}(\phi(x)w(x)) = \psi(x)w(x) \ ,$$

for deg $\phi \leq 2$ and deg $\psi = 1$. However if the deg $\phi > 2$ and\or deg $\psi > 1$, then the weight function produces a class of semi-classical orthogonal polynomials.

To illustrate this point we choose to write the Pearson equation in the alternate way

$$\frac{1}{w(x)}\frac{dw(x)}{dx} = \frac{\psi - \phi'}{\phi} = \frac{V(x)}{W(x)},$$

where $W(x)$ and $V(x)$ are polynomials. In this case the classical weights satisfy the equation if deg $V \le 1$ and deg $W \le 2$ and if deg $W > 2$ and\or deg $V > 1$, then we have semi-classical weights.

Then as an example, if we alter the Hermite weight function $w_0(x) = e^{-x^2}$ to $w_1(x) = e^{-x^2-x^4}$, then from the Pearson equation we have

$$e^{x^2+x^4}(-2x-4x^3)e^{-x^2-x^4} = -2x-4x^3$$

a polynomial of degree 3. Therefore we refer to this weight as a semi-classical weight.

A consequence of this change is that while the semi-classical orthogonal polynomials still satisfy the three point recurrence relation, the recurrence coefficients often satisfy interesting nonlinear relations. It is these relations between the coefficients which interests us from the perspective of integrable systems.

3.1 The Markov-Stieltjes Function

We introduce the formal semi-classical orthogonal polynomials $P_n(z)$, $n = 0, \ldots, \infty$ which are orthogonal with respect to some weight function $w(z)$ on a support S

$$\int_S P_n(z) P_m(z) w(z) dz = \langle P_n, P_m \rangle,$$

with a corresponding recurrence relation

$$P_{n+1}(z) = (z - S_n) P_n(z) + R_n P_{n-1}(z).$$

Then given the Markov-Stieltjes function

$$f(z) = \int_S \frac{w(x)}{z - x} dx \ ,$$

equations for P_n can be summarized as

$$f(z) P_n(z) = P^{(1)}_{n-1}(z) + \epsilon_n(z),$$

where $P^{(1)}_{n-1}(z)$ and $\epsilon_n(z)$ are given by:

$$P^{(1)}_{n-1}(z) = \int_S \frac{P_n(z) - P_n(x)}{z - x} w(x) dx \quad , \ \epsilon_n(z) = \int_S \frac{P_n(x)}{z - x} w(x) dx.$$

Since both $P_n(z)$ and $\epsilon_n(z)$ satisfy the recurrence relation we can give an explicit form of $P_n(z)$ and $\epsilon_n(z)$ defined in terms of the recurrence relation's coefficients:

$$\begin{aligned} P_n(z) &= z^n - \left(\sum_{j=0}^{n-1} S_j\right) z^{n-1} + \sum_{j=1}^{n-1}\left(\sum_{k=0}^{j-1} S_j S_k - R_j\right) z^{n-2} + \cdots \\ \epsilon_n(z) &= h_n \left(\frac{1}{z^{n+1}} + \left(\sum_{j=0}^{n} S_j\right)\frac{1}{z^{n+2}} + \sum_{j=0}^{n}\left(R_{j+1} + \sum_{i=0}^{j} S_j S_i\right)\frac{1}{z^{n+3}} + \cdots\right) \quad . \end{aligned}$$

The relation for $P_n(z)$ is derived using $P_n(z) = z^n + p_{n,n-1}z^{n-1} + p_{n,n-2}z^{n-2} + \ldots$ and the recurrence relation. However, since $\epsilon_n(z)$ is not polynomial we expand it

$$\epsilon_n(z) = \int_S \frac{P_n(x)}{z-x} w(x)dx = h_n \left(\frac{1}{z^{n+1}} + \frac{e_{n,n+2}}{z^{n+2}} + \frac{e_{n,n+3}}{z^{n+3}} + \cdots\right)$$

and since it also satisfies the monic recurrence relation:

$$\begin{aligned} \int_S \frac{xP_n}{z-x} dw(x) &= \int_S \frac{P_{n+1} + S_n P_n + R_n P_{n-1}}{z-x}\ dw(x) \\ \Rightarrow \quad z\epsilon_n(z) - h_n\delta_{n0} &= \epsilon_{n+1}(z) + S_n\epsilon_n(z) + R_n\epsilon_{n-1}(z) \end{aligned}$$

we look at compatibility between the two.

Additionally we have the following relations between $P_n, P_n^{(1)}$ and ϵ_n

$$\begin{aligned} P_n P_{n-2}^{(1)} - P_{n-1} P_{n-1}^{(1)} &= -h_{n-1} \\ P_{n-1}\epsilon_n - P_n \epsilon_{n-1} &= -h_{n-1} \end{aligned}$$

which can be easily found using the Christoffel-Darboux identity:

$$\sum_{j=0}^{n} \frac{P_j(x)P_j(y)}{h_j} = \frac{(P_{n+1}(x)P_n(y) - P_{n+1}(y)P_n(x))}{h_n(x-y)} \ .$$

We also have that $f(z)$ satisfies a first order differential equation

$$W(z)\partial_z f(z) = V(z)f(z) + U(z) \ ,$$

which we derive using $f(z)$, to get expressions for V and U (which are polynomials in z).

$$\begin{aligned} W(z)(\partial_z f(z)) = -\int_S \frac{W(z)w(x)}{(z-x)^2}dx &= -\int_S \frac{d}{dx}\left(\frac{1}{z-x}W(z)w(x)\right)dx + \int_S \frac{W(z)}{z-x}\partial_x w(x) \\ &= \int_S \frac{W(z)}{W(x)}V(x)\frac{1}{z-x}w(x)dx \\ &= V(z)f(z) + W(z)\int_S \left(\frac{V(x)}{W(x)} - \frac{V(z)}{W(z)}\right)\frac{w(x)}{z-x}dx \end{aligned}$$

3.2 The Fundamental Linear System for Semi-Classical Orthogonal Polynomials

We look at the compatibility between the equation for fP_n and $W(z)(\partial_z f(z))$ in order to identify a general differential equation for P_n

$$\begin{aligned} Wf\partial_z P_n + (Vf+U)P_n &= W(\partial_z P^{(1)}_{n-1} + \partial_z \epsilon_n) \\ W\partial_z P_n(P^{(1)}_{n-1}+\epsilon_n) + VP_n(P^{(1)}_{n-1}+\epsilon_n) + UP_n^2 &= W(\partial_z P^{(1)}_{n-1} + \partial_z\epsilon_n)P_n \end{aligned}$$

We then go about separating the polynomial expression $P^{(1)}_{n-1}$ and ϵ_n so we get the following two equivalent expressions, which we denote Θ_n

$$\begin{aligned} \Theta_n &= W((\partial_z P^{(1)}_{n-1})P_n - (\partial_z P_n)P^{(1)}_{n-1}) - UP_n^2 - VP_nP^{(1)}_{n-1} \ , \\ &= W((\partial_z P_n)\epsilon_n - (\partial_z\epsilon_n)P_n) + VP_n\epsilon_n \ , \end{aligned}$$

where Θ_n is a polynomial bounded by a constant.

We consider the same method again except we use fP_{n-1}.

$$\begin{aligned} (Vf+U)P_{n-1} + Wf(\partial_z P_{n-1}) &= W(\partial_z P^{(1)}_{n-2} + \partial_z\epsilon_{n-1}) \\ VP_{n-1}(P^{(1)}_{n-1}+\epsilon_n) + UP_nP_{n-1} + W\partial_z P_{n-1}(P^{(1)}_{n-1}+\epsilon_n) &= W(\partial_z P^{(1)}_{n-2} + \partial_z\epsilon_{n-1})P_n \end{aligned}$$

Again we separate the polynomial expression $P^{(1)}_{n-1}$ and ϵ_n to get a second object, which will be called Ω_n:

$$\begin{aligned} \Omega_n &= W(P_n(\partial_z P^{(1)}_{n-2}) - P^{(1)}_{n-1}(\partial_z P_{n-1})) - VP_{n-1}P^{(1)}_{n-1} - UP_nP_{n-1} \ , \\ &= W(\epsilon_n(\partial_z P_{n-1}) - P_n(\partial_z\epsilon_{n-1})) + V\epsilon_nP_{n-1} \ . \end{aligned}$$

Since the recurrence relation can be expressed in a matrix form

$$\psi_{n+1}(z) = \begin{pmatrix} z - S_n & -R_n \\ 1 & 0 \end{pmatrix} \psi_n(z), \text{ where } \psi_n(z) = \begin{pmatrix} P_n(z) \\ P_{n-1}(z) \end{pmatrix}$$

we also express our two polynomial expressions for Θ_n and Ω_n in matrix form:

$$\begin{pmatrix} P_{n-1} & -P^{(1)}_{n-2} \\ P_n & -P^{(1)}_{n-1} \end{pmatrix} \begin{pmatrix} W\partial_z P^{(1)}_{n-1}(z) \\ W\partial_z P_n(z) \end{pmatrix} = \begin{pmatrix} \Omega_n + VP_{n-1}P^{(1)}_{n-1} + UP_nP_{n-1} \\ \Theta_n + VP_nP^{(1)}_{n-1} + UP_n^2 \end{pmatrix},$$

which can easily be solved to give:

$$\begin{pmatrix} W\partial_z P^{(1)}_{n-1} \\ W\partial_z P_n \end{pmatrix} = \frac{1}{h_{n-1}} \begin{pmatrix} P^{(1)}_{n-1} & -P^{(1)}_{n-2} \\ P_n & -P_{n-1} \end{pmatrix} \begin{pmatrix} \Omega_n + VP_{n-1}P^{(1)}_{n-1} + UP_nP_{n-1} \\ \Theta_n + VP_nP^{(1)}_{n-1} + UP_n^2 \end{pmatrix},$$

where we have two differential equations:

$$\begin{aligned} W\partial_z P_n &= \frac{1}{h_{n-1}}(\Omega_n P_n - \Theta_n P_{n-1}) \ , \\ W\partial_z P^{(1)}_{n-1} &= (\Omega_n P^{(1)}_{n-1} - \Theta_n P^{(1)}_{n-2} + V h_{n-1} P^{(1)}_{n-1} + U h_{n-1} P_n). \end{aligned}$$

However this is a differential system for P_n and $P^{(1)}_{n-1}$, whereas we are interested in a differential system for P_n only.

Looking for a second differential relation for P_n, we consider $W\partial_z P_{n-1}$ and use the recurrence relation to keep a similar order

$$W(\partial_z P_{n-1}) = \frac{1}{h_{n-2}}\left(\Omega_{n-1}P_{n-1} - \frac{\Theta_{n-1}}{R_{n-1}}((z-S_{n-1})P_{n-1} - P_n)\right) \ .$$

However we have no expression to remove the z from the equation, so we consider the problematic part of the expression, which we can expand:

$$\begin{aligned}(z-S_n)\Theta_n &= (z-S_n)\left(W(\epsilon_n\partial_z(P_n) - \partial_z(\epsilon_n)P_n) + V\epsilon_n P_n\right)\\ &= W(-\partial_z\epsilon_n(P_{n+1} + R_nP_{n-1}) + \partial_z P_n(\epsilon_{n+1} + R_n\epsilon_{n-1})) + VP_n(\epsilon_{n+1} + R_n\epsilon_{n-1})\\ &= \Omega_{n+1} + R_n\Omega_n + Vh_n\end{aligned}$$

to give a second differential equation.

$$W\partial_z P_{n-1} = \frac{1}{h_{n-1}}(\Theta_{n-1}P_n - \Omega_n P_{n-1}) - VP_{n-1}$$

We now have a *differential system*

$$W\partial_z\psi(z) = \frac{1}{h_{n-1}}\begin{pmatrix}\Omega_n(z) & -\Theta_n(z)\\ \Theta_{n-1}(z) & -(\Omega_n(z) + V(z)h_{n-1})\end{pmatrix}\psi(z) \ ,$$

where $\psi(z) = \begin{pmatrix}P_n(z)\\ P_{n-1}(z)\end{pmatrix}$.

4 Compatibility Relations

If we present the recurrence and differential equations in a semi-discrete Lax representation we have $\psi_{n+1}(z) = L_n(z)\psi_n(z)$ and $\partial_z\psi_n(z) = M_n(z)\psi_n(z)$, where

$$L_n = \begin{pmatrix} z - S_n & -R_n \\ 1 & 0 \end{pmatrix} \quad , \quad M_n = \frac{1}{Wh_{n-1}} \begin{pmatrix} \Omega_n(z) & -\Theta_n(z) \\ \Theta_{n-1}(z) & -(\Omega_n(z) + V(z)h_{n-1}) \end{pmatrix}$$

and leads to the semi-discrete Lax equation $\partial_z L_n = M_{n+1}L_n - L_n M_n$. Equating this expression gives us

$$\begin{aligned}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad = \quad & \frac{1}{Wh_n} \begin{pmatrix} \Omega_{n+1}(z) & -\Theta_{n+1}(z) \\ \Theta_n(z) & -(\Omega_{n+1}(z) + V(z)h_n) \end{pmatrix} \begin{pmatrix} z - S_n & -R_n \\ 1 & 0 \end{pmatrix} \\ & - \frac{1}{Wh_{n-1}} \begin{pmatrix} z - S_n & -R_n \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \Omega_n(z) & -\Theta_n(z) \\ \Theta_{n-1}(z) & -(\Omega_n(z) + V(z)h_{n-1}) \end{pmatrix}\end{aligned}$$

we can identity two distinct relations.

$$\begin{aligned}(z - S_n)\left(\frac{\Omega_{n+1}}{h_n} - \frac{\Omega_n}{h_{n-1}}\right) \quad &= \quad R_{n+1}\frac{\Theta_{n+1}}{h_{n+1}} - R_n\frac{\Theta_{n-1}}{h_{n-1}} + W \\ (z - S_n)\frac{\Theta_n}{h_n} \quad &= \quad \frac{\Omega_{n+1}}{h_n} + \frac{\Omega_n}{h_{n-1}} + V\end{aligned}$$

To fully utilize the compatibility relations we need to express both Θ_n and Ω_n in terms of the recurrence coefficients. This is achieved by substituting the expressions for P_n and ϵ_n into Θ_n and Ω_n, hence:

$$\begin{aligned}
\Theta_n &= W(z)h_n\left\{\left[\frac{1}{z^{n+1}}+\left(\sum_{j=0}^{n}S_j\right)\frac{1}{z^{n+2}}+\cdots\right]\times\left[nz^{n-1}-\left(\sum_{j=0}^{n-1}S_j\right)(n-1)z^{n-2}+\cdots\right]\right.\\
&\quad\left.+\left[\frac{n+1}{z^{n+2}}+\left(\sum_{j=0}^{n}S_j\right)\frac{n+2}{z^{n+3}}+\cdots\right]\times\left[z^{n}-\left(\sum_{j=0}^{n-1}S_j\right)z^{n-1}+\cdots\right]\right\}\\
&\quad+V(z)\times h_n\left[\frac{1}{z^{n+1}}+\left(\sum_{j=0}^{n}S_j\right)\frac{1}{z^{n+2}}+\cdots\right]\times\left[z^{n}-\left(\sum_{j=0}^{n-1}S_j\right)z^{n-1}+\cdots\right]
\end{aligned}$$

$$
\begin{aligned}
\Omega_n \;=\; & W(z)\Bigg\{h_n\left[\frac{1}{z^{n+1}}+\left(\sum_{j=0}^{n}S_j\right)\frac{1}{z^{n+2}}+\sum_{j=0}^{n}\left(R_{j+1}+\sum_{k=0}^{j}S_jS_k\right)\frac{1}{z^{n+3}}+\cdots\right] \\
& \times\left[(n-1)z^{n-2}-(n-2)\left(\sum_{j=0}^{n-2}S_j\right)z^{n-3}+(n-3)\sum_{j=1}^{n-2}\left(\sum_{k=0}^{j-1}S_jS_k-R_j\right)z^{n-4}+\cdots\right] \\
& +h_{n-1}\left[z^n-\left(\sum_{j=0}^{n-1}S_j\right)z^{n-1}+\sum_{j=1}^{n-1}\left(\sum_{k=0}^{j-1}S_jS_k-R_j\right)z^{n-2}+\cdots\right] \\
& \times\left[\frac{n}{z^{n+1}}+\left(\sum_{j=0}^{n-1}S_j\right)\frac{(n+1)}{z^{n+2}}+\sum_{j=0}^{n-1}\left(R_{j+1}+\sum_{k=0}^{j}S_jS_k\right)\frac{(n+2)}{z^{n+3}}+\cdots\right]\Bigg\} \\
& +V(z) \\
& \times h_n\left[\frac{1}{z^{n+1}}+\left(\sum_{j=0}^{n}S_j\right)\frac{1}{z^{n+2}}+\sum_{j=0}^{n}\left(R_{j+1}+\sum_{k=0}^{j}S_jS_k\right)\frac{1}{z^{n+3}}+\cdots\right] \\
& \times\left[z^{n-1}-\left(\sum_{j=0}^{n-2}S_j\right)z^{n-2}+\sum_{j=1}^{n-2}\left(\sum_{k=0}^{j-1}S_jS_k-R_j\right)z^{n-3}+\cdots\right].
\end{aligned}
$$

These definitions will be particularly useful when we are looking at examples of specific semi-classical weights.

5.1 Semi-classical Hermite weights - Example 1

We choose the semi-classical Hermite weight $e^{-\frac{a}{2}z^2-\frac{b}{4}z^4}$ then from the Pearson equation we have

$$V(z) = -(az + bz^3) \quad , \quad W(z) = 1.$$

When we substitute $V(z), W(z)$ into the relations above and then make use of the consistency relations, we must be reminded that a weight function of this form satisfies a simplified recurrence relation, specifically one where $S_n = 0$. We find that Θ_n and Ω_n have the following forms respectively

$$\frac{\Theta_n}{h_n} = -(bz^2 + (R_{n+1} + R_n)b + a) \quad , \quad \frac{\Omega_n}{h_{n-1}} = -bR_n z$$

and consequently there is only one non-trivial equation

$$(R_{n+1}(R_{n+2} + R_{n+1}) - R_n(R_n + R_{n-1}))\, b + a(R_{n+1} - R_n) = 1\ .$$

Then after integrating up we are left with

$$R_n(b(R_{n+1} + R_n + R_{n-1}) + a) = \alpha n + \beta$$

which is a discrete form of Painlevé I, d-$\mathrm{P_I}$.

Example 2

Alternatively we consider the semi-classical Hermite weight $e^{-a_1 z-\frac{a_2}{2}z^2-\frac{a_3}{3}z^3}$, then from the Pearson equation we have

$$V(z) = -(a_1 + a_2 z + a_3 z^2) \quad , \quad W(z) = 1.$$

From these values of $V(z), W(z)$ we have the following forms for Θ_n and Ω_n respectively

$$\frac{\Theta_n}{h_n} = -(a_3 z + a_2 + S_n a_3) \quad , \quad \frac{\Omega_n}{h_{n-1}} = -a_3 R_n.$$

Then in the consistency relations we have two non-trivial equations

$$\begin{aligned}&R_{n+1}(a_3(S_{n+1} + S_n + a_2)) - R_n(a_3(S_n + S_{n-1}) + a_2) = 1\\ &S_n(a_2 + S_n a_3) = -a_3(R_{n+1} + R_n) - a_1\end{aligned}$$

of which the first is a pure difference equation and implies that

$$R_n = \frac{\alpha n + \beta}{a_3(S_n + S_{n-1}) + a_2}$$

hence we have

$$S_n^2 a_3 + S_n a_2 + a_1 = -a_3 \left(\frac{\alpha(n+1) + \beta}{a_3(S_{n+1} + S_n) + a_2} + \frac{\alpha n + \beta}{a_3(S_n + S_{n-1}) + a_2} \right)$$

which is an alternate expression for discrete $\mathrm{P_I}$. This can be seen from the continuum limits $S = \varepsilon^2 u$, $\alpha = \alpha_1 \varepsilon^5$ and $\beta = \frac{a_1 a_2}{2 a_3}$.

5.2 Semi-classical Laguerre weights - Example 1

We first consider the semi-classical weight $l_0(z) = (z-t)^{b_1} e^{-(a_1 z + \frac{a_2}{2} z^2)}$ with $b_1, a_1, a_2 > 0$ and where the support S is an arc from $(t \to \infty)$. Then in the consistency relations we have two non-trivial equations

$$\begin{aligned}
&a_2(R_{n+1} + R_n) = -S_n(a_2 S_n + (a_1 - a_2 t)) + (2n + 1 + a_1 t + b_1),\\
&R_{n+1}(a_2(S_{n+1} + S_n) + (a_1 - a_2 t)) - R_n(a_2(S_n + S_{n-1}) - (a_1 - a_2 t)) = S_n - t.
\end{aligned}$$

Example 2

Alternatively we consider the weight function $l_1(z) = z^{b_1}(t-z)^{b_2} e^{-z}$ with $b_1, b_2 > 0$ and where the support S joins the points $0, t$ and ∞ in some way, such as an arc from $0 \to \infty$. Then in the consistency relations we have two non-trivial equations

$$\begin{aligned}
&S_n(S_n - t) - R_{n+1}(S_{n+1} + S_n) + R_n(S_n + S_{n-1})\\
&= -R_{n+1}(2n + 3 + t + b_1 + b_2) + R_n(2n - 1 + t + b_1 + b_2),\\
&2\sum_{j=0}^{n-1} S_j - S_n^2 + S_n(2n + 2 + t + b_1 + b_2) = R_{n+1} + R_n + (2n + 1 + b_1)t.
\end{aligned}$$

Example 3

Finally we can deform both parts of the weight and have a weight function of the form $l_2(z) = z^{b_1}(t-z)^{b_2}e^{-(a_1 z+\frac{a_2}{2}z^2)}$ with $b_1, b_2, a_1, a_2 > 0$ and where the support S joins the points $0, t$ and ∞ in some way, such as an arc from $0 \to \infty$. Then from the consistency relations we have the non-trivial equations

$$R_{n+1}(a_1 + a_2(S_{n+1} + 2S_n - t)) + R_n(a_1 + a_2(2S_n + S_{n-1} - t)) + (2n + 1 + b_1 t)$$
$$= 2\sum_{j=0}^{n-1} S_j - S_n a_1(S_n - t) - a_2 S_n^2(S_n - t) + S_n(2n + 2 + b_1 + b_2),$$

$$R_{n+1}(a_1(S_{n+1} + S_n - t) + a_2(R_{n+2} + R_n + S_{n+1}^2 + S_n^2 + S_{n+1}S_n$$
$$-t(S_{n+1} + S_n)) - (2n + 3 + b_1 + b_2))$$
$$-R_n(a_1(S_n + S_{n-1} - t) + a_2(R_{n+1} + R_{n-1} + S_n^2 + S_{n-1}^2 + S_n S_{n-1}$$
$$-t(S_n + S_{n-1})) - (2n + 1 + b_1 + b_2))$$
$$= S_n(S_n - t) + 2R_n.$$

5.3 Semi-classical Jacobi weight

Our choice of deformation to the semi-classical case, consists of rewriting the weight function as $w(x) = (1-x)^\alpha x^\beta (t-x)^\gamma$, where a second variable t has been included with addition of another parameter γ.

Then (like the Laguerre case) we get a coupled system of nonlinear difference equations:

$$-S_{n+1}\left(2\sum_{j=0}^{n} S_j - (1+t)(2n+4) + (2n+5+\alpha+\beta+\gamma)S_{n+1} - (\alpha t + \beta(t+1) + \gamma)\right)$$
$$+S_n\left(2\sum_{j=0}^{n} S_j - (1+t)2n + (2n-1+\alpha+\beta+\gamma)S_n - (\alpha t + \beta(t+1) + \gamma)\right)$$
$$= (2n+5+\alpha+\beta+\gamma)R_{n+2} + 2R_{n+1} - (2n-1+\alpha+\beta+\gamma)R_n + 2t\ ,$$

$$R_{n+1}\left(2\sum_{j=0}^{n} S_j - (1+t)(2n+3) + (2n+4+\alpha+\beta+\gamma)S_{n+1} - (\alpha t + \beta(t+1) + \gamma)\right)$$
$$-R_n\left(2\sum_{j=0}^{n-1} S_j - (1+t)(2n-1) + (2n-2+\alpha+\beta+\gamma)S_{n-1} - (\alpha t + \beta(t+1) + \gamma)\right)$$
$$= -S_n\left((2n+3+\alpha+\beta+\gamma)R_{n+1} - (2n-1+\alpha+\beta+\gamma)R_n + t + S_n^2 - (1+t)S_n\right)$$

6 Remarks

1. Given specific weights it is easy to derive relations from the Lax equation - however classifying these relations (such as through a continuum limit) is not

2. The method described here is only applicable for deformations of the classical families Hermite, Laguerre and Jacobi. If it was to be used with other types of orthogonal polynomials then the appropriate analogue of the Pearson equation would be required.