Solutions to 2×2 iso-monodromy problems for the second and third $q-$discrete Painlevé equations

Nalini Joshi   Yang Shi

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University of Sydney

Examples of some Painlevé equations

- 2nd order nonlinear integrable equations

$$
\begin{aligned}
\text{PII} \quad \frac{d^2y}{dz^2} &= 2y^3 + zy + \alpha \\
\text{PIII} \quad \frac{d^2y}{dz^2} &= \frac{1}{y}\left(\frac{dy}{dz}\right)^2 - \frac{1}{z}\frac{dy}{dz} + \frac{\alpha y^2 + \beta}{z} + \gamma y^3 + \frac{\delta}{y}
\end{aligned}
$$

- Iso-monodromy deformation (Lax Pair)
- Symmetry reduction from well known partial differential equation
- Transendental general solutions
- Special solutions
- Backlünd transformation

q-discrete Painlevé equations

- Second order q-discrete nonlinear equations

$$\text{q-PII} \quad g(q^{-1}z)g(qz) = \frac{-\gamma_0 g(z)z + \alpha_0 z^2}{g(z)(g(z) - \gamma_0)}$$

$$\text{q-PIII} \quad g(qz)g(q^{-1}z) = \frac{cd(g(z) - az)(g(z) - bz)}{(g(z) - c)(g(z) - d)}$$

- Resulting from singularity confinement
- Symmetry reduction from well known q-partial discrete equations
- Iso-monodromy deformation (Lax Pair)
- Transendental general solutions
- Special solutions
- Backlünd transformation

Iso-monodromy problem for q-PIII

- Spectral Problem (monodromy data is computed)

$$\Psi(qx) = A(x,z)\Psi(x)$$

$$A(x,z) = \begin{pmatrix} 0 & a_1(z) \\ a_2(z) & 0 \end{pmatrix} x^2 + \begin{pmatrix} m_1(z) & 0 \\ 0 & m_2(z) \end{pmatrix} x + \begin{pmatrix} 0 & b_1(z) \\ b_2(z) & 0 \end{pmatrix}$$

- Deformation

$$\begin{aligned} \Psi(x,qz) &= B(x,z)\Psi(x,z) \\ &= \begin{pmatrix} \frac{g(qz)}{\sqrt{q}g(z)} & -\frac{z^{3/2}\delta}{\sqrt{q}g(z)}x \\ -\sqrt{z}\delta g(qz)x & 1 \end{pmatrix} \Psi(x,z) \end{aligned}$$

- Compatibility Condition: $A(x,qz)B(x,z) = B(qx,z)A(x,z)$

q-PIII

$A(x, qz)B(x, z) = B(qx, z)A(x, z) \Rightarrow$

$$g(qz)g(q^{-1}z) = \frac{\alpha_0 g(z)^2 + \frac{\gamma_0}{q^2} z^2}{g(z)^2 + \alpha_0}$$

- This is a special case of the full q-PIII

$$g(qz)g(q^{-1}z) = \frac{cd(g(z) - az)(g(z) - bz)}{(g(z) - c)(g(z) - d)}$$

with

$$a = -b \qquad c = -d$$

Why study the iso-monodromy problem of q-Painlevé equations?

- Associated linear problem for q-PIII

$$\begin{aligned}\Psi(qx) &= A(x,z)\Psi(x)\\ &= \left(A_0(z) + A_1(z)x + A_2(z)x^2\right)\Psi(x)\end{aligned}$$

- Inverse iso-monodromy deformation method for differential system (Flaschka and Newell (1980))
- Availibility of simple 2×2 Lax pairs (M.Hay (2007))
- General theory for q-discrete Linear system
- q-discrete Special functions
- Some known special solutions to q-Painlevé equations

Birkhoff's formula (1913)

q-Discrete Linear System

$$\Psi(qx) = A(x)\Psi(x)$$

$$A(x) = A_0 + A_1 x + A_2 x^2 + \ldots + A_\mu x^\mu$$

where Ψ, A_i are $n \times n$ matrices.

Systems of solutions

$$\begin{cases} \Psi_0(x) = (x^{\rho_j} a_{ij}(x)) \\ \Psi_\infty(x) = q^{\frac{\mu}{2}(t^2-t)}(x^{-\sigma_j} b_{ij}(x)) \\ \Psi_0(x) = \Psi_\infty(x)P(x) \end{cases}$$

q^{ρ_j}, the n eigenvalues of A_0

$q^{-\sigma_j}$, the n eigenvalues of A_μ

$$t = \frac{\ln x}{\ln q}$$

Two cases of when a q-linear system can be explicitly solved interms of elementary known functions

$$\Psi(qx) = (A_0 + A_1 x + A_2 x^2 + ... + A_\mu x^\mu)\Psi(x)$$

where Ψ, A_i are $n \times n$ matrices.

$n = 1,\ \mu = 1 \quad \psi(qx) = (1 - x)\psi(x)$

- solved by q-Gamma function

$n = 2,\ \mu = 1 \quad \Psi(qx) = (A_0 + A_1 x)\Psi(x)$

- solved by q-Hypergeometric function

Associated Linear System for q-Painlevé equations $n = 2,\ \mu > 1$

- $\Psi(qx) = (A_0 + A_1 x + A_2 x^2)\Psi(x)$ for qPIII
- $\Psi(qx) = (A_0 + A_1 x + A_2 x^2 + A_3 x^3)\Psi(x)$ for qPII

When does linear system simplifies?

q-PIII: $$g(q^2 z)g(z) = \frac{\alpha_0 g(qz)^2 + \gamma_0 z^2}{g(qz)^2 + \alpha_0}$$

$$\begin{aligned}\Psi(qx) &= (A_2(z)x^2 + A_1(z)x + A_0(z))\Psi(x) \\ &= \left\{ \begin{pmatrix} 0 & a_1(z) \\ a_2(z) & 0 \end{pmatrix} x^2 + \begin{pmatrix} m_1(z) & 0 \\ 0 & m_2(z) \end{pmatrix} x + \begin{pmatrix} 0 & b_1(z) \\ b_2(z) & 0 \end{pmatrix} \right\} \Psi(x)\end{aligned}$$

- $A_0(z)$, $A_1(z)$, $A_2(z)$ commute $\Rightarrow g(z) = (\frac{\gamma_0}{q^2})^{1/4}\sqrt{z}$

Linear system simplifies in case of q-PIII having algebraic special solutions $g(z) = (\frac{\gamma_0}{q^2})^{1/4}\sqrt{z}$

let $\Psi = C\Psi_1$,

$$\begin{aligned} \Psi_1(qx) &= C^{-1}A(x,z)C\Psi_1(x) \\ &= (C^{-1}A_2Cx^2 + C^{-1}A_1Cx + C^{-1}A_0C)\Psi_1 \end{aligned}$$

$$\begin{pmatrix} u(qx) \\ v(qx) \end{pmatrix}$$

$$= \begin{pmatrix} \gamma_0^{1/4}a_2(z)x^2 + m_1(z)x + \gamma_0^{1/4}b_2(z) & 0 \\ 0 & -\gamma_0^{1/4}a_2(z)x^2 + m_1(z)x - \gamma_0^{1/4}b_2(z) \end{pmatrix} \begin{pmatrix} u(x) \\ v(x) \end{pmatrix}$$

Linear system simplifies in case of q-PIII having algebraic special solutions $g(z) = (\frac{\gamma_0}{q^2})^{1/4}\sqrt{z}$

$$
\begin{aligned}
u(qx) &= (\gamma_0^{1/4} a_2(z)x^2 + m_1(z)x + \gamma_0^{1/4} b_2(z))u(x) \\
&= c_1(x - \frac{q}{z})(x - \frac{\gamma_0}{\alpha_0})u(x)
\end{aligned}
$$

- Let $u = c_1^t u_1 u_2$, $t = \frac{\ln x}{\ln q}$

$$
\begin{aligned}
u(qx) &= c_1^{t+1} u_1(qx)u_2(qx) \\
&= \left(c_1(x - \frac{q}{z})(x - \frac{\gamma_0}{\alpha_0})\right) c_1^t u_1(x)u_2(x)
\end{aligned}
$$

- $u_1(qx) = (x - \frac{q}{z})u_1(x), \qquad u_2(qx) = (x - \frac{\gamma_0}{\alpha_0})u_2(x)$

Lax Pair for q-PII

$$\begin{aligned}\Psi(qx,z) &= A(x,z)\Psi(x,z)\\ &= \left\{ \begin{pmatrix} e_1(z) & 0 \\ 0 & e_2(z) \end{pmatrix} x^3 + \begin{pmatrix} 0 & m_1(z) \\ m_2(z) & 0 \end{pmatrix} x^2 \right.\\ &\quad \left. + \begin{pmatrix} n_1(z) & 0 \\ 0 & n_2(z) \end{pmatrix} x + \begin{pmatrix} 0 & b_1(z) \\ b_2(z) & 0 \end{pmatrix} \right\} \Psi(qx,z)\end{aligned}$$

$$\begin{aligned}\Psi(x,qz) &= B(x,z)\Psi(x,z)\\ &= \begin{pmatrix} q^{-1/2} g(z/q)\sqrt{z} & -\frac{\sqrt{z}g(z/q)g(z)}{x} \\ -\frac{z^{-3/2}}{xg(qz)g(z)} & \frac{q^{-1/2}}{\sqrt{z}g(z/q)} \end{pmatrix} \Psi(x,z)\end{aligned}$$

Associated linear system for q-PII

q-PII

$$g(q^{-1}z)g(qz) = \frac{-\gamma_0 g(z)z + \alpha_0 z^2}{g(z)(g(z) - \gamma_0)}$$

Associated linear system

$$\begin{aligned}\Psi(qx) &= A(x,z)\Psi(x) \\ &= \left\{ \begin{pmatrix} e_1(z) & 0 \\ 0 & e_2(z) \end{pmatrix} x^3 + \begin{pmatrix} 0 & m_1(z) \\ m_2(z) & 0 \end{pmatrix} x^2 \right. \\ &\quad \left. + \begin{pmatrix} n_1(z) & 0 \\ 0 & n_2(z) \end{pmatrix} x + \begin{pmatrix} 0 & f_1(z) \\ f_2(z) & 0 \end{pmatrix} \right\} \Psi(qx)\end{aligned}$$

Asymptotic expandsions as $x \to \infty$

$$\Psi(x,z) = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \frac{1}{x} \begin{pmatrix} a_1(z) & b_1(z) \\ c_1(z) & d_1(z) \end{pmatrix} + \frac{1}{x^2} \begin{pmatrix} a_2(z) & b_2(z) \\ c_2(z) & d_2(z) \end{pmatrix} + \ldots \right\} \times q^{\frac{3}{2}(t^2-t)} \begin{pmatrix} e_1^t & 0 \\ 0 & e_2^t \end{pmatrix}$$

$$\begin{aligned} b_n(e_2/q^n - e_1) &= F(a_{n-1}, b_{n-1}, c_{n-1}, d_{n-1}) \\ c_n(e_1/q^n - e_2) &= F(a_{n-1}, b_{n-1}, c_{n-1}, d_{n-1}) \end{aligned}$$

$$e_2 = q^n e_1 \quad \Rightarrow \quad \alpha_0 = \frac{1}{q^n}$$

$$e_1 = q^n e_2 \quad \Rightarrow \quad \alpha_0 = q^n$$

for n odd

- Two simple examples:

$$\begin{aligned} e_2 &= q e_1 \Rightarrow \alpha_0 = 1/q \\ e_2 &= q^{-1} e_1 \Rightarrow \alpha_0 = q \end{aligned}$$

$$\Psi(x,z) = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \frac{1}{x}\begin{pmatrix} 0 & b_1(z) \\ c_1(z) & 0 \end{pmatrix} + \frac{1}{x^2}\begin{pmatrix} a_2(z) & 0 \\ 0 & d_2(z) \end{pmatrix} + \dots \right\} \times q^{\frac{3}{2}(t^2-t)} \begin{pmatrix} e_1^t & 0 \\ 0 & e_2^t \end{pmatrix}$$

- Equating coefficients of the same order in x we get conditions on $b_1(z)$, $c_1(z)$

$$b_1(z)(e_2/q - e_1) = m_1(z) \quad \text{if} \quad \alpha_0 = \frac{1}{q} \quad \Rightarrow g(qz) = \frac{-z}{g(z) - \gamma_0}$$

$$c_1(z)(e_1/q - e_2) = m_2(z) \quad \text{if} \quad \alpha_0 = q \quad \Rightarrow g(qz) = \frac{\gamma_0 - qz}{g(z)}$$

Special Solutions for q-PII

- q-PII

$$g(q^{-1}z)g(qz) = \frac{-\gamma_0 g(z)z + \alpha_0 z^2}{g(z)(g(z)-\gamma_0)}$$

Riccati special solution

$$g(qz) = \frac{-z}{g(z)-\gamma_0}, \quad \alpha_0 = \frac{1}{q}$$

- Linearise ⇒ q-Airy equation

$$g(z) = \frac{u(z)}{v(z)} \qquad u(qz) = -zv(z)$$

$$v(q^2z) + \gamma_0 v(qz) + zv(z) = 0$$

Summary

- Follow the analogous behaviors of the differential and q-discrete systems, especially the iso-monodromy problems for q-PII and q-PIII
- Analyze the spectral half the iso-monodromy problems in the frame work of q-discrete linear systems developed by Birkhoff and Adam et al.
- Special cases which the associated linear system can be simplified correspond to q-Painelevé equation having special solutions
- The q-discrete Lax pair contains all the information and properties of its associated q-Painlevé equation, hence will provide a way of studying these very special q-nonlinear equations.

THANK YOU!