Integrability of discrete systems via multiscale expansions

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Introduction

Multiscale perturbation techniques [TN][1]:

- Important tools for *approximate solutions* to many physical problems by reducing a given PDE to a simpler equation;
- Suitable for a *regularization* of the spurious diverging *secular terms*;
- Applicable to both integrable and non-integrable systems;
- Zakharov-Kuznetsov's [ZK][2]: *"if a nonlinear dispersive equation is integrable then its lowest order multiscale reduction is an integrable NLS equation"*;
- Orders beyond NLS order [DMS][3] provide *necessary conditions for integrability* [C][4];
- Notion of *asymptotic integrability* [DP][5], [KM] (normal forms theory)[6].

[1] T. Taniuti, K. Nishihara, *Nonlinear waves* (Pitman, Boston, 1983).

[2] V.E. Zakharov, E.A. Kuznetsov, Phys. D **18**, 455 (1986).

[3] A. Degasperis, S.V. Manakov, P.M. Santini, Phys. D **100**, 187 (1997).

[4] F. Calogero, W. Eckhaus, Inv. Prob. **3**/2, L27 (1987); **3**/2, 229 (1987); **4**/1, 11 (1987); F. Calogero, A. Degasperis, X-D. Ji, Jour. Math. Phys. **42**/6, 2635 (2001); **41**/9, 6399 (2000); F. Calogero, A. Maccari, in *Inverse problems: an interdisciplinary study* (Academic Press, London, 1987) 463.

[5] A. Degasperis, M. Procesi in *SPT98*, (World Scientific, Singapore, 1999) 23.

[6] Y. Kodama, A. V. Mikhailov in *Algebraic aspects of integrable systems*, (Birkhauser, Boston, 1996) 173.

The reduction of the nonintegrable discrete *NLS* (*DNLS*)

- Consider the *DNLS* equation:

$$\mathrm{i}\partial_t f_n(t) + \frac{f_{n+1}(t) - 2f_n(t) + f_{n-1}(t)}{2\sigma^2} = \alpha |f_n(t)|^2 f_n(t),$$

$\alpha = \pm 1$;

- *Bose-Einstein condensates* in *optical lattices* [AB][7] and *semiconductors* [HT][8];
- *numerical* emergence of *chaos* [AO][9];
- Set $f_n(t) \doteq \nu_n(t)^{1/2} e^{\mathrm{i}\phi_n(t)}$:

$$\partial_t \nu_n(t) + \frac{1}{\sigma^2}\left[\delta_+^{1/2} \sin\beta_+ + \delta_-^{1/2} \sin\beta_-\right] = 0,$$

$$\partial_t \phi_n(t) + \frac{1}{\sigma^2} - \frac{1}{2\sigma^2}\left[\gamma_+^{1/2} \cos\beta_+ + \gamma_-^{1/2} \cos\beta_-\right] + \alpha\nu_n(t) = 0,$$

$\beta_\pm \doteq \phi_{n\pm1}(t) - \phi_n(t)$, $\gamma_\pm \doteq \nu_n(t)^{-1}\nu_{n\pm1}(t)$, $\delta_\pm \doteq \nu_n(t)\nu_{n\pm1}(t)$;

[7] F.Kh. Abdullaev, B.B. Baizakov, S.A. Darmanyan, V.V. Konotop, M. Salerno, Phys. Rev. A **64**, 043606 (2001).

[8] D. Hennig, G. Tsironis, Phys. Reports **307**, 333 (1999).

[9] M. J. Ablowitz, Y. Ohta, A. D. Trubatch, Chaos Sol. and Frac. **11**, 158 (2000).

- Seek for solutions of the form:

$$\nu_n(t) = 1 + \sum_{\kappa=1}^{+\infty} \varepsilon^{2\kappa} \, \nu_{n_1}^{(\kappa)}(\{t_j\}_{j\geq 1}),$$

$$\phi_n(t) = -\alpha t + \sum_{\kappa=1}^{+\infty} \varepsilon^{2\kappa-1} \, \phi_{n_1}^{(\kappa)}(\{t_j\}_{j\geq 1});$$

Expansion Parameters

1. $0 \leq \varepsilon \ll 1$: perturbative parameter;
2. $n_1 \doteq \varepsilon \xi n$: slow space variable ($\xi$ constant);
3. $t_j = \varepsilon^{2j-1} t$, $j \geq 1$ slow times variables;

- Suppose $\nu_{n_1}^{(\kappa)}(\{t_j\}_{j\geq 1})$ and $\phi_{n_1}^{(\kappa)}(\{t_j\}_{j\geq 1})$ analytic in n_1:

$$\nu_{n\pm 1}(t) = 1 + \sum_{\kappa=2}^{+\infty} \sum_{j=1}^{[\kappa/2]} \varepsilon^\kappa \mathcal{A}_{\kappa-2j}^{\pm} \left[\nu_{n_1}^{(j)}(\{t_j\}_{j\geq 1}) \right],$$

$$\phi_{n\pm 1}(t) = -\alpha t + \sum_{\kappa=1}^{+\infty} \sum_{j=1}^{[(\kappa+1)/2]} \varepsilon^\kappa \mathcal{A}_{\kappa-2j+1}^{\pm} \left[\phi_{n_1}^{(j)}(\{t_j\}_{j\geq 1}) \right];$$

Expansion Operators

1. $\mathcal{A}_\kappa^{\pm} \doteq (\pm\xi\delta_{n_1})^\kappa/\kappa!$: from shift operators as series of derivatives;
2. δ_{n_1}: derivative operator w. r. t. n_1 (continuos through analyticity) with derivatives calculated in $n_1 = \varepsilon\xi n$;
3. $[\rho]$ integer part of ρ;

- Similar expansions for the time derivatives:

$$\partial_t \nu_n(t) = \sum_{\kappa=2}^{+\infty} \sum_{j=1}^{\kappa-1} \varepsilon^{2\kappa-1} \, \partial_{t_j} \nu_{n_1}^{(\kappa-j)}(\{t_j\}_{j\geq 1}),$$

$$\partial_t \phi_n(t) = -\alpha + \sum_{\kappa=1}^{+\infty} \sum_{j=1}^{\kappa} \varepsilon^{2\kappa} \, \partial_{t_j} \phi_{n_1}^{(\kappa-j+1)}(\{t_j\}_{j\geq 1}).$$

The reduced equations

Plug all the expansion in the *DNLS* equation:

- Order ε^2:

$$\nu^{(1)} = -\alpha \partial_{t_1} \phi^{(1)}.$$

- Order ε^3:

$$\left(\partial_{t_1}^2 - c^2 \delta_{n_1}^2\right) \phi^{(1)} = 0, \qquad c \doteq \pm \frac{\xi}{\sigma} (\alpha)^{1/2};$$

1. c real: $\alpha = 1$;
2. c finite in the continuous limit $\sigma \to 0$: $\xi = \sigma$ so that $c = \pm 1$;
3. choose $\phi^{(1)}$ depending on $x \doteq n_1 - c t_1$:

$$\left(\partial_{t_1} + c \partial_x\right) \phi^{(1)} = 0.$$

- Orders ε^4, ε^5:

$$\left(\partial_{t_1}^2 - \delta_{n_1}^2\right)\phi^{(2)} = 2c\partial_x\left[\partial_{t_2}\phi^{(1)} - \rho_1\partial_{x1}^3\phi^{(1)} + \frac{3}{4}\left(\partial_x\phi^{(1)}\right)^2\right],$$

$$\rho_1 \doteq \frac{1}{8}\left[c - \frac{\sigma^2}{3c}\right];$$

Secularity condition

- The right-hand side solves the homogeneous equation: secularity!

$$\left(\partial_{t_1}^2 - \delta_{n_1}^2\right)\phi^{(2)} = 0,$$

$$\partial_{t_2}\phi^{(1)} = K_2\left[\phi^{(1)}\right];$$

$$K_2\left[\phi^{(1)}\right] \doteq \rho_1\partial_x^3\phi^{(1)} - \frac{3}{4}\left(\partial_x\phi^{(1)}\right)^2$$: *potential KdV equation!*

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ρ_2, ρ_3, ρ_4 coefficients;

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$$+\partial_{t_3}\phi^{(1)} + c\rho_2 \partial_x^5 \phi^{(1)} - \frac{c\sigma^2}{12}\left(\partial_x \phi^{(1)}\right)^3 +$$

$$\left. +\rho_3 \left(\partial_x \phi^{(1)}\right)\left(\partial_x^3 \phi^{(1)}\right) + \rho_4 \left(\partial_x^2 \phi^{(1)}\right)^2 \right],$$

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$$\left(\partial_{t_1}^2 - \delta_{n_1}^2\right)\phi^{(3)} = 0,$$

Secularity condition

$$\partial_{t_2}\phi^{(2)} - \rho_1\partial_x^3\phi^{(2)} + \frac{3}{2}\left(\partial_x\phi^{(1)}\right)\left(\partial_x\phi^{(2)}\right) = -\partial_{t_3}\phi^{(1)} - c\rho_2\partial_x^5\phi^{(1)} +$$

$$+\frac{c\sigma^2}{12}\left(\partial_x\phi^{(1)}\right)^3 - \rho_3\left(\partial_x\phi^{(1)}\right)\left(\partial_x^3\phi^{(1)}\right) - \rho_4\left(\partial_x^2\phi^{(1)}\right)^2; \quad (*)$$

Conjecture

Degasperis Procesi's [DP] (*NLS* hierarchy): *a given nonlinear equation is integrable iff:*

1. $\partial_{t_n}\phi^{(1)} = K_n\left[\phi^{(1)}\right]$, $K_n\left[\phi^{(1)}\right]$ *n-th flow of a hierarchy of commuting flows* (our case: *pKdV hierarchy*), $\forall n \geq 2$;
2. $M_n\phi^{(j)} = f_n^{(j)}$, $M_n \doteq \partial_{t_n} - K_n'\left[\phi^{(1)}\right]$, $K_n'[u]\,v$ *Frechet derivative of* $K_n[u]$ *along* v, $\forall n,\ j \geq 2$;
3. $f_n^{(j)}$ *nonlinear differential polynomial forcing term in* $\phi^{(\kappa)}$, $\kappa \leq j-1$;

- *Necessary* part of the conjecture proved;

Secularity condition

$$\partial_{t_2}\phi^{(2)} - \rho_1\partial_x^3\phi^{(2)} + \frac{3}{2}\left(\partial_x\phi^{(1)}\right)\left(\partial_x\phi^{(2)}\right) = -\partial_{t_3}\phi^{(1)} - c\rho_2\partial_x^5\phi^{(1)} +$$
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- *Necessary* part of the conjecture proved;

Notations/Definitions

1. *degree:* $deg\left[\partial_x^p \phi^{(j)}\right] \doteq p + 2j - 1$;
2. *vector space* $\mathcal{P}_n$: all differential polynomials in $\partial_x^p \phi^{(j)}$ with $p \geq 1$ and total $deg = n$;
3. *vector subspace* $\mathcal{P}_n(m)$: $\partial_x^p \phi^{(j)}$, $j \leq m$;
4. *asymptotic integrability degree* A_n: if eq. admits expansion till slow time t_{n+1};

- $f_n^{(j)} \in \mathcal{P}_{2(n+j-1)}(j-1)$;
- $\partial_{t_3}\phi^{(1)} = \lambda \partial_x^5 \phi^{(1)} + \frac{5\lambda}{8\rho_1^2}\left(\partial_x \phi^{(1)}\right)^3 - \frac{5\lambda}{4\rho_1}\left(\partial_x^2 \phi^{(1)}\right)^2 - \frac{5\lambda}{2\rho_1}\left(\partial_x \phi^{(1)}\right)\left(\partial_x^3 \phi^{(1)}\right)$, *quintic pKdV equation*, λ free parameter; $(**)$
- Insert $(**)$ into $(*)$ ($\partial_x^5 \phi^{(1)}$ *secular*: fix λ so as coefficient be 0):

$$\partial_{t_2}\phi^{(2)} - K_2'\left[\phi^{(1)}\right]\phi^{(2)} = f_2^{(2)},$$

$$f_2^{(2)} \doteq \theta_1\left(\partial_x \phi^{(1)}\right)^3 + \theta_2\left(\partial_x \phi^{(1)}\right)\left(\partial_x^3 \phi^{(1)}\right) + \theta_3\left(\partial_x^2 \phi^{(1)}\right)^2,$$

θ_1, θ_2, θ_3 constants;

Integrability conditions

1. $\partial_{t_3}\phi^{(2)} - K_3'\left[\phi^{(1)}\right]\phi^{(2)} = f_3^{(2)}$;
2. $f_3^{(2)} \in \mathcal{P}_8(1)$, $dim[\mathcal{P}_8(1)] = 6$: lin. comb. ξ_j, $j = 1, \ldots, 6$;
3. *Compatibility condition:* $\left(\partial_{t_3} - K_3'\left[\phi^{(1)}\right]\right) f_2^{(2)} = \left(\partial_{t_2} - K_2'\left[\phi^{(1)}\right]\right) f_3^{(2)}$: *no conditions* on θ_i, $i = 1, \ldots, 3$, only $\xi_i(\theta_j)$: *DNLS* equation *A_2-asymptotically integrable!*

- Orders ε^8, ε^9 (after caring for secularities):

$$\left(\partial_{t_1}^2 - \delta_{n_1}^2\right)\phi^{(4)} = 0,$$

$$\partial_{t_4}\phi^{(1)} = \chi\partial_x^7\phi^{(1)} - \frac{7\chi}{2\rho_1}\left(\partial_x\phi^{(1)}\right)\left(\partial_x^5\phi^{(1)}\right) + \frac{35\chi}{8\rho_1^2}\left(\partial_x\phi^{(1)}\right)\left(\partial_x^2\phi^{(1)}\right)^2 + \\ + \frac{35\chi}{8\rho_1^2}\left(\partial_x\phi^{(1)}\right)^2\left(\partial_x^3\phi^{(1)}\right) - \frac{35\chi}{64\rho_1^3}\left(\partial_x\phi^{(1)}\right)^4 - \frac{21\chi}{4\rho_1}\left(\partial_x^3\phi^{(1)}\right)^2 - \\ - \frac{7\chi}{\rho_1}\left(\partial_x^2\phi^{(1)}\right)\left(\partial_x^4\phi^{(1)}\right),$$

septic pKdV equation, χ free, fixed so as *secular term $\partial_x^7\phi^{(1)}$* removed;

- $\partial_{t_2}\varphi^{(3)} - H_2'\left[\varphi^{(1)}\right]\varphi^{(3)} = g_2^{(3)}$. Results *not integrated:* $\varphi^{(j)} \doteq \partial_x \phi^{(j)}$, $H_n\left[\varphi^{(1)}\right]$ n-th flow of the *KdV hierarchy;*
- $g_2^{(3)} \in \mathcal{P}_9(2)$, $dim[\mathcal{P}_9(2)] = 14$, lin. comb. μ_i, $i = 1, \ldots, 14$;

Integrability conditions

1. $\partial_{t_3}\varphi^{(3)} - H_3'\left[\varphi^{(1)}\right]\varphi^{(3)} = g_3^{(3)}$;
2. $g_3^{(3)} \in \mathcal{P}_{11}(2)$, $dim[\mathcal{P}_{11}(2)] = 31$: lin. comb. δ_i, $i = 1, \ldots, 31$;
3. *Compatibility condition:* $\left(\partial_{t_3} - H_3'\left[\varphi^{(1)}\right]\right) g_2^{(3)} = \left(\partial_{t_2} - H_2'\left[\varphi^{(1)}\right]\right) g_3^{(3)}$: *5 integrability constraints* on μ_i, $i = 1, \ldots, 14$, and $\delta_i(\mu_j)$. μ_i *doesn't respect them: DNLS equation not integrable!*

Constraints are satisfied:

- in the continuous limit;
- in the case of $A. - L.$ (integrable) NLS equation

$$i\partial_t f_n(t) + \frac{f_{n+1}(t) - 2f_n(t) + f_{n-1}(t)}{2\sigma^2} = \alpha|f_n(t)|^2 \frac{f_{n+1}(t) + f_{n-1}(t)}{2}.$$

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The dispersive case

Similar reduction procedure for: (double expansion in harmonics and pert. param.)

- offcentrically discretized KdV equation : A_0-asymptotically integrable!

$$u_2 - u_{-2} = \frac{\alpha}{4}\left[u_{111} - 3u_1 + 3u_{-1} - u_{-1-1-1}\right] - \frac{b}{2}\left[u_1^2 - u^2\right];$$

- symmetrically discretized KdV equation : A_2-asymptotically integrable!

$$u_2 - u_{-2} = \frac{\alpha}{4}\left[u_{111} - 3u_1 + 3u_{-1} - u_{-1-1-1}\right] - \frac{b}{2}\left[u_1^2 - u_{-1}^2\right];$$

- lpKdV equation (integrable: A_∞):

$$\alpha\,(u_{12} - u) + \beta\,(u_1 - u_2) - (u_1 - u_2)\,(u_{12} - u) = 0;$$

- linearizable equations (Hietarinta).

Conclusions/Open problems

- DNLS equation: *analytical* evidence of its nonintegrability proposed;
- *Integrability test* suitable for a large variety of nonlinear systems developed;
- Do the same without relying on the *analyticity* of solutions: *new discrete integrable systems*;