Do integrable mappings have the Painlevé property?

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Discrete systems are the fundamental entities
but everybody is more familiar with continuous systems

Poincaré:
to integrate a differential equation is to find for the general solution a finite expression, possibly multivalued, in terms of a finite number of functions

"finite" → integrability *global* rather than *local*

For ODEs in the complex domain to find a global solution

start with a local one and use analytically continuation

Problem: branch points!

How to treat branch points?

Uniformization procedures *provided* the branch points are fixed

Linear ODE's are OK

Nonlinear ODE's: positions of singularities depend on initial conditions: singularities are movable!

The Painlevé approach
(Fuchs, Kovalevskaya and the challenge of Picard)

Look for nonlinear ODEs with solutions

free from movable branch points

Painlevé property not just a *predictor* of integrability but practically a definition of integrability (a tautology rather than a criterion?)

Distinguish: Painlevé property and algorithm for its investigation!

Ablowitz-Ramani-Segur algorithm
necessary condition for the absence of movable branch points
(movable essential singularities cannot be detected)

$$w_i' = F_i(w_1, w_2, \ldots, w_n; z) \qquad i = 1, \ldots, n$$

Assumption

$$w_i \sim \alpha_i (z - z_0)^{p_i}, \qquad z \to z_0$$

One must find all possible dominant behaviours

Painlevé expansion
Laurent series

$$w_i = (z - z_0)^{p_i} \sum_0^\infty a_i^{(m)} (z - z_0)^m$$

The powers m where free constants arise are the “resonances”

The constants of integration
Truncated expansion

$$w_i = \alpha_i \tau_i^{p_i} + \sum_1^{r_s} a_i^{(m)} \tau^{p_i+m}$$

substitute into full equation $\rightarrow$ compatibility condition

$$Q(m) a^{(m)} = R^{(m)}(z_0; a^{(j)}), \qquad j = 1, \ldots, m-1$$

When det $Q(m)$=0, resonance condition! If not satisfied then

$$w_i = \sum_0^{r-1} a_i^{(m)} \tau^{p_i+m} + (a_i^{(r)} + b_i^{(r)} \ln\tau) \tau^{p_i+r} + \ldots$$

2-D Hamiltonian with quartic potential

$$H = \frac{1}{2}\left(p_x^2 + p_y^2\right) + y^4 + ax^2y^2 + bx^4$$

three simple integrable cases:
a) $a = 0$: V separable in x and y
b) $a = 6,\ b = 1$: V separable in $x \pm y$
c) $a = 2,\ b = 1$: V separable in polar coordinates.

Painlevé analysis
Equations of motion
$\ddot{x} = -2axy^2 - 4bx^3 \quad \ddot{y} = -4y^3 - 2ax^2y$

Leading behaviours
i) $x \propto (t - t_0)^s, \quad y \propto \gamma(t - t_0)^{-1}, \quad s > -1$
ii) $x \propto \delta(t - t_0)^{-1}, \quad y \propto (t - t_0)^q, \quad q > -1$
iii) $x \propto \alpha(t - t_0)^{-1}, \quad y \propto \beta(t - t_0)^{-1}$

From (i) and (ii) $s(s-1)=a$, $q(q-1)=a/b$
(iii) $2\beta^2+a\alpha^2+1=0$, $a\beta^2+2b\alpha^2+1=0$.

Resonances' equation

$$(N-6)(N+6+(12+2a)\beta^2+(2a+12b)\alpha^2)=0$$

where $N=(r-1)(r-2)$.

$N=6$ leads to $r=-1,4$
second possibility
$N=-\big(6+(12+2a)\beta^2+(2a+12b)\alpha^2\big)$
must be the product of two consecutive integers

Integrable cases:

d) $s = 3/2,\ q = 4,\ (r = 5, -2)$ V “separable” in parabolic coordinates

$V = y^4 + 3x^2y^2/4 + x^4/16$

Invariant

$$I = p_x(xp_y - yp_x) + x^2y^3/2 + x^4y/4$$

e) $s = 3/2,\ q = 3,\ (r = 8, -5)$

$V = y^4 + 3x^2y^2/4 + x^4/8$

Invariant

$$I = p_x^4 + \frac{1}{2}(6x^2y^2 + x^4)p_x^2 - 2x^3yp_xp_y + \frac{1}{2}x^4p_y^2 + \frac{1}{4}(x^4y^4 + x^6y^2) + \frac{x^8}{16}$$

Integrability *without* the Painlevé property

$$H = \frac{1}{2}p_x^2 + \frac{1}{2}p_y^2 + y^5 + y^3x^2 + \frac{3}{16}yx^4$$

Invariant $I = -yp_x^2 + xp_xp_y + \frac{1}{2}y^4x^2 + \frac{3}{8}y^2x^4 + \frac{1}{32}x^6$
Movable singularities $y \approx \alpha(t-t_0)^{-2/3}$, $x \approx \beta(t-t_0)^{-1/3}$
Taking the cube is *not* sufficient: *all* powers of $(t-t_0)^{1/3}$ present

"Weak Painlevé" property

Hamiltonian member of a class with potential

$$V = (F(\sqrt{x^2+y^2}+y) + G(\sqrt{x^2+y^2}-y))/\sqrt{x^2+y^2}$$

Since F and G are free the singularities are arbitrary

Solution by quadratures

$$H = \frac{1}{2}(p_x^2 + p_y^2) + F(\rho) + \frac{1}{\rho^2}G(\phi)$$

with

$$I = (xp_y - yp_x)^2 + 2G(\phi)$$

leads to

$$\dot{\rho}^2 = 2H_0 - 2F(\rho) - \frac{1}{\rho^2}I_0$$

and then

$$\int \frac{d\phi}{\sqrt{I_0 - 2G(\phi)}} = \pm \int \frac{dt}{\rho^2(t)}$$

Here also any kind of singularities!

ARS approach not failsafe: movable essential critical singularities

Mixmaster Universe model

$$\dot{X} = X(p_x - p_y - p_z)$$

$$\dot{Y} = Y(p_y - p_z - p_x)$$

$$\dot{Z} = Z(p_z - p_x - p_y)$$

$$\dot{p}_x = X(Y + Z - X)$$

$$\dot{p}_y = Y(Z + X - Y)$$

$$\dot{p}_z = Z(X + Y - Z)$$

ARS singularity analysis not conclusive

No logarithmic singularity found

Change of variables $Y - Z = \delta$, $Y + Z = \sigma$, $p_y - p_z = q$, $p_y + p_z = p$

New equations

$$\dot{X} = X(p_x - p) \qquad \dot{\sigma} = -p_x\sigma + \delta q \qquad \dot{\delta} = -p_x\delta + \sigma q$$

$$\dot{p}_x = X(\sigma - X) \qquad \dot{p} = X\sigma - \delta^2 \qquad \dot{q} = (X - \sigma)\delta$$

We find that δ and q vanish at all orders!

Beyond-all-orders behaviour of δ and q:

$$\ddot{\delta} - 2\frac{\dot{\sigma}}{\sigma}\,\dot{\delta} + \left(\sigma^2 + 2(\frac{\dot{\sigma}}{\sigma})^2 - \frac{\ddot{\sigma}}{\sigma}\right)\delta = 0$$

Dominant terms

$$\ddot{\delta} + \frac{4}{\tau}\dot{\delta} + \left(\frac{4}{A^2\tau^4} - \frac{8C}{A\tau^3}\right)\delta = 0$$

General solution

$$\delta = c_1\tau^{2iC-1}e^{\frac{2i}{A\tau}} + c_2\tau^{-2iC-1}e^{-\frac{2i}{A\tau}}$$

transcendental branching point!

Three kinds of integrability

Integrability by spectral methods: Painlevé property required
(system reduced to linear integrodifferential equations)

Integrability by quadratures: no relation to Painlevé property

but also
Integrability by linearisation
(system reduced to linear differential equations)

again, no relation to Painlevé property

Linearisable systems without the Painlevé property

Start from linear 2nd order equation (K=cnst, α, etc. free functions)

$$\frac{\alpha x'' + \beta x' + \gamma x + \delta}{\epsilon x'' + \zeta x' + \eta x + \theta} = K$$

and a *nonlinear* 2nd order equation (M is constant)

$$f(x'', x', x) = M$$

f polynomial in x and derivatives of second degree, but linear in x''

Take derivative for K and $M \to$ *same* 3rd order (nonlinear) equation

Both nonlinear equations violate the Painlevé property

No paradox! Linear equation can have a critical singularity
at a position *fixed by its parameter* K.

But K is not a parameter but a cnst of integration of the other equations!

For them the critical singularity is *movable*

Example:

Case where nonlinear eq. contains a term $x''x'$

→ violates the Painlevé property

Choose

$$\frac{tx'' + (at - 1/2)x' + btx}{x'' + ax' + bx} = K$$

$$x''x' + 2ax'^2 + 3bx'x + (2ab - b')x^2 = M$$

with $b = a^2 - a'/2$ and a given by

$$a''' = 6a''a + 7a'^2 - 16a'a^2 + 4a^4$$

(equation XII of the Chazy classification)

Essentially, the linear equation has a **fixed** “bad” singularity at $t = K$....
but for the nonlinear equation K not explicit, just integration constant:

→ for it, the “bad” singularity is “movable”

Singularity confinement criterion

Lattice KdV equation

$$x_j^{i+1} = x_{j+1}^{i-1} + \frac{1}{x_j^i} - \frac{1}{x_{j+1}^i}$$

"what if a singularity appears spontaneously?"

$x = 0$ at (i, j)
$x = \infty$ at both $(i+1, j-1)$ and $(i+1, j)$
and $x = 0$ at $(i+2, j-1)$

At $(i+3, j-2)$ and $(i+3, j-1)$ finite values!

Sing. does not propagate beyond a few lattice points: it is **confined**

"Discrete Painlevé property" ?

An example

$$x_{n+1} + x_{n-1} = \frac{a}{x_n} + \frac{1}{x_n^2}$$

Singularity, whenever x_n=0

Iterate $\rightarrow$ sequence $\{0, \infty, 0\}$

and then *indeterminate form* $\infty - \infty$

Kruskal:

The real problem is the indeterminate form not the simple infinity

Solution

Use continuity with respect to the initial conditions

Introduce a small parameter ϵ

Start from $x_n = \epsilon$, obtain: $x_{n+1} \approx 1/\epsilon^2$, $x_{n+2} \approx -\epsilon$

Compute carefully x_{n+3}

Finite and depends on initial condition x_{n-1}

The singularity has disappeared!

Discrete Painlevé equations

Laguerre? (1885)
Shohat (1939)

$$x_{n+1} + x_{n-1} + x_n = \frac{z_n}{x_n} + 1$$

with $z_n = \delta n + z_0$
(many years later was recognised as d-$\mathrm{P_I}$)

Jimbo & Miwa (1981), contiguity relations of c-P's
From $\mathrm{P_{II}}$:

$$w'' = 2w^3 + tw + \alpha$$

contiguity relation, $x_n = w(t, \alpha_n)$, $\alpha_n = n + \alpha_0$:

$$\frac{\alpha_n + 1/2}{x_{n+1} + x_n} + \frac{\alpha_n - 1/2}{x_n + x_{n-1}} = -(2x_n^2 + t)$$

d-$\mathrm{P_I}$ from singularity confinement

$$x_{n+1} + x_n + x_{n-1} = a(n) + \frac{b(n)}{x_n}$$

Assume: x_n regular and x_{n+1} vanishes

$x_{n+1} = \epsilon$
$x_{n+2} = \frac{b_{n+1}}{\epsilon} + a_{n+1} - x_n + \mathcal{O}(\epsilon)$
$x_{n+3} = -\frac{b_{n+1}}{\epsilon} + a_{n+2} - a_{n+1} + x_n + \mathcal{O}(\epsilon)$
x_{n+4} diverges unless $a_{n+3} - a_{n+2}$=0 (for confinement a=constant)
For x_{n+5} finite, second condition: $b_{n+1} - b_{n+2} - b_{n+3} + b_{n+4} = 0$
Solution $b_n = \alpha n + \beta + \gamma(-1)^n$

Ignore even-odd dependence *with hindsight: very bad idea!*
put $b_n \equiv z_n = \alpha n + \beta$:

$$x_{n+1} + x_n + x_{n-1} = a + \frac{z_n}{x_n}$$

Continuous limit is $\mathrm{P_I}$ (full case: $\mathrm{P_{II}}$)

Discovery of q-Painlevé equations

$$x_{n+1}x_{n-1} = \frac{ab(x_n - cq_n)(x_n - dq_n)}{(x_n - a)(x_n - b)}$$

where a, b, c, and d are constants
(neglecting even-odd dependence, another very bad idea)
From singularity confinement

$$q_n = q_0\lambda^n$$

Not a difference equation, but a q- (multiplicative) mapping

Continuous limit is $\mathrm{P_{III}}$ (full case: $\mathrm{P_{VI}}$!)

The Hietarinta-Viallet (H&V) discovery:
Confinement is not sufficient for integrability

Integrability related to low–growth properties
(notion of *algebraic entropy*)

Mapping of degree d
$\rightarrow$ n-th iterate: degree d^n, unless there exist simplifications

Integrable mappings: massive simplifications
$\rightarrow$ polynomial degree growth

Non-integrable mappings:
even if a lot of simplifications (sing. conf. guarantees **some** amount)
$\rightarrow$ still exponential degree growth

Example:

$$x_{n+1} + x_{n-1} = \frac{a}{x_n} + \frac{1}{x_n^2}$$

Homogeneous coordinates $x_0 = p$, $x_1 = q/r(\deg p\!=\!0, \deg q\!=\!\deg r\!=\!1)$
Obtain the degrees:
0, 1, 2, 5, 8, 13, 18, 25, 32, 41, ..., $d_{2m} = 2m^2$ and $d_{2m+1} = 2m^2\!+\!2m\!+\!1$
Degree growth is polynomial: integrable mapping (QRT)

Nonintegrable mapping:

$$x_{n+1} + x_{n-1} = x_n + \frac{1}{x_n^2}$$

Singularity pattern is $\{0, \infty, \infty, 0\}$ but chaotic behaviour
Degrees: 0, 1, 3, 8, 23, 61, 162, 425, ...,
$d_{n+4} = 3(d_{n+3} - d_{n+1}) + d_n$
Exponential growth with ratio $(3 + \sqrt{5})/2$.

Algebraic entropy is *not necessary*

Simplest example

$$x_{n+1}x_{n-1} = x_n^3$$

Put $\omega_n = \log x_n$ and find for ω a linear equation

$$\omega_{n+1} + \omega_{n-1} = 3\omega_n$$

Algebraic entropy $\epsilon = \log((3 + \sqrt{5})/2)$

Another example

$$x_{n+1} = \frac{3x_n - x_n^3 + x_{n-1}(1 - 3x_n^2)}{1 - 3x_n^2 + x_{n-1}(3x_n - x_n^3)}$$

Put $\omega_n = \tan x_n$ and find for ω the same linear equation

But this should be called “solvability”, not “integrability”:
essentially it is the “baker’s transformation”

Nevanlinna theory

Growth conditions integrability

Formal identity between discrete systems and delay equations
Nevanlinna theory: value distribution of meromorphic functions

Notion of *order*
Infinite order for very-fast-growing functions
Finite order indicates a moderate growth
Ablowitz, Herbst, Halburd
Infinite order is an indication of nonintegrability *for discrete systems*

Main tool (for a function f)
Nevanlinna characteristic $T(r; f)$
measures the 'affinity' of f for the value ∞

Order: $\sigma = \limsup_{r\to\infty} \log T(r;f)/\log r$

f is rational, $T(r;f) \propto \log r$ and $\sigma = 0$

f of type $e^{P_n(z)} \to T \propto r^n$ and $\sigma = n$

Fast growth $e^{e^z} \to T \propto e^r$ and $\sigma = \infty$

Conjecture:
If the order σ of x (considered as a continuous function) is infinite
the mapping cannot be integrable

Ablowitz, Herbst, Halburd:

technique to find sufficient conditions for $\sigma = \infty$

Autonomous : precise control of corrective terms in inequalities for T

Three-tiered approach

- First step: consider *only autonomous* mappings
  and apply Nevanlinna techniques
  $\rightarrow$ some constraints, but not restrictive enough
- Second step: apply singularity confinement
  $\rightarrow$ non-confining autonomous mappings are rejected
- Third step: deautonomisation (using again singularity confinement)

Algebraic entropy is failsafe, while sing. conf. is not
But algebraic entropy not always efficacious to "sift" for "good" cases
Sing. conf. treats singularities *one by one*: tractable
If one wants, doublecheck afterwards

The main conclusion

For discrete systems growth conditions integrability

Infinite order is an indication of nonintegrability *for discrete systems* (though it is compatible with "exact solvability")

This is *not* the case for continuous systems

For continuous systems

Painlevé property sufficient for integrabillity
(almost tautological, if we look at Painlevé's definition of integrability)

while, for discrete systems

Singularity confinement is not sufficient

Finite order is also necessary

Singularity confinement is not necessary either

In analogy to continuous systems, expect confinement
only for systems integrable with spectral methods

What about integrability without singularity confinement?

a priori expect mappings integrable through linearisation
do not need singularity confinement

Linearisable mappings

Gambier mapping

$$y_{n+1} = \frac{\alpha y + \beta}{\gamma y + \delta}$$

$$x_{n+1} = \frac{ayx + bx + cy + d}{fyx + gx + hy + k}$$

Not confining in general

Still linear degree growth!

Simplifications are even more massive than in some confined cases (typically, Painlevé have *quadratic* degree growth)

but simplifies at the "wrong" places... no confinement!

Another linearisability case

Linear equation (K a constant)

$$\frac{\alpha x_{n+1} + \beta x_n + \gamma x_{n-1} + \delta}{\epsilon x_{n+1} + \zeta x_n + \eta x_{n-1} + \theta} = K$$

Nonlinear mapping

$$f(x_{n-1}, x_n, x_{n+1}; n) = M$$

f is globally polynomial of degree two in all the x's but not more than linear separately in each of x_{n-1} and x_{n+1}

Require that both lead to same mapping after discrete derivation

Example

$$\left(\frac{x_{n+1}+x_n-a}{z_{n+1}}-\frac{x_n}{\zeta_n}\right)\left(\frac{x_{n-1}+x_n-a}{z_n}-\frac{x_n}{\zeta_n}\right)-\frac{x_n^2}{\zeta_n^2}=M$$

with a a constant, where z and ζ are defined from a single arbitrary function g of n through $z_n = g_{n+1} + g_{n-1}$, $\zeta_n = g_{n+1} + g_n$

It can be solved through the linear equation:

$$\frac{A_n x_{n+1} + B_n(x-a) + A_{n+1}x_{n-1}}{z_n x_{n+1} + (z_{n+1}+z_n)(x_n-a) + z_{n+1}x_{n-1}} = K$$

where $A_n = g_n^2(g_{n+1} + g_{n-1})$
and $B_n = -(g_{n+1}+g_n)g_{n+2}g_{n-1} - (g_{n+2}+g_{n-1})g_{n+1}g_n$

The mapping is generically non-confining unless g is a constant

Ultra-discrete systems and integrability detection

What is the u-d equivalent of the Painlevé property?

Joshi & Lafortune proposed analogue to SC for ud mappings

Nonlinearity mediated by terms involving the max operator

Typically we have $\max(X_n, 0)$
When X_n crosses zero, the derivative becomes discontinuous

This discontinuity plays the role of the singularity

Put $X = \epsilon$
A term $\epsilon_+ = \max(\epsilon, 0)$ propagates with the iterations
unless by some coincidence it disappears

Disappearance equivalent to singularity confinement for ud systems

Complication: must examine all possible singular sectors

Here limit ourselves to just one representative sector

$$w_n^{k+1} - 2w_n^k + w_n^{k-1} = (w_{n+1}^k)_+ - 2(w_n^k)_+ + (w_{n-1}^k)_+$$

Assumption: $w_{n\pm1}^k$ large and negative
$\rightarrow$ their contribution may be neglected (ODE)

When $w < 0$ the $(w_n^k)_+$ term is negligible
Solution as $w^k = k$ for $k < 0$ and discontinuity at $k = 0$
Take $w^0 = \epsilon$ and iterate
$w^1 = 1 + \epsilon - 2(\epsilon)_+$ (this is the discontinuous value)
$w^2 = -\epsilon$
$w^3 = -1 - \epsilon$
In general $w^k = 2 - k - \epsilon$ for $k \geq 2$
The singularity which appeared at $k = 1$ is confined

Nonautonoumous reduction is more interesting

$$X^{k+1} - 2X^k + X^{k-1} = 2(A^k - X^k)_+ - 2(X^k)_+$$

Find proper form of A with singularity analysis

Examine sector $A > 0$ and assume $2A^k - A^{k-1} > 0$

Singularity when X crosses the value of A

Start from $X^0 > 0$ and assume $X^1 = A^1 - \epsilon$
$X^2 = -X^0 + 2(\epsilon)_+$
$X^3 = 2A^2 - A^1 + \epsilon$
$X^4 = X^0 - 2(\epsilon)_+ + 2(A^3 - 2A^2 + A^1 - \epsilon)_+$

By inspection X^4 can be regular provided $A^3 - 2A^2 + A^1 = 0$
The $(\epsilon)_+$ cancel exactly and the singularity is confined
Constraint on A: $A^k = mk + p$
Precisely the form corresponding to the ultradiscrete Painlevé III

Just as in the discrete case we have:

Nonintegrable systems with confined singularities
and
Integrable systems with unconfined singularities

Example of a mapping passing the confinement test and having a positive algebraic entropy

$$x_{n+1} = x_{n-1}\left(x_n + \frac{1}{x_n}\right)$$

Advantage over HV: multiplicative → ultradiscretisation

$$X_{n+1} = X_{n-1} + |X_n|$$

(absolute value of X instead of its equivalent $\max(X,0) + \max(-X,0)$)

Singularity appearing at $n = 1$ where $X_1 = \epsilon$, while X_0 is regular
Two different sectors $X_0 < 0$ and $X_0 > 0$

For $(X_0 < 0)$ we find the sequence

$$\vdots$$

$$X_{-3} = 3X_0$$
$$X_{-2} = 2X_0 - \epsilon$$
$$X_{-1} = X_0 + \epsilon$$
$$X_0$$
$$X_1 = \epsilon$$
$$X_2 = X_0 - \epsilon + 2(\epsilon)_+$$
$$X_3 = -X_0 + 2\epsilon - 2(\epsilon)_+$$
$$X_4 = \epsilon$$
$$X_5 = -X_0 + \epsilon$$

Singularity is confined (to X_2 and X_3 only)

Turning to the case $X_0 > 0$ we find the sequence

$$\begin{aligned}
&\vdots \\
&X_{-4} = -X_0 + 2(\epsilon)_+ + \epsilon \\
&X_{-3} = -X_0 + 2(\epsilon)_+ \\
&X_{-2} = \epsilon \\
&X_{-1} = -X_0 + \epsilon \\
&X_0 \\
&X_1 = \epsilon \\
&X_2 = X_0 + 2(\epsilon)_+ - \epsilon \\
&X_3 = -X_0 + 2(\epsilon)_+ \\
&X_4 = 2X_0 + 4(\epsilon)_+ - \epsilon
\end{aligned}$$

Weakly confined solution

Regular part (around $n = 0$) surrounded by unconfined singularities both for large positive and large negative n's.

Converse situation

Example to be sought among “multiplicative” linearisable mappings

$$\frac{x_{n+1}}{x_{n-1}} = a\frac{x_n + a}{x_n + 1}$$

linearisable through

$$y_{n+1} - ay_n + a(a-1) = 0 \qquad \text{where} \quad y_n = (x_{n-1}+1)(x_n + a)$$

Ultradiscretise to

$$X_{n+1} = X_{n-1} + A + \max(X_n, A) - \max(X_n, 0)$$

where we can always take $A > 0$
Many sectors exit
But enough to exhibit *one* sector with unconfined singularities

Case where X_0 has a large negative value

$X_{-4} = -X_0 - 4A$

$X_{-3} = -4A + \epsilon$

$X_{-2} = X_0 - 2A$

$X_{-1} = -2A + \epsilon$

X_0

$X_1 = \epsilon$

$X_2 = X_0 + 2A - (\epsilon)_+$

$X_3 = 2A + \epsilon$

$X_4 = X_0 + 3A - (\epsilon)_+$

$X_5 = 4A + \epsilon$

$X_6 = X_0 + 4A - (\epsilon)_+$

$X_7 = 6A + \epsilon$

For negative indices the solution is regular

For positive n's the singularity is never confined

Final conclusion

Singularity confinement: necessary for "IST"-type integrability
not necessary for linearisablity
similar to (continuous) Painlevé property

but *not sufficient* while Painlevé property tautologically sufficient

Still should be part of the definition of "discrete Painlevé property"

Integrability detection is a *very* delicate business