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Singularity Analysis for Ultra-discrete Equations and Cellular Automata

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Singularity Analysis: Continuous

- *Painlevé property for ODEs: no movable branching points*

- *Painlevé test: necessary condition for the property*

$$y'' = f(x, y, y') \leftrightarrow y = \sum_{i=-p}^{\infty} a_i (x - x_0)^i$$

Singularity Analysis: Continuous

- *Practicality: good integrability detector.*
- *Has been successfully extended to PDEs*
- *Several formal results that relates singularity analysis to constant of motion, Hamiltonian structure, Lax pair.*

Singularity Confinement

Singularity Confinement

- *Tool to study singularities of difference equations*
- *Heuristic connections with: first integrals, Lax Pairs, integrable continuous limit*
- *Formal connection with first integral* (*S.L., Goriely, JMP, 2004*)
- *Formal connection with Backlund transformation* (*Cresswell, Joshi, LMP, 2002*)

Ref: Ramani, Grammaticos, Hietarinta, Papageorgiou, PRL, 1991

Singularity Confinement

$$x_{n+1}x_{n-1} = k + \frac{1}{x_n}$$

Singularity if $x_n = 0$ $x_n = \varepsilon$

Actually: $x_n = -1/k + \varepsilon$

Singularity Confinement

$$x_{n+1}x_{n-1} = k + \frac{1}{x_n}$$

$$\begin{aligned}
x_{n+1} &= -\frac{k^2}{x_{n-1}}\epsilon + \mathcal{O}(\epsilon^2), \\
x_{n+2} &= \frac{x_{n-1}}{\epsilon\, k} + \mathcal{O}(1), \\
x_{n+3} &= -\frac{x_{n-1}}{\epsilon\, k} + \mathcal{O}(1), \\
x_{n+4} &= \frac{k^2}{x_{n-1}}\epsilon + \mathcal{O}(\epsilon^2), \\
x_{n+5} &= -\frac{1}{k} + \mathcal{O}(\epsilon), \\
x_{n+6} &= y + \mathcal{O}(\epsilon).
\end{aligned}$$

Singularity Confinement

$$x_{n+1}x_{n-1} = \phi_n + \frac{1}{x_n}$$

$$\begin{aligned}
x_{n+1} &= -\frac{\phi_n^2}{x_{n-1}}\,\epsilon + \mathcal{O}(\epsilon^2), \\
x_{n+2} &= \frac{x_{n-1}}{\epsilon\,\phi_n} + \mathcal{O}(1), \\
x_{n+3} &= -\frac{\phi_{n+2}\,x_{n-1}}{\epsilon\,\phi_n^2} + \mathcal{O}(1), \\
x_{n+4} &= \frac{\phi_n\,\phi_{n+3}}{x_{n-1}}\,\epsilon + \mathcal{O}(\epsilon^2), \\
x_{n+5} &= -\frac{\phi_n}{\phi_{n+3}\,\phi_{n+2}} + \mathcal{O}(\epsilon),
\end{aligned}$$

Singularity Confinement

Condition

$$\frac{\phi_n \, \phi_{n+5}}{\phi_{n+3} \, \phi_{n+2}} = 1$$

$$\phi_n = \left(\alpha + \beta \left(-1\right)^n\right) \lambda^n$$

- Painleve I equation

Ultra-discrete limit: KdV

KdV

$$\frac{c_j^{t+1}}{c_j^t} = \frac{1 + \delta c_{j-1}^t}{1 + \delta c_{j+1}^{t+1}}$$

$$c_i^j = e^{d_i^j/\varepsilon}, \qquad \delta = e^{\Delta/\varepsilon}$$

$$\lim_{\varepsilon \to 0^+} \varepsilon \ln\left(e^{A/\varepsilon} + e^{B/\varepsilon}\right) = \max(A, B)$$

Ref: Tokihiro, Takahashi, Matsukidaira, Satsuma, PRL, 1996

Ultra-discrete limit: KdV

KdV

$$\frac{c_j^{t+1}}{c_j^t} = \frac{1 + \delta c_{j-1}^t}{1 + \delta c_{j+1}^{t+1}}$$

Additions become “max” Multiplications become sums

$$d_j^{t+1} - d_j^t = \max(\Delta + d_{j-1}^t, 0) - \max(\Delta + d_{j-1}^{t+1}, 0)$$

Ref: Tokihiro, Takahashi, Matsukidaira, Satsuma, PRL, 1996

Ultra-discrete limit: P eqs

d-P I

$$x_{n+1}\, x_{n-1} = \phi_n + \frac{1}{x_n} \qquad \phi_n = \alpha \lambda^n$$

$$x_{n+1} x_n x_{n-1} = \alpha \lambda^n x_n + 1$$

$$x_n = e^{X_n/\varepsilon}, \qquad \alpha = e^{A/\varepsilon}$$

$$X_{n+1} + X_n + X_{n-1} = \max(A + n + X_n, 0)$$

Ref: Ramani, Takahashi, Grammaticos, Ohta, Phys. D, 1998

Singularity Analysis: Ultra-D.

- *First Example: start with a QRT mapping*

$$x_{n+1}x_{n-1} = \frac{1}{x_n} + k$$

$$I = x_n + x_{n-1} + k\left(\frac{1}{x_n} + \frac{1}{x_{n-1}}\right) + \frac{1}{x_n x_{n-1}}$$

Ref: Quispel, Roberts, Thompson, PLA ‘88

Singularity Analysis: Ultra-D.

- *Ultra-discretize it*

$$x_{n+1}x_{n-1} = \frac{1}{x_n} + k$$

$$x_n = e^{X_n/\varepsilon}, \quad k = e^{K/\varepsilon}$$

$$X_{n+1} + X_n + X_{n-1} = \max(X_n + K, 0)$$

Singularity Analysis: Ultra-D.

- *Ultra-discretize it*

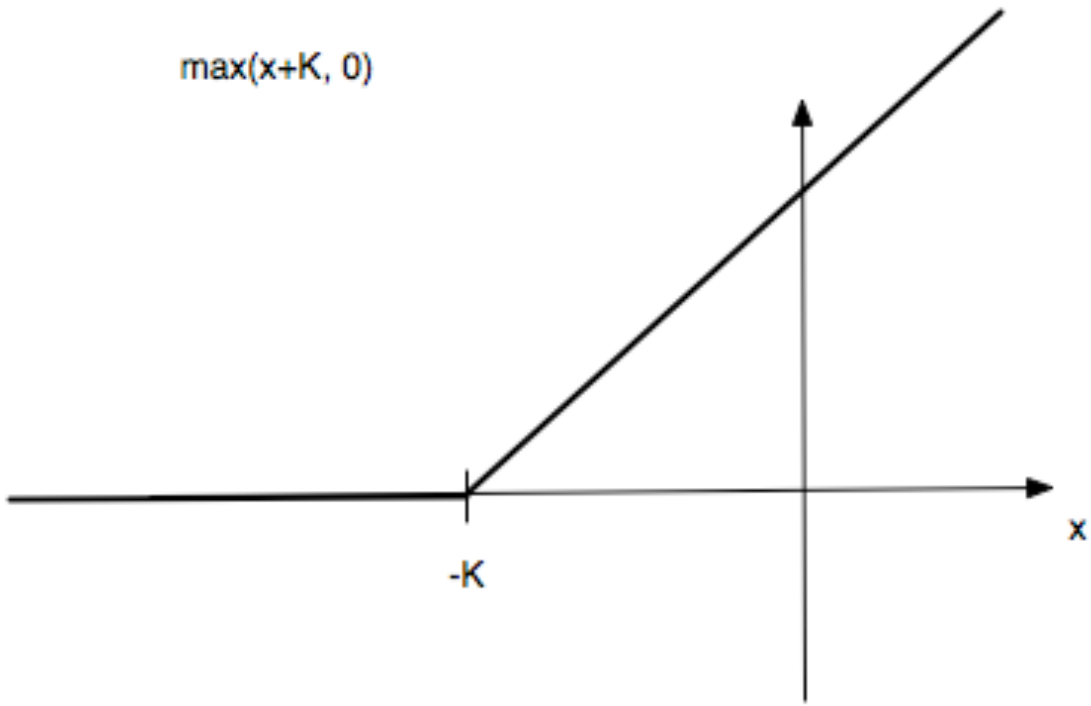
$$I = x_n + x_{n-1} + k\left(\frac{1}{x_n} + \frac{1}{x_{n-1}}\right) + \frac{1}{x_n x_{n-1}}$$

$$x_n = e^{X_n/\varepsilon}, \quad k = e^{K/\varepsilon}$$

$$I = \max(X_n, X_{n-1}, -X_n + K, -X_{n-1} + K, -X_{n-1} - X_n)$$

Singularity Analysis: Ultra-D.

- *Singularity?* $X_{n+1} + X_n + X_{n-1} = \max(X_n + K, 0)$



- *Mapping not differentiable at* $X=-K$*!*

Singularity Analysis: Ultra-D.

- *Confinement?* $X_{n+1} + X_n + X_{n-1} = \max(X_n + K, 0)$

	$\epsilon > 0$	$\epsilon < 0$
X_n	$-K + \epsilon$	$-K + \epsilon$
X_{n+1}	$K - X_{n-1}$	$K - X_{n-1} - \epsilon$
X_{n+2}	$X_{n-1} - \epsilon$	X_{n-1}
X_{n+3}	X_{n-1}	$X_{n-1} + \epsilon$
X_{n+4}	$K - X_{n-1} + \epsilon$	$K - X_{n-1}$
X_{n+5}	$-K - \epsilon$	$-K - \epsilon$
X_{n+6}	X_{n-1}	X_{n-1}

$X_{n-1} > 2|K|.$

Singularity Analysis: Ultra-D.

- *Deautonomization!*

$$X_{n+1} + X_n + X_{n-1} = \max(X_n + \phi_n, 0)$$

	$\epsilon > 0$	$\epsilon < 0$
X_n	$-\phi_n + \epsilon$	$-\phi_n + \epsilon$
X_{n+1}	$\phi_n - X_{n-1}$	$\phi_n - X_{n-1} - \epsilon$
X_{n+2}	$X_{n-1} - \epsilon$	X_{n-1}
X_{n+3}	$X_{n-1} - \phi_n + \phi_{n+2}$	$X_{n-1} - \phi_n + \phi_{n+2} + \epsilon$
X_{n+4}	$\phi_{n+3} - X_{n-1} + \epsilon$	$\phi_{n+3} - X_{n-1}$
X_{n+5}	$-\phi_{n+3} - \phi_{n+2} + \phi_n - \epsilon$	$-\phi_{n+3} - \phi_{n+2} + \phi_n - \epsilon$

Singularity Analysis: Ultra-D.

- *Confinement condition*

$$\phi_{n+5} - \phi_{n+3} - \phi_{n+2} + \phi_n = 0$$

$$\phi_n = \alpha + \beta n + \gamma(-1)^n$$

- *Ultra-discrete PI equation and its asymmetric version*

Singularity Analysis: Ultra-D.

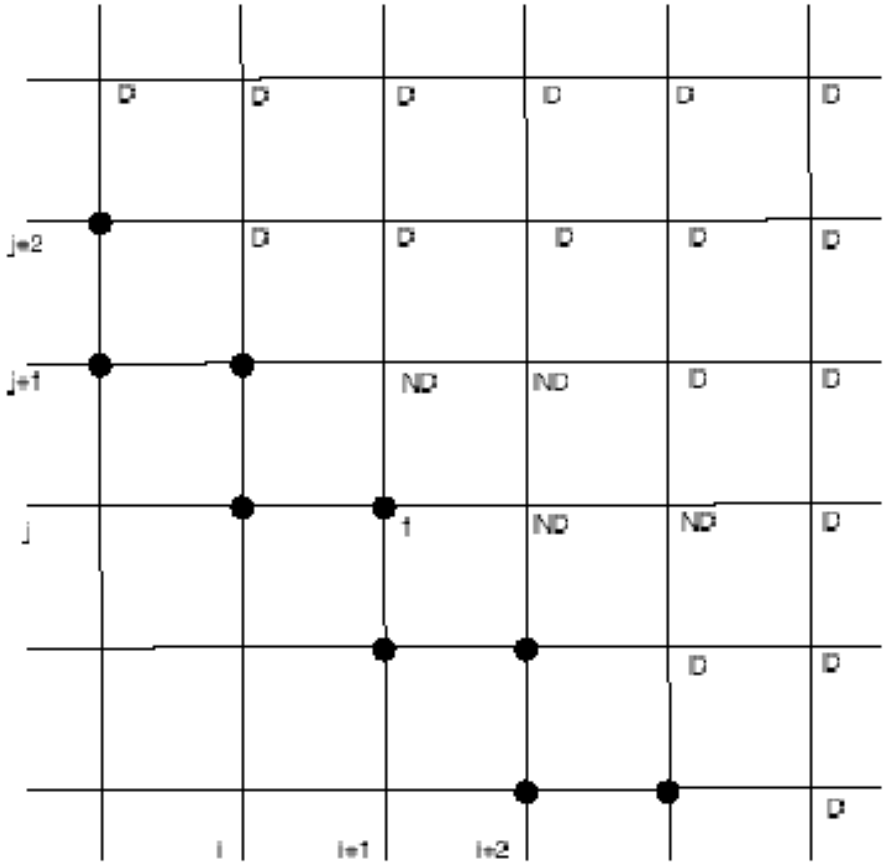
- *Partial difference case*

$$u_{j+1}^{i+1} = u_j^i + \max\left(u_j^{i+1} - 1, 0\right) - \sigma \max\left(u_{j+1}^i - 1, 0\right)$$

$\sigma = 1$ corresponds to KdV

Singularity Analysis: Ultra-D.

- *Singularity Pattern*



Confinement for σ=1

Singularity Analysis: Ultra-D.

- *Complete classification of Painleve equations*
- *1+1-D KdV and sine-Gordon*
- *Cellular automata associated with KdV*

Bilinearisation P equations and singularity analysis

- Continuous case: Hietarinta and Kruskal, ‘92

- Discrete case: Ramani, Grammaticos, and Satsuma, ‘95

Bilinearisation of d-PI

$$x_{n+1}x_{n-1} = \alpha\lambda^n + \frac{1}{x_n}$$

$$\{0, \infty, \infty, 0\}$$

Introduce an entire function in such a way that the singularity Pattern is reproduced

Ref:Ramani, Grammaticos, Satsuma, JPA ’95

Bilinearisation of d-PI

$$x_{n+1}x_{n-1} = \alpha\lambda^n + \frac{1}{x_n}$$

$$\{0,\infty,\infty,0\}$$

Introduce an entire function in such a way that the singularity Pattern is reproduced

$$\tau - \text{function } \tau_{\mathrm{n}}$$

It is entire but goes through zeroes

Ref:Ramani, Grammaticos, Satsuma, JPA ’95

Bilinearisation of d-PI

$$x_{n+1}x_{n-1} = \alpha\lambda^n + \frac{1}{x_n}$$

$$\{0, \infty, \infty, 0\}$$

$$x_n = \tau_n D$$

Ref:Ramani, Grammaticos, Satsuma, JPA ’95

Bilinearisation of d-PI

$$x_{n+1}x_{n-1} = \alpha\lambda^n + \frac{1}{x_n}$$

$$\{0, \infty, \infty, 0\}$$

$$x_n = \frac{\tau_n}{\tau_{n-1}} D_2$$

Ref:Ramani, Grammaticos, Satsuma, JPA ’95

Bilinearisation of d-PI

$$x_{n+1}x_{n-1} = \alpha\lambda^n + \frac{1}{x_n}$$

$$\{0, \infty, \infty, 0\}$$

$$x_n = \frac{\tau_n}{\tau_{n-1}\tau_{n-2}} D_3$$

Ref:Ramani, Grammaticos, Satsuma, JPA ’95

Bilinearisation of d-PI

$$x_{n+1}x_{n-1} = \alpha\lambda^n + \frac{1}{x_n}$$

$$\{0, \infty, \infty, 0\}$$

$$x_n = \frac{\tau_n \tau_{n-3}}{\tau_{n-1}\tau_{n-2}}$$

Ref:Ramani, Grammaticos, Satsuma, JPA ’95

Bilinearisation of d-PI

$$x_{n+1}x_{n-1} = \alpha\lambda^n + \frac{1}{x_n}$$

$$\{0,\infty,\infty,0\}$$

$$x_n = \frac{\tau_n \tau_{n-3}}{\tau_{n-1}\tau_{n-2}}$$

$$\tau_{n+1}\tau_{n-4} = \alpha\lambda^n \tau_n \tau_{n-3} + \tau_{n-1}\tau_{n-2}$$

Ref:Ramani, Grammaticos, Satsuma, JPA ’95

Bilinearisation of U-PI

$$X_{n+1} + X_n + X_{n-1} = \max(X_n + \phi_n, 0)$$

$$\Phi_n = \alpha n + \beta$$

	$\epsilon > 0$	$\epsilon < 0$
X_n	$-\phi_n + \epsilon$	$-\phi_n + \epsilon$
X_{n+1}	$\phi_n - X_{n-1}$	$\phi_n - X_{n-1} - \epsilon$
X_{n+2}	$X_{n-1} - \epsilon$	X_{n-1}
X_{n+3}	$X_{n-1} - \phi_n + \phi_{n+2}$	$X_{n-1} - \phi_n + \phi_{n+2} + \epsilon$
X_{n+4}	$\phi_{n+3} - X_{n-1} + \epsilon$	$\phi_{n+3} - X_{n-1}$
X_{n+5}	$-\phi_{n+3} - \phi_{n+2} + \phi_n - \epsilon$	$-\phi_{n+3} - \phi_{n+2} + \phi_n - \epsilon$

Bilinearisation of U-PI

$$X_{n+1} + X_n + X_{n-1} = \max(X_n + \phi_n, 0)$$

$$\Phi_n = \alpha n + \beta$$

$$X_n = \left(\mathrm{Y}_n\right)_+ + D$$

Bilinearisation of U-PI

$$X_{n+1} + X_n + X_{n-1} = \max(X_n + \phi_n, 0)$$

$$\Phi_n = \alpha n + \beta$$

$$X_n = \left(\mathrm{Y}_n\right)_+ - \left(\mathrm{Y}_{n-1}\right)_+ + D_2$$

Bilinearisation of U-PI

$$X_{n+1} + X_n + X_{n-1} = \max(X_n + \phi_n, 0)$$

$$\Phi_n = \alpha n + \beta$$

$$X_n = \left(\mathrm{Y}_n\right)_+ - \left(\mathrm{Y}_{n-1}\right)_+ - \left(\mathrm{Y}_{n-2}\right)_+ + D_3$$

Bilinearisation of U-PI

$$X_{n+1} + X_n + X_{n-1} = \max(X_n + \phi_n, 0)$$

$$\Phi_n = \alpha n + \beta$$

$$X_n = \left(\mathrm{Y}_n\right)_+ - \left(\mathrm{Y}_{n-1}\right)_+ - \left(\mathrm{Y}_{n-2}\right)_+ + \left(\mathrm{Y}_{n-3}\right)_+$$

Bilinearisation of U-PI

$$X_{n+1} + X_n + X_{n-1} = \max(X_n + \phi_n, 0)$$

$$\Phi_n = \alpha n + \beta$$

$$X_n = (\mathrm{Y}_n)_+ - (\mathrm{Y}_{n-1})_+ - (\mathrm{Y}_{n-2})_+ + (\mathrm{Y}_{n-3})_+$$

$$(\mathrm{Y}_{n-4})_+ + (\mathrm{Y}_{n+1})_+ = \max\left(\Phi_n + (\mathrm{Y}_n)_+ + (\mathrm{Y}_{n-3})_+,\ (\mathrm{Y}_{n-1})_+ + (\mathrm{Y}_{n-2})_+\right)$$

Conclusions

- *Integrability detector for U-D equations based on singularity analysis*
- *What about Cellular Automata?*
- *Classification of Painleve equations*
- *Bilinearisation of Painleve equations*