

# A rep theoretic construction of Baxter's Q operator and solutions to the discrete Liouville equation

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5th July 2008

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# Quantum Yang-Baxter equation and algebras

Quantum integrable lattice models are constructed from solns of the quantum YBE,

$$R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12} \in \text{End}(V_1 \otimes V_2 \otimes V_3)$$

Solns are obtained from the rep theory of a “quantum algebra”  $\mathcal{A}$  (i.e. Yangian,  $q$ -deformed env algebra, elliptic algebra).

## Definition

Let  $V, W$  be  $\mathcal{A}$ -modules and  $R_{VW}$  the intertwiner of the tensor product

$$R_{VW} : V \otimes W \rightarrow V \otimes W.$$

Then the corresponding transfer matrix is defined as

$$T_V = \text{Tr}_V R_{VW} \in \text{End } W,$$

where  $V$  is called “auxiliary space” and  $W$  is the “quantum” (physical state) space.

# Quantum Integrability and Hirota-Miwa equation

“Quantum Integrability” now follows from construction:

## Theorem

*Let  $V, V'$  be two  $\mathcal{A}$ -modules. If  $R_{VV'}$  exists then it solves the QYBE and one has  $[T_V, T_{V'}] = 0$ .*

The set of all transfer matrices constitutes the fusion hierarchy.

## $s/k$ fusion relation

Let  $V \rightarrow V_s^a(u)$  be labelled by rectangular Young diagram:

$$T_s^a(u+1)T_s^a(u-1) - T_{s+1}^a(u)T_{s-1}^a(u) = T_s^{a+1}(u)T_s^{a-1}(u),$$

3D integrable system: Hirota-Miwa (discrete KP) equation!

c.f. [Kuniba et al 1994], [Krichever et al 1997]

# $k = 2$ : Truncation of the Hirota-Miwa equation

Specialize to  $sl_2$ . Consider evaluation modules  $V \rightarrow V_s(u)$  of  $\mathcal{A} = Y(sl_2)$ ,  $U_q(\widehat{sl_2})$  with  $u \in \mathbb{C}$  spectral parameter such that  $V_s = V_s(u=0)$  is an  $sl_2$  (resp.  $U_q(sl_2)$ ) module of “spin”  $s/2$ .

H-M equation  $\rightarrow$  discrete Liouville equation

Boundary conditions:  $T_s^a(u) = 0$  for  $a < 0$  and  $a > 2$

Remaining non-trivial relation:

$$T_s^{a=1}(u+1)T_s^{a=1}(u-1) - T_{s+1}^{a=1}(u)T_{s-1}^{a=1}(u) = T_s^0(u)T_s^2(u).$$

Identify

$$T_s^{a=1}(u) = T_{V_s(u)}, \quad T_s^{a=0}(u) = \varphi(u-s), \quad T_s^{a=2}(u) = \varphi(u+2+s)$$

and

$$\varphi(u) = T_{s=0}^{a=1}(u-1)$$

is a scalar function associated with the trivial representation  $s = 0$  depending on the quantum/physical space.

# Bäcklund transformation: auxiliary linear problems

[Krichever et al 1997]: Given a solution  $T_s^a(u)$  consider the linear set of equations determining  $Q_s^a(u)$ ,

$$\begin{aligned}T_{s+1}^{a+1}(u)Q_s^a(u) - T_s^{a+1}(u+1)Q_{s+1}^a(u-1) &= T_s^a(u)Q_{s+1}^{a+1}(u), \\T_{s+1}^a(u+1)Q_s^a(u) - T_s^a(u)Q_{s+1}^a(u+1) &= T_s^{a+1}(u+1)Q_{s+1}^{a-1}(u)\end{aligned}$$

Auto-Bäcklund transformation:  $T_s^a(u) \rightarrow Q_s^a(u)$

New solution satisfies different b.c.:

$$Q_s^a(u) = 0 \text{ for } a < 0 \text{ and } a > 1.$$

For  $k = 2$  these b.c. lead to Baxter's TQ-equation. If we assume  $Q_s^{a=0,1}(u)$  to be analytic in the spectral parameter, then

$$\rightsquigarrow Q_s^0(u) = Q(u-s) \text{ and } Q_s^1(u) = \bar{Q}(u+s)$$

Dependence on “light cone coordinates” implies the following invariant of the  $s$ -dynamics:

$$A(u) = \frac{\varphi(u+2)T_{s+1}(u+s) + \varphi(u)T_{s-1}(u+2+s)}{T_s(u+1+s)}$$

which satisfies the equations

$$Q(u)A(u) = \varphi(u+2)Q(u-2) + \varphi(u)Q(u+2)$$

Exploiting the initial conditions  $T_{-1}(u) = 0$  and  $T_0(u) = \varphi(u+1)$  the quantity  $A(u)$  specializes at  $s = 0$  to  $A(u) = T_1(u)$  implying

- alternative version of the fusion relation

$$T_1(u)T_s(u+1+s) = \varphi(u+2)T_{s+1}(u+s) + \varphi(u)T_{s-1}(u+2+s)$$

- Baxter's TQ-equation [1972]  $\Rightarrow$  Bethe ansatz eqns [1933]

$$Q(u)T_1(u) = \varphi(u+2)Q(u-2) + \varphi(u)Q(u+2) .$$

# The XXZ Heisenberg quantum spin-chain

Prototype of a quantum integrable model:

$$H_{\text{XXZ}} = \frac{1}{2} \sum_{m=1}^M \left\{ \sigma_m^x \sigma_{m+1}^x + \sigma_m^y \sigma_{m+1}^y + \frac{q + q^{-1}}{2} (\sigma_m^z \sigma_{m+1}^z - 1) \right\}$$

quasi-periodic b.c. :  $\sigma_{M+1}^{\pm} \equiv q^{\pm 2\alpha} \sigma_1^{\pm}$ ,  $\sigma_{M+1}^z \equiv \sigma_1^z$

Quantum space:  $W = \bigotimes_{i=1}^M V_1(z_i)$

In the homogeneous case  $z_i = 1$  we have

$$H_{\text{XXZ}} \propto z \frac{d}{dz} \ln T_1(z), \quad T_1(z) = \text{Tr}_{V_1(z)} q^{\alpha h \otimes 1} R_{V_1(z), W}$$

Note: All transfer matrices commute with Hamiltonian.



# Quantum group reminder

Quantum algebra  $\mathcal{A} = U_q(\widehat{sl}_2)$  and evaluation modules  $V_s(z = q^u)$ . For  $z = 0$  these reduce to the  $U_q(sl_2)$ -modules  $V_s$ .

Algebra relations:

$$q^h e = e q^{h+2}, \quad q^h f = f q^{h-2}, \quad [e, f] = [h]_q$$

Modules  $V_s$ :

$$\begin{aligned} (e, v_k) &\mapsto [s - k + 1]_q [k]_q v_{k-1}, & (e, v_0) &\mapsto 0, \\ (f, v_k) &\mapsto v_{k+1}, & (q^h, v_k) &\mapsto q^{s-2k} v_k, & k = 0, 1, 2, 3, \dots \end{aligned}$$

If  $s \in \mathbb{N}_{\geq 0}$  then  $V_s \cong \mathbb{C}^{s+1}$  and the module truncates

$$(f, v_s) \mapsto 0$$

otherwise the module is infinite-dimensional.

# Fusion hierarchy and relation

Consider first  $s \in \mathbb{N}_{\geq 0}$ .

Theorem (Chari, Pressley 1990)

*Decomposition of tensor module:*

$$0 \rightarrow V_{s-1}(q^{u+s+2}) \hookrightarrow V_s(q^{u+s+1}) \otimes V_1(q^u) \rightarrow V_{s+1}(q^{u+s}) \rightarrow 0$$

Setting

$$T_s(u) \rightarrow T_s(z = q^u) = \text{Tr}_{V_s(z)} q^{\alpha h \otimes 1} R_{V_s(z), W}, \quad s \in \mathbb{N}_{\geq 0}$$

we obtain the fusion relation

$$T_s(q^{u+s+1}) T_1(q^u) = T_0(q^{u+1}) T_{s+1}(q^{u+s}) + T_0(q^{u-1}) T_{s-1}(q^{u+s+2})$$

with "quantum determinant" (spin-1/2 chain)

$$T_0(q^{u-1}) = \varphi(u) = \prod_{m=1}^M (1 - q^u z_m)$$

# The Q-operator

Let  $s$  be generic.

The corresponding “transfer matrix” is NOT the analytic continuation w.r.t. the continuum limit of the discrete Liouville equation. Instead “generating matrix” for 2 lin indep solutions to Baxter’s TQ equation.

## Definition

$$Q(z; s) = \operatorname{Tr}_{V_s(z)} q^{\alpha h \otimes 1} R_{V_s(z), W}, \quad s \in \mathbb{C}/\mathbb{N}_{\geq 0}$$

Note:  $V_s(z)$  is now infinite-dimensional. Trace is defined through analytic continuation in  $\alpha$ . For instance,

$$Q(0; s) = \operatorname{Tr}_{V_s(z)} q^{\alpha h \otimes 1 - h \otimes S^z} := \frac{(-1)^M q^{\alpha - S^z}}{1 - q^{2(S^z - \alpha)}}, \quad S^z = \sum_m \frac{\sigma_m^z}{2}$$

## Theorem (Bazhanov et al 1999, CK 2008)

We have the decomposition  $Q(z; s) = Q(0)Q^+(z)Q^-(zq^{s+1})$  with

$$Q^+(z) = \lim_{q^{s+1} \rightarrow 0} Q(z; s)/Q(0), \quad Q^-(z) = \lim_{q^{s+1} \rightarrow \infty} Q(z; s)/Q(0)$$

Proof [CK 2008].

$$0 \rightarrow V_{s+t}(q^{u-2s}) \hookrightarrow V_s(q^{u-2s}) \otimes V_t(q^u) \rightarrow V_{s+t}(q^{u+s}) \otimes V_{-1}(q^{u+2}) \rightarrow 0$$

## Solutions to the TQ equation:

$$T_1(q^u)Q^\pm(q^u) = T_0(q^{u+1})Q^\pm(q^{u-2}) + T_0(q^{u+1})Q^\pm(q^{u+2})$$

with eigenvalues

$$Q^\pm(q^u) = q^{\mp \frac{\alpha - S^z}{2} u} \prod_{i=1}^{n_\pm} (1 - q^u/x_i^\pm), \quad n_\pm = \frac{M}{2} \mp S^z.$$

# The Quantum Wronskian

The TQ-equation is a 2nd order  $\Delta$  equation. The two lin indep solutions  $Q^\pm$  satisfy non-trivial Wronskian type relation ( $s = 1$ ):

$$\left( q^{S^z - \alpha} - q^{\alpha - S^z} \right) T_{s-1}(u) = \begin{vmatrix} Q^+(q^{u+s}) & Q^-(q^{u+s}) \\ Q^+(q^{u-s}) & Q^-(q^{u-s}) \end{vmatrix}$$

## Conjecture

*Provided  $\alpha \neq 0$  is generic, there exist precisely  $\dim W$  solutions.  
 $\Rightarrow$  Bethe ansatz is complete.*

*For  $\alpha = 0$  and  $M$  odd, the above continues to hold true.*

*For  $\alpha = 0$  and  $M$  even, there do not exist solutions with the required analyticity requirements.*

Decomposition at periodic b.c. [CK 2005]:

$$M \in 2\mathbb{N} : \quad \lim_{\alpha \rightarrow 0} T_{s-1}(u) = f(q^u, q^s) + s g(q^u, q^s).$$

## Summary

- One can realize complete (= dim  $W$ ) set of solutions to the discrete Liouville equation (subject to analyticity) as spectrum of explicitly constructed  $Q$ -operator.
- Representation theory yields functional relations.

Note: If  $q$  is root of 1 and  $\alpha \in \mathbb{Z}$ , then number of solns  $\leq \dim W$ .

$\tilde{sl}_2$ -symmetry [Deguchi et al '00][CK, McCoy '01][CK '04]. The number is obtained by counting paths on restricted Bratelli diagrams (combinatorial problem).

Degeneracies also occur for XXX:  $sl_2$ -symmetry.

Modified quantum Wronskian [Pronko, Stroganov 1998]

$$u^M = Q^+(u-1)Q^-(u) - Q^+(u)Q^-(u-1) \quad \text{with}$$

$$Q^+(u) = \prod_{i=1}^{M/2-S^z} (u-v_i^+) \quad \text{and} \quad Q^-(u) = \frac{1}{2S^z+1} \prod_{i=1}^{M/2+S^z+1} (u-v_i^-) !$$

# The continuum limit

Discrete Liouville equation can be recast into

$$Y_s(u+1)Y_s(u-1) = (1 + Y_{s+1}(u))(1 + Y_{s-1}(u)) .$$

with

$$Y_s(u) = \frac{T_{s+1}(u)T_{s-1}(u)}{\varphi(u-s)\varphi(u+2+s)}$$

The solution  $\phi$  to the continuous Liouville equation

$$\phi_{tt} - \phi_{xx} = 2e^\phi$$

is then obtained by making the identification

$$e^{-\phi(x,t)} = \lim_{\delta \rightarrow 0} \delta^2 Y_{t/\delta}(x/\delta) .$$

# Complex dimension and the trace functional

Consider scaling limit:  $U_q(\hat{sl}_2) \rightarrow Y(sl_2)$ . The corresponding model is the isotropic quantum Heisenberg spin-chain (XXX).

XXX: rational solutions [CK 2005]

$$\varphi(u) = \prod_{m=1}^M (u - u_m), \quad Q^\pm(u) = \omega^{\mp u/2} \prod_{i=1}^{n_\pm} (u - v_i^\pm).$$

[Boos, Jimbo, Miwa, Smirnov, Takeyama'02] Define trace functional

$$\mathrm{Tr}_x : U(sl_2) \otimes \mathbb{C}[x] \rightarrow \mathbb{C}[x]$$

such that

- for  $x \in \mathbb{N}$ : ordinary trace,  $\mathrm{Tr}_x(a) = \mathrm{Tr}_{V_{x-1}} a$ ,  $\forall a \in U(sl_2)$
- action on the Cartan element  $h \in sl_2$ ,

$$\mathrm{Tr}_x e^{zh} = \frac{\sinh(zx)}{\sinh(x)} = x + \frac{x(x^2-1)}{6} z^2 + \frac{x(7-10x^2+3x^4)}{360} z^4 + \dots$$



## Definition

Define for complex  $x$  the transfer matrix (periodic b.c.)

$$T(u, x) := \text{Tr}_x R(u)_W, \quad R_W(u) \in U(\mathfrak{sl}_2) \otimes \text{End}W.$$

Note: for  $x = s + 1 \in \mathbb{N}$ ,  $T(u, s + 1) = T_s(u) = \text{Tr}_{V_s} R_{V_s(u), W}$ .

## Theorem (CK 2005)

$$T(u, x) = \lim_{\omega \rightarrow 1} \frac{\omega Q^+(u-x)Q^-(u+x) - \omega^{-1}Q^+(u+x)Q^-(u-x)}{\omega - \omega^{-1}}$$

Setting

$$Y_s(u) = \frac{T(u, s+2)T(u, s)}{T(u-s-1, 1)T(u+s+1, 1)}$$

yields single soliton solution in the continuum limit.

## Example

Homogeneous spin-1/2 XXX chain. Choose  $M = 4$  then one eigenvalue of  $T(u, s + 1)$  gives rise to the solution

$$\phi(x, t) = - \lim_{\delta \rightarrow 0} \delta^2 Y_{t/\delta}(x/\delta) = - \log \frac{(t^5 + 10t^3x^2 + 5tx^4)^2}{25(t^2 - x^2)^4}$$

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- $n$  soliton solutions? 2D Toda lattice?
- Construction for  $k > 2$ ? H-M eqn has solutions as Casoratian determinants [Ohta et al 1993][Nimmo 1997]
- Classification of solns to H-M eqn in terms of (nested) Bethe ansatz and vice versa? String hypothesis and thermodynamic Bethe ansatz?
- What about other, "continuum" quantum integrable models, e.g. QNLS, Liouville CFT?
- Ultra-discrete and crystal limit?
- Elliptic case, Baxter's 8-vertex and Belavin's model? Bethe roots = discrete integrable model? [Krichever et al 1997]