Conservation laws for difference equations

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Question: Why not use equivalent expressions? This one is useful:

$$\prod_i S_i A^i(\mathbf{n}, [\mathbf{u}]) = \prod_i A^i(\mathbf{n}, [\mathbf{u}]).$$

Answer: The Euler operator **E** annihilates all divergences; indeed, the variational complex is exact.

$$\dots \Lambda^{n-2,0} \xrightarrow{\mathrm{d_H}} \Lambda^{n-1,0} \xrightarrow{\mathrm{d_H}} \Lambda^{n,0} \xrightarrow{\mathbf{E}} \Lambda_*^{n,1} \dots$$

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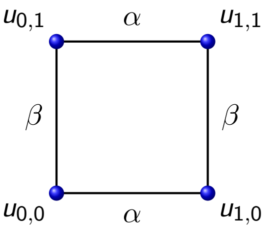
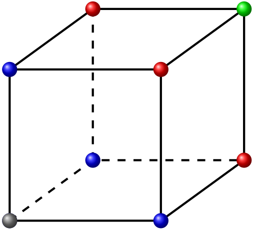
Difference operators also produce a variational complex; now $\ker(\mathbf{E})$ is the set of all standard-form expressions.

Quad-graph equations





Independent variables: $k, l \in \mathbb{Z}$; dependent variable $u \in \mathbb{R}$ (or $\mathbb{C}$).
Quad-graph equations depend on $u_{i,j} = u(k+i, l+j)$, $i, j \in \{0, 1\}$.

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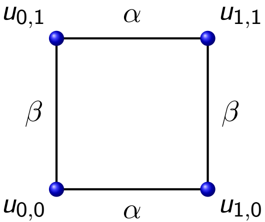

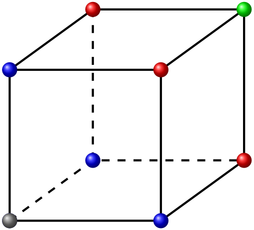

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Consistency-on-a-cube implies integrability (ABS classification and others); $\alpha = \alpha(k),\ \beta = \beta(l)$.

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ABS classification is up to Möbius transformations of u and point transformations of parameters.

Q1 : $$\alpha(u_{0,0}-u_{0,1})(u_{1,0}-u_{1,1})-\beta(u_{0,0}-u_{1,0})(u_{0,1}-u_{1,1})+\delta^2\alpha\beta(\alpha-\beta)=0,$$

Q2 : $$\alpha(u_{0,0}-u_{0,1})(u_{1,0}-u_{1,1})-\beta(u_{0,0}-u_{1,0})(u_{0,1}-u_{1,1})+\alpha\beta(\alpha-\beta)(u_{0,0}+u_{1,0}+u_{0,1}+u_{1,1})$$
$$-\alpha\beta(\alpha-\beta)(\alpha^2-\alpha\beta+\beta^2)=0,$$

Q3 : $$(\beta^2-\alpha^2)(u_{0,0}u_{1,1}+u_{1,0}u_{0,1})+\beta(\alpha^2-1)(u_{0,0}u_{1,0}+u_{0,1}u_{1,1})-\alpha(\beta^2-1)(u_{0,0}u_{0,1}+u_{1,0}u_{1,1})$$
$$-\delta^2(\alpha^2-\beta^2)(\alpha^2-1)(\beta^2-1)/(4\alpha\beta)=0,$$

Q4 : $$\mathrm{sn}(\alpha)(u_{0,0}u_{1,0}+u_{0,1}u_{1,1})-\mathrm{sn}(\beta)(u_{0,0}u_{0,1}+u_{1,0}u_{1,1})-\mathrm{sn}(\alpha-\beta)(u_{0,0}u_{1,1}+u_{1,0}u_{0,1})$$
$$+\mathrm{sn}(\alpha-\beta)\mathrm{sn}(\alpha)\mathrm{sn}(\beta)(1+K^2u_{0,0}u_{1,0}u_{0,1}u_{1,1})=0,$$

H1 : $$(u_{0,0}-u_{1,1})(u_{1,0}-u_{0,1})+\beta-\alpha=0,$$

H2 : $$(u_{0,0}-u_{1,1})(u_{1,0}-u_{0,1})+(\beta-\alpha)(u_{0,0}+u_{1,0}+u_{0,1}+u_{1,1})+\beta^2-\alpha^2=0,$$

H3 : $$\alpha(u_{0,0}u_{1,0}+u_{0,1}u_{1,1})-\beta(u_{0,0}u_{0,1}+u_{1,0}u_{1,1})+\delta^2(\alpha^2-\beta^2)=0,$$

A1 : $$\alpha(u_{0,0}+u_{0,1})(u_{1,0}+u_{1,1})-\beta(u_{0,0}+u_{1,0})(u_{0,1}+u_{1,1})-\delta^2\alpha\beta(\alpha-\beta)=0,$$

A2 : $$(\beta^2-\alpha^2)(u_{0,0}u_{1,0}u_{0,1}u_{1,1}+1)+\beta(\alpha^2-1)(u_{0,0}u_{0,1}+u_{1,0}u_{1,1})-\alpha(\beta^2-1)(u_{0,0}u_{1,0}+u_{0,1}u_{1,1})=0.$$

Three-point conservation laws

Definition: A *conservation law* (CLaw) of a quad-graph equation is an expression

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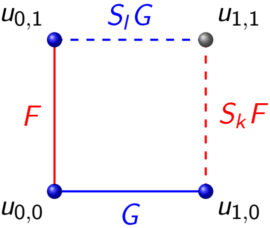
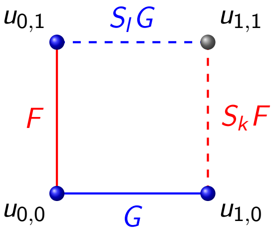
Too hard — start small.



The simplest nontrivial CLaws have

$$F = F(k, l, u_{0,0}, u_{0,1}), \qquad G = G(k, l, u_{0,0}, u_{1,0}).$$



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The determining equation for $u_{1,1} = \omega$ is

$$F(k+1, l, u_{1,0}, \omega) - F(k, l, u_{0,0}, u_{0,1}) + G(k, l+1, u_{0,1}, \omega) - G(k, l, u_{0,0}, u_{1,0}) = 0.$$

Solution Method

Apply the commuting operators

$$L_1 = \frac{\partial}{\partial u_{1,0}} - \frac{\omega_{1,0}}{\omega_{0,0}} \frac{\partial}{\partial u_{0,0}}\,, \qquad L_2 = \frac{\partial}{\partial u_{0,1}} - \frac{\omega_{0,1}}{\omega_{0,0}} \frac{\partial}{\partial u_{0,0}}\,,$$

to get

$$-L_1 L_2 \big\{ F(k,l,u_{0,0},u_{0,1}) + G(k,l,u_{0,0},u_{1,0}) \big\} = 0.$$

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Elimination can be made faster for some quad-graph equations.

Example: For the dpKdV equation (**H1**)

$$u_{1,1} = u_{0,0} + \frac{\beta - \alpha}{u_{1,0} - u_{0,1}} \, ,$$

the set of three-point CLaws (up to equivalence) is spanned by

$$F_1 = -(-1)^{k+l} \left(2u_{0,0}u_{0,1} - \beta\right) , \quad G_1 = (-1)^{k+l} \left(2u_{0,0}u_{1,0} - \alpha\right) ,$$

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Other ABS equations:

4 CLaws :	**Q1**$_{\delta=0}$(cross-ratio), **A1**$_{\delta=0}$, **H3**$_{\delta=0}$(dpmKdV);
1 CLaw :	**Q2**, **Q3**$_{\delta=1}$, **Q4**$_{K^2\neq 1}$;
2 CLaws :	all other equations.

The same process can be used to find higher conservation laws for

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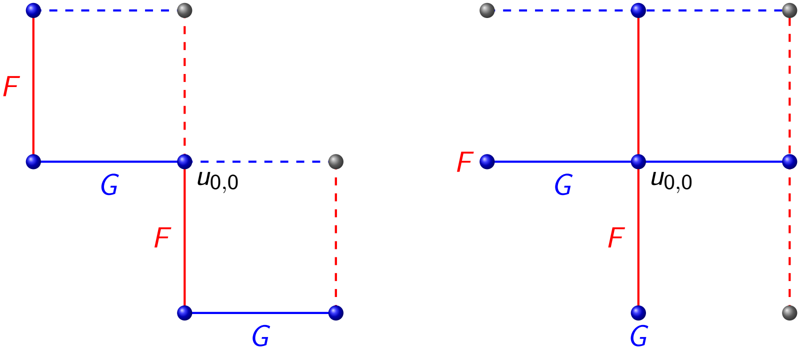
Potential problems: Expression swell, difficulty solving ΔEs.

Five-point conservation laws



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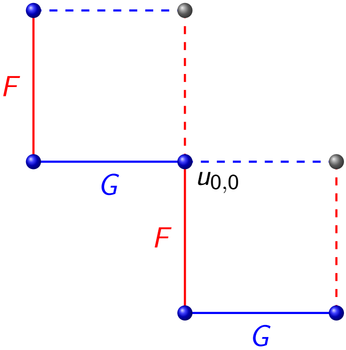

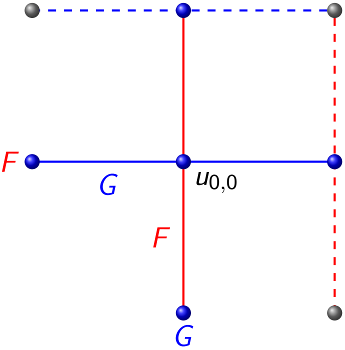

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For ABS, the cross is more efficient.

Example: The dpKdV equation has the following five-point CLaws (modulo three-point and trivial CLaws):

$$F_1 = -\ln\left(u_{0,1} - u_{-1,0}\right), \quad G_1 = \ln\left(u_{1,0} - u_{-1,0}\right),$$

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For constant α, β:

every ABS equation has something similar (at least).

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to

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to get (up to equivalence) the new CLaw

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Warning: apparent novelty must be checked; use characteristics.

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Drawback: The resulting CLaws found so far are quite messy!

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- infinite hierarchies of symmetries;
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However, there are no five-point symmetries or CLaws if neither α nor β are constant.

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- Does each of the ABS equations have infinitely many CLaws (up to equivalence)? If so, is this a property shared by all integrable quad graphs?
- Is there a one-to-one correspondence between variational symmetries and CLaws (up to equivalence)?

The End