Soliton solutions to quadrilateral lattice equations

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in collaboration with
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The setting

We consider lattice maps defined on an elementary square:

$$x_{n,m} = x_{00} = x$$

$$x_{n+1,m} = x_{10} = x_{[1]} = \widetilde{x}$$

$$x_{n,m+1} = x_{01} = x_{[2]} = \widehat{x}$$

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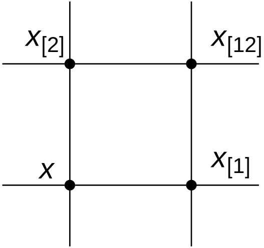

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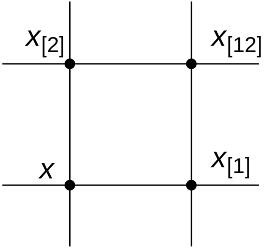


The four corner values are related by a multi-linear equation:

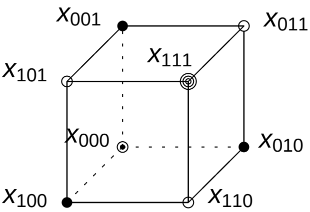
$$Q(x, x_{[1]}, x_{[2]}, x_{[12]}; p, q) = 0$$

This allows propagation from initial data given on a staircase or on a corner.

CAC - Consistency Around a Cube

One definition of integrability for such lattices is that they should allow consistent extensions from 2D to 3D (and higher)

Adjoin a third direction $x_{n,m} \to x_{n,m,k}$ and construct a cube.



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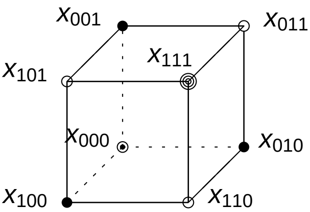
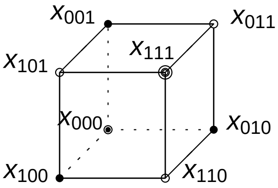


Map at the bottom $Q_{12}(x, \tilde{x}, \hat{x}, \hat{\tilde{x}}; p, q) = 0$,
on the sides $Q_{23}(x, \hat{x}, \bar{x}, \hat{\bar{x}}; q, r) = 0$, $Q_{31}(x, \bar{x}, \tilde{x}, \bar{\tilde{x}}; r, p) = 0$,
shifted maps on parallel shifted planes.



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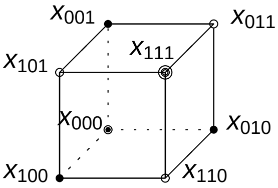
Given values at black disks, we can compute values at open disks uniquely.

But x_{111} can be computed in 3 different ways! They must agree!



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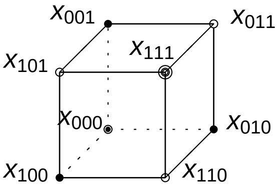

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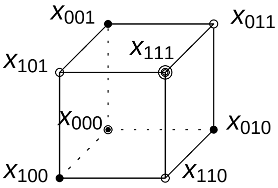

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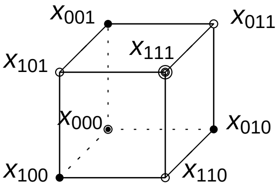

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- It allows construction of Lax presentation [Nijhoff and Walker, Glasgow Math. J. **43A** (2001) 109].
- It can be used as a method to classify integrable equation. [Adler, Bobenko and Suris, Commun. Math. Phys. **233** 513 (2003)].

ABS classification

Assume CAC with 2 additional assumptions:

- symmetry ($\varepsilon, \sigma = \pm 1$):

$$
\begin{aligned}
Q(x_{000}, x_{100}, x_{010}, x_{110}; p, q) &= \varepsilon Q(x_{000}, x_{010}, x_{100}, x_{110}; q, p) \\
&= \sigma Q(x_{100}, x_{000}, x_{110}, x_{010}; p, q)
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Result: complete classification under these assumptions, 9 models.

ABS results:

List H:

(H1) $$(x-\hat{\tilde{x}})(\tilde{x}-\hat{x})+q-p=0,$$

(H2) $$(x-\hat{\tilde{x}})(\tilde{x}-\hat{x})+(q-p)(x+\tilde{x}+\hat{x}+\hat{\tilde{x}})+q^2-p^2=0,$$

(H3) $$p(x\tilde{x}+\hat{x}\hat{\tilde{x}})-q(x\hat{x}+\tilde{x}\hat{\tilde{x}})+\delta(p^2-q^2)=0.$$

List A:

(A1) $$p(x+\hat{x})(\tilde{x}+\hat{\tilde{x}})-q(x+\tilde{x})(\hat{x}+\hat{\tilde{x}})-\delta^2pq(p-q)=0,$$

(A2)

$$(q^2-p^2)(x\tilde{x}\hat{x}\hat{\tilde{x}}+1)+q(p^2-1)(x\hat{x}+\tilde{x}\hat{\tilde{x}})-p(q^2-1)(x\tilde{x}+\hat{x}\hat{\tilde{x}})=0.$$

Main list:

(Q1) $$p(x-\hat{x})(\tilde{x}-\hat{\tilde{x}})-q(x-\tilde{x})(\hat{x}-\hat{\tilde{x}})+\delta^2 pq(p-q)=0,$$

(Q2)
$$p(x-\hat{x})(\tilde{x}-\hat{\tilde{x}})-q(x-\tilde{x})(\hat{x}-\hat{\tilde{x}})+pq(p-q)(x+\tilde{x}+\hat{x}+\hat{\tilde{x}})$$
$$-pq(p-q)(p^2-pq+q^2)=0,$$

(Q3)
$$(q^2-p^2)(x\hat{\tilde{x}}+\tilde{x}\hat{x})+q(p^2-1)(x\tilde{x}+\hat{x}\hat{\tilde{x}})-p(q^2-1)(x\hat{x}+\tilde{x}\hat{\tilde{x}})$$
$$-\delta^2(p^2-q^2)(p^2-1)(q^2-1)/(4pq)=0,$$

(Q4) (the root model from which others follow)
$$a_0 x\tilde{x}\hat{x}\hat{\tilde{x}}+a_1(x\tilde{x}\hat{x}+\tilde{x}\hat{x}\hat{\tilde{x}}+\hat{x}\hat{\tilde{x}}x+\hat{\tilde{x}}x\tilde{x})+a_2(x\hat{\tilde{x}}+\tilde{x}\hat{x})+$$
$$\bar{a}_2(x\tilde{x}+\hat{x}\hat{\tilde{x}})+\tilde{a}_2(x\hat{x}+\tilde{x}\hat{\tilde{x}})+a_3(x+\tilde{x}+\hat{x}+\hat{\tilde{x}})+a_4=0,$$

where the a_i depend on the lattice directions and are given in terms of Weierstrass elliptic functions. This was first derived by Adler as a superposition rule of BT's for the Krichever-Novikov equation. [Adler, Intl. Math. Res. Notices, # 1 (1998) 1-4]

J.H., JNMP **12** Suppl 2, 223 (2005). a simpler Jacobi form for (Q4) of ABS:

$$(h_1 f_2 - h_2 f_1)[(x x_{[1]} x_{[12]} x_{[2]} + 1) f_1 f_2 - (x x_{[12]} + x_{[1]} x_{[2]})] + (f_1^2 f_2^2 - 1)[(x x_{[1]} + x_{[12]} x_{[2]}) f_1 - (x x_{[2]} + x_{[1]} x_{[12]}) f_2] = 0,$$

$h_i^2 = f_i^4 + \delta f_i^2 + 1$, parametrizable by Jacobi elliptic functions.

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Hirota's bilinear form is well suited for constructing soliton solutions, because the new dependent variable is a polynomial of exponentials with linear exponents.

What is a discrete Hirota bilinear form?

The key property of an equation in Hirota's bilinear form is invariance under a gauge transformation

$$f_i \to f_i' := e^{ax+bt} f_i.$$

or in the discrete case

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We say an equation is in **Hirota bilinear (HB) form** if it can be written as

$$f_j(n,m) \to \sum_j c_j f_j(n+\nu_j^+, m+\mu_j^+) g_j(n+\nu_j^-, m+\mu_j^-) = 0$$

where $\nu_i^+ + \nu_i^- = \nu_k^+ + \nu_k^-, \mu_i^+ + \mu_i^- = \mu_k^+ + \mu_k^-, \forall i, k$, that is, the equation is a sum of quadratic terms, and the index sums are the same in each component of the sum.

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$$\mathrm{H1}\equiv(u-\widehat{\widetilde{u}})(\widetilde{u}-\widehat{u})-(p-q)=0,$$

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For convenience we reparametrize $(p,q)\to(a,b)$ by

$$p=r-a^2,\quad q=r-b^2.$$

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From consistency $\theta,\in\{n,0\}$,$\chi,\in\{m,0\}$.
The set of possible background solution turns out to be

$$\begin{aligned} an+bm+\gamma,\\ \tfrac{1}{2}(-1)^n a+bm+\gamma,\\ an+\tfrac{1}{2}(-1)^m b+\gamma,\\ \tfrac{1}{2}(-1)^n a+\tfrac{1}{2}(-1)^m b+\gamma. \end{aligned}$$

1SS

[Atkinson, JH and Nijhoff, JPhysA **41** (2008) 142001.]
The BT generating 1SS for H1 is

$$(u - \bar{\widetilde{u}})(\widetilde{u} - \bar{u}) = p - \varkappa,$$
$$(u - \bar{\widehat{u}})(\bar{u} - \widehat{u}) = \varkappa - q.$$

- u is the background solution $an + bm + \gamma$,
- $\bar{u}$ is the new 1SS,
- $\varkappa$ is the soliton parameter (the parameter in the bar-direction).

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We search for a new solution $\bar{u}$ of the form

$$\bar{u} = \bar{u}_0 + v,$$

where $\bar{u}_0$ is the bar-shifted background solution

$$\bar{u}_0 = an + bm + k + \lambda.$$

For v we then find (in the case of H1)

$$\widetilde{v} = \frac{Ev}{v+F}, \quad \widehat{v} = \frac{Gv}{v+H},$$

where

$$E = -(a+k), \quad F = -(a-k), \quad G = -(b+k), \quad H = -(b-k),$$

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Introducing $v = f/g$ and $\Phi = (g, f)^T$ we can write the v-equations as matrix equations

$$\Phi(n+1, m) = \mathcal{N}(n,m)\Phi(n,m), \quad \Phi(n, m+1) = \mathcal{M}(n,m)\Phi(n,m),$$

where

$$\mathcal{N}(n,m) = \Lambda \begin{pmatrix} E & 0 \\ 1 & F \end{pmatrix}, \quad \mathcal{M}(n,m) = \Lambda' \begin{pmatrix} G & 0 \\ 1 & H \end{pmatrix},$$

In this case E, F, G, H are constants and we can choose $\Lambda = \Lambda' = 1$.

Since the matrices $\mathcal{N}, \mathcal{M}$ commute it is easy to find

$$\Phi(n,m) = \begin{pmatrix} E^nG^m & 0 \\ \frac{E^nG^m - F^nH^m}{-2k} & F^nH^m \end{pmatrix} \Phi(0,0).$$

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$$\rho_{n,m} = \left(\frac{E}{F}\right)^n \left(\frac{G}{H}\right)^m \rho_{0,0} = \left(\frac{a+k}{a-k}\right)^n \left(\frac{b+k}{b-k}\right)^m \rho_{0,0},$$

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(ρ takes the role of e^η) then we obtain

$$v_{n,m} = \frac{-2k\,\rho_{n,m}}{1+\rho_{n,m}}.$$

Finally we obtain the 1SS for H1:

$$u_{n,m}^{(1SS)} = (an + bm + \lambda) + k + \frac{-2k\,\rho_{n,m}}{1+\rho_{n,m}}.$$

Bilinearizing transformation

In an explicit form the 1SS is

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Then it is easy to show that

$$\begin{aligned} \mathrm{H1} &\equiv (u - \widehat{\widetilde{u}})(\widetilde{u} - \widehat{u}) - p + q \\ &= -\left[\mathcal{H}_1 + (a-b) f \widehat{\widetilde{f}}\right]\left[\mathcal{H}_2 + (a+b)\widehat{f}\widetilde{f}\right]/(f \widehat{f} \widetilde{f} \widehat{\widetilde{f}}) + (a^2 - b^2), \end{aligned}$$

where

$$\begin{aligned} \mathcal{H}_1 &\equiv \widehat{g}\widetilde{f} - \widetilde{g}\widehat{f} + (a-b)(\widehat{f}\widetilde{f} - f\widehat{\widetilde{f}}) = 0, \\ \mathcal{H}_2 &\equiv g\widehat{\widetilde{f}} - \widehat{\widetilde{g}} f + (a+b)(f\widehat{\widetilde{f}} - \widehat{f}\widetilde{f}) = 0. \end{aligned}$$

Casoratians

For given functions $\varphi_i(n,m,h)$ we define the column vectors

$$\varphi(n,m,h) = (\varphi_1(n,m,h), \varphi_2(n,m,h), \cdots, \varphi_N(n,m,h))^T,$$

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and then compose the $N \times N$ Casorati matrix from columns with different shifts h_i, and then the determinant

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For example

$$\begin{aligned} C^1_{n,m}(\varphi) &:= |\varphi(n,m,0), \varphi(n,m,1), \cdots, \varphi(n,m,N-1)| \\ &\equiv |0, 1, \cdots, N-1| \equiv |\widehat{N-1}|, \\ C^2_{n,m}(\varphi) &:= |\varphi(n,m,0), \cdots, \varphi(n,m,N-2), \varphi(n,m,N)| \\ &\equiv |0, 1, \cdots, N-2, N| \equiv |\widehat{N-2}, N|, \end{aligned}$$

Main result

The bilinear equations $\mathcal{H}_i$ are solved by Casoratians
$f = |\widehat{N-1}|$, $g = |\widehat{N-2}, N|$, with ψ given by

$$\psi_i(n,m,h;k_i) = \varrho_i^+ k_i^h (a+k_i)^n (b+k_i)^m + \varrho_i^- (-k_i)^h (a-k_i)^n (b-k_i)^m.$$

Main result

The bilinear equations $\mathcal{H}_i$ are solved by Casoratians $f = |\widehat{N-1}|$, $g = |\widehat{N-2}, N|$, with ψ given by

$$\psi_i(n,m,h;k_i) = \varrho_i^+ k_i^h (a+k_i)^n (b+k_i)^m + \varrho_i^- (-k_i)^h (a-k_i)^n (b-k_i)^m.$$

Similar results exist for H2,H3,A1,Q1
[JH and Da-jun Zhang, to appear],

and for Q3 [Atkinson, JH and Nijhoff]

The structure of the soliton solution is similar to those of the Hirota-Miwa equation

Conclusions

We have considered the construction of multisoliton solution for the quadrilateral lattice equations in the ABS list.

A detailed straightforward construction exists han has been done for the low members in the ABS list (H1,H2,H3,A1,Q1). (JH and Da-jun Zhang, work in progress)

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The result for Q4 is less explicit. This is expected since elliptic functions enter big time.
(Atkinson, JH and Nijhoff, JPhysA **40** (2007) F1.)