Symbolic Computation of Lax Pairs of Nonlinear Partial Difference Equations

Willy Hereman

Department of Mathematical and Computer Sciences

Colorado School of Mines

Golden, Colorado, U.S.A.

Reinout Quispel

Department of Mathematical and Statistical Sciences

La Trobe University

Bundoora, Victoria 3083, Australia

SIDE8 International Conference

Ste.-Adèle, Québec, Canada

Saturday, June 28, 2008, 9:30-10:00

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Collaborators

Prof. Reinout Quispel

Dr. Peter van der Kamp

Miss Dinh Tran

Mr. Omar Rojas

This presentation was made in TeXpower

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Outline

- Lax pair of nonlinear PDEs
- Lax pair of nonlinear PΔEs on quad-graphs
- Examples: discrete potential KdV, modified KdV, H and Q equations from ABS classification (except Q4), and the discrete sine-Gordon equation
- Algorithm (Nijhoff 2001, Bobenko & Suris 2001)
- Software demonstration
- Additional examples
- Conclusions and future work

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Lax Pair of Nonlinear PDEs

- Historical example: Korteweg-de Vries equation

$$\boxed{u_t + \alpha u u_x + u_{xxx} = 0}$$

- Lax equation: $\mathrm{L}_t + [\mathrm{L}, \mathrm{M}] = 0$ (on PDE)

  with commutator $[\mathrm{L}, \mathrm{M}] = \mathrm{LM} - \mathrm{ML}$

- Lax operators:

$$\boxed{L = \frac{\partial^2}{\partial x^2} + \frac{\alpha}{6} u}$$

$$\boxed{M = -4\frac{\partial^3}{\partial x^3} - \frac{\alpha}{2}\left(u\frac{\partial}{\partial x} + \frac{\partial}{\partial x}u\right) + A(t)}$$

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- Note: $\mathrm{L}_t\phi + [\mathrm{L},\mathrm{M}]\phi = \frac{\alpha}{6}\left(u_t + \alpha u u_x + u_{xxx}\right)\phi$
- Linear problem
  - ⋆ Sturm-Liouville equation: $\mathrm{L}\psi = \lambda\psi$

    For the KdV equation

$$\psi_{xx} + \left(\frac{\alpha}{6}u - \lambda\right)\psi = 0$$

  - ⋆ Time evolution of data: $\psi_t = \mathrm{M}\psi$
  - ⋆ Eigenvalues of L are constant: $\lambda_t = 0$

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⋆ Compatibility of $\mathrm{L}\psi = \lambda\psi$ and $\psi_t = \mathrm{M}\psi$ gives

$$\mathrm{L}_t\psi + \mathrm{L}\psi_t = \lambda\psi_t$$

$$\begin{aligned} \mathrm{L}_t\psi + \mathrm{LM}\psi &= \lambda\mathrm{M}\psi \\ &= \mathrm{M}\lambda\psi \\ &= \mathrm{ML}\psi \end{aligned}$$

Thus,

$$\mathrm{L}_t\psi + (\mathrm{LM} - \mathrm{ML})\psi = \mathrm{O}$$

or

$$\mathrm{L}_t + [\mathrm{L}, \mathrm{M}] = 0 \ \ (\text{on PDE})$$

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- Eliminate the third order term in M :

Recall that $\mathrm{L}\psi = 0 \longrightarrow \psi_{xx} = (\lambda - \frac{\alpha}{6}u)\psi$

$$\boxed{M = -4\frac{\partial^3}{\partial x^3} - \frac{\alpha}{2}\left(u\frac{\partial}{\partial x} + \frac{\partial}{\partial x}u\right) + A(t)}$$

$$\longrightarrow \boxed{M = \left(A(t) + \frac{\alpha}{6}u_x\right) - \left(4\lambda + \frac{\alpha}{3}u\right)\frac{\partial}{\partial x}}$$

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Reasons to compute a Lax pair

- Replace nonlinear PDE by linear scattering problem and apply the IST
- Describe the time evolution of the scattering data
- Confirm the complete integrability of the PDE
- Zero-curvature representation of the PDE
- Compute conservation laws of the PDE
- Discover families of completely integrable PDEs

Question: How to find a Lax pair of a completely integrable PDE?

Answer: There is no completely systematic method

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Nonlinear Partial Difference Equations (PΔEs)

- Discrete Potential Korteweg-de Vries (pKdV) equation

$$\boxed{(v_{n,m} - v_{n+1,m+1})(v_{n+1,m} - v_{n,m+1}) - p^2 + q^2 = 0}$$

- Notation:

v is dependent variable or field (others use u or x)

n and m are lattice points (others use ℓ and m)

p and q are parameters (others use α and β)

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- For brevity, denote
  $(v_{n,m}, v_{n+1,m}, v_{n,m+1}, v_{n+1,m+1}) = (x, x_1, x_2, x_{12})$
  Alternatives (in the literature): $(v, \tilde{v}, \hat{v}, \hat{\tilde{v}})$ or (x, u, v, y) or $(v_{00}, v_{10}, v_{01}, v_{11})$
- discrete pKdV equation:

$$\boxed{(x - x_{12})(x_1 - x_2) - p^2 + q^2 = 0}$$

- Alternatives (in the literature):

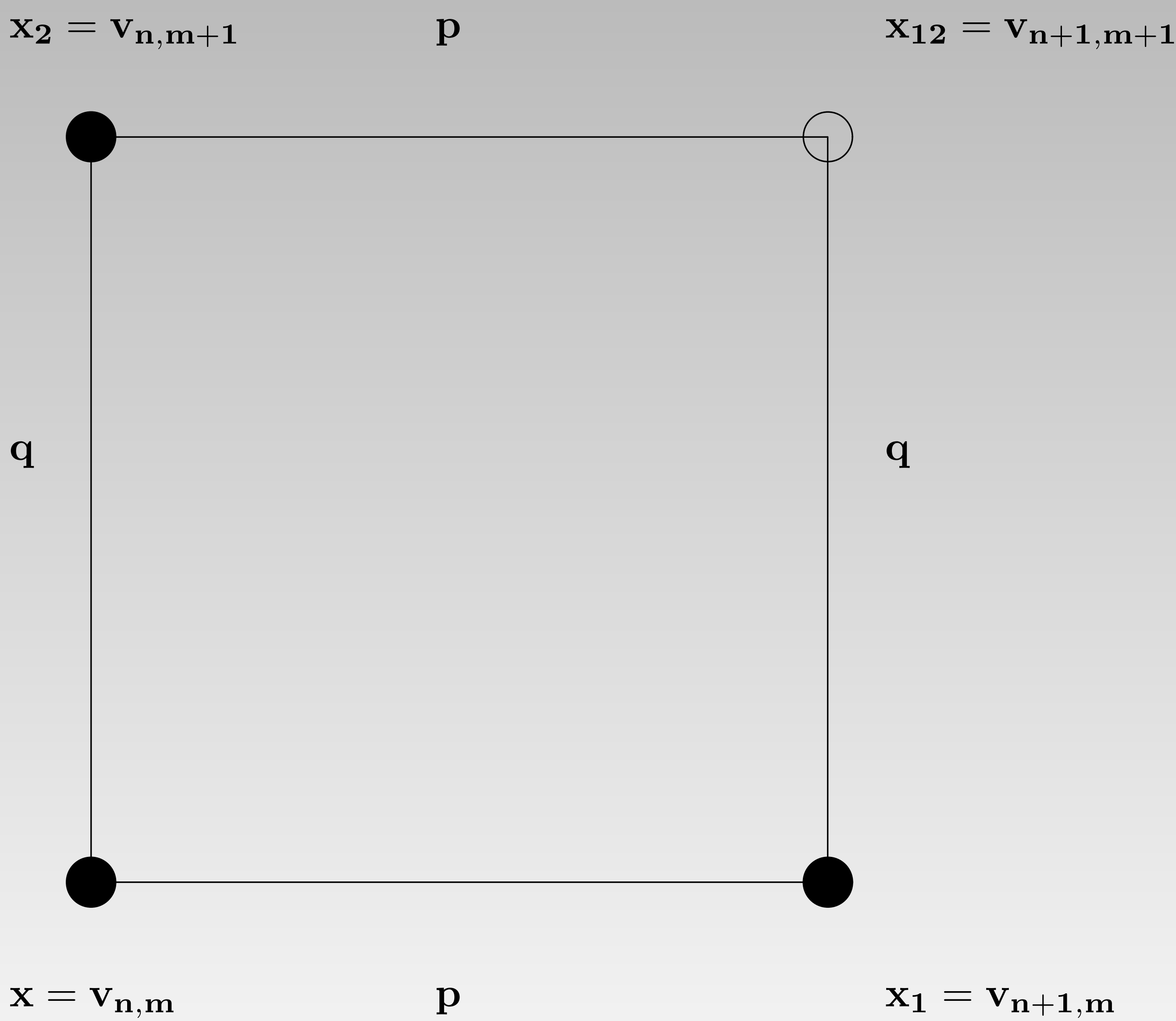
$$\boxed{(v - \hat{\tilde{v}})(\tilde{v} - \hat{v}) - p^2 + q^2 = 0}$$

or

$$\boxed{(x - y)(u - v) - p^2 + q^2 = 0}$$

or

$$\boxed{(v_{00} - v_{11})(v_{10} - v_{01}) - p^2 + q^2 = 0}$$


$\mathbf{x_2 = v_{n,m+1}}$
$\mathbf{p}$
$\mathbf{x_{12} = v_{n+1,m+1}}$
$\mathbf{q}$
$\mathbf{q}$
$\mathbf{x = v_{n,m}}$
$\mathbf{p}$
$\mathbf{x_1 = v_{n+1,m}}$

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Lax Pair of Nonlinear $P\Delta Es$

- Require that $\psi_1 = \mathrm{L}\psi, \quad \psi_2 = \mathrm{M}\psi$
- Compatibility:

$$\psi_{12} = \mathrm{L}_2\psi_2 = \mathrm{L}_2\mathrm{M}\psi$$

$$\psi_{12} = \mathrm{M}_1\psi_1 = \mathrm{M}_1\mathrm{L}\psi$$

- Hence, Lax equation: $\mathrm{L}_2\mathrm{M} - \mathrm{M}_1\mathrm{L} = 0$ (on $P\Delta E$)

- Example 1: Discrete pKdV equation

$$\boxed{(x - x_{12})(x_1 - x_2) - p^2 + q^2 = 0}$$

H1 via extended Möbius transformation:

$$x \longrightarrow \frac{1}{x}, x_1 \longrightarrow x_1, x_2 \longrightarrow \frac{1}{x_2}, x_{12} \longrightarrow x_{12}, p \longrightarrow \frac{1}{p}, q \longrightarrow q$$

- Lax operators:

$$\boxed{L = t \begin{bmatrix} x & p^2 - k^2 - x x_1 \\ 1 & -x_1 \end{bmatrix}}$$

$$\boxed{M = s \begin{bmatrix} x & q^2 - k^2 - x x_2 \\ 1 & -x_2 \end{bmatrix}}$$

with $t = s = 1$ or $t = \frac{1}{\sqrt{k^2 - p^2}}$ and $s = \frac{1}{\sqrt{k^2 - q^2}}$

Note: $\frac{t_2}{t} \frac{s}{s_1} = 1$

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- Note: $\mathrm{L_2M} - \mathrm{M_1L} = \left((x - x_{12})(x_1 - x_2) - p^2 + q^2 \right) \mathrm{N}$

  with

$$L = \begin{bmatrix} x & p^2 - k^2 - x x_1 \\ 1 & -x_1 \end{bmatrix}$$

$$M = \begin{bmatrix} x & q^2 - k^2 - x x_2 \\ 1 & -x_2 \end{bmatrix}$$

$$N = \frac{1}{\sqrt{(p^2 - k^2)(q^2 - k^2)}} \begin{bmatrix} -1 & x_1 + x_2 \\ 0 & 1 \end{bmatrix}$$

- Example 2: Discrete modified KdV equation

$$p(xx_2 - x_1x_{12}) - q(xx_1 - x_2x_{12}) = 0$$

H3 for $\delta = 0$ and $x_{12} \to -x_{12}$

- Lax operators:

$$L = t\begin{bmatrix} -px & kxx_1 \\ k & -px_1 \end{bmatrix}$$

$$M = s\begin{bmatrix} -qx & kxx_2 \\ k & -qx_2 \end{bmatrix}$$

with $t = \frac{1}{x_1}$ and $s = \frac{1}{x_2}$, or $t = s = \frac{1}{x}$

or $t = \frac{1}{\sqrt{(p^2-k^2)xx_1}}$ and $s = \frac{1}{\sqrt{(q^2-k^2)xx_2}}$

Note: $\frac{t_2}{t}\,\frac{s}{s_1} = \frac{x\,x_1}{x\,x_2} = \frac{x_1}{x_2}$

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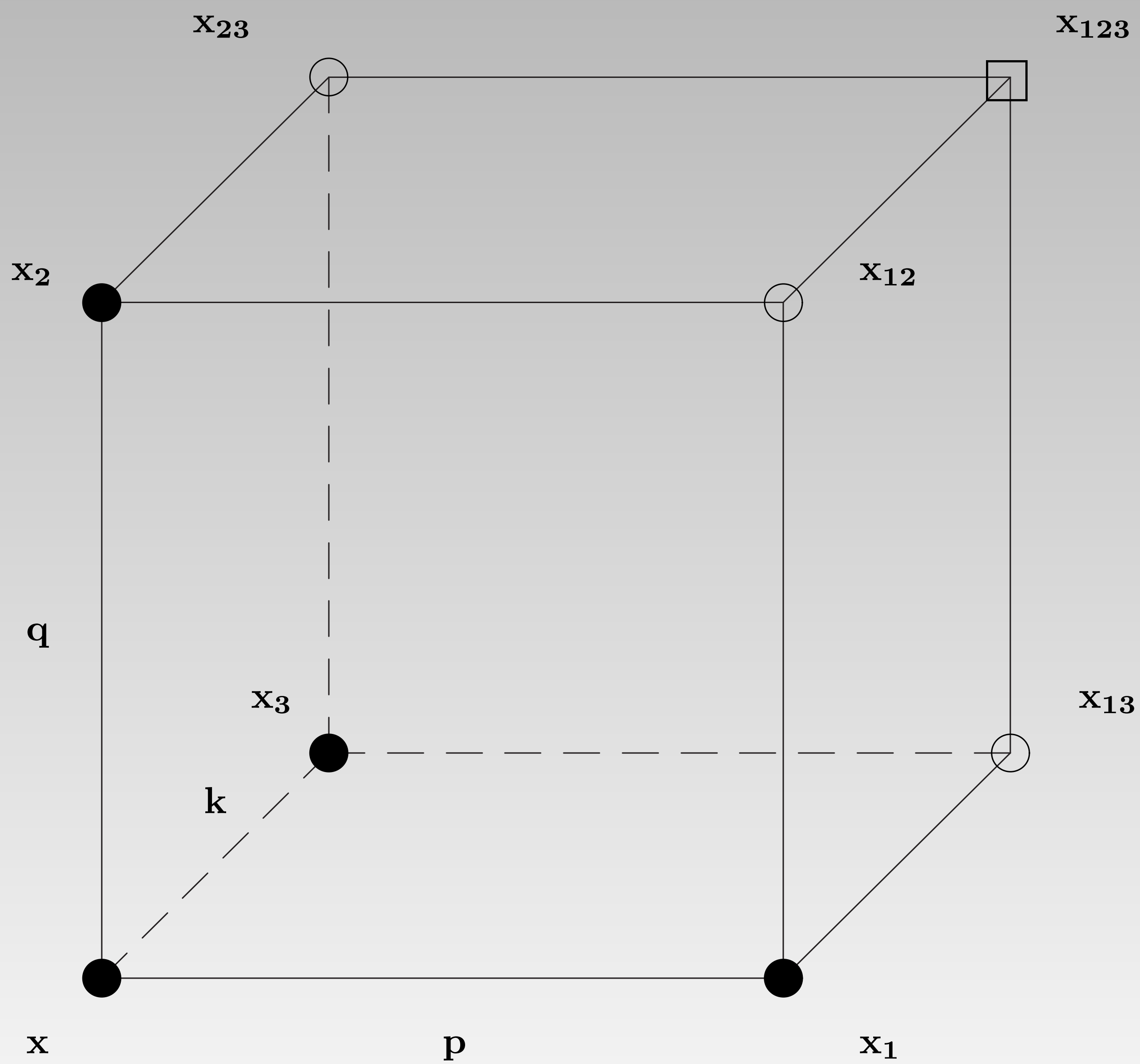
Algorithm to Compute a Lax Pair

(Nijhoff 2001, Bobenko & Suris 2001)

Applies to equations that are consistent on cube

Example: Discrete pKdV equation

- Step 1: Verify the consistency around the cube


$\mathbf{x_{23}}$
$\mathbf{x_{123}}$
$\mathbf{x_2}$
$\mathbf{x_{12}}$
$\mathbf{q}$
$\mathbf{x_3}$
$\mathbf{x_{13}}$
$\mathbf{k}$
$\mathbf{x}$
$\mathbf{p}$
$\mathbf{x_1}$

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⋆ Equation on front face of cube:

$$(x - x_{12})(x_1 - x_2) - p^2 + q^2 = 0$$

Solve for $x_{12} = x - \frac{p^2 - q^2}{x_1 - x_2}$

Compute x_{123} : $\quad x_{12} \longrightarrow x_{123} = x_3 - \frac{p^2 - q^2}{x_{13} - x_{23}}$

⋆ Equation on floor of cube:

$$(x - x_{13})(x_1 - x_3) - p^2 + k^2 = 0$$

Solve for $x_{13} = x - \frac{p^2 - k^2}{x_1 - x_3}$

Compute x_{123} : $\quad x_{13} \longrightarrow x_{123} = x_2 - \frac{p^2 - k^2}{x_{12} - x_{23}}$

⋆ Equation on left face of cube:

$$(x - x_{23})(x_3 - x_2) - k^2 + q^2 = 0$$

Solve for $x_{23} = x - \frac{q^2 - k^2}{x_2 - x_3}$

Compute x_{123} : $\quad x_{23} \longrightarrow x_{123} = x_1 - \frac{q^2 - k^2}{x_{12} - x_{13}}$

⋆ Verify that

$$x_{123} = x_1 - \frac{q^2 - k^2}{x_{12} - x_{13}} = x_2 - \frac{p^2 - k^2}{x_{12} - x_{23}} = x_3 - \frac{p^2 - q^2}{x_{13} - x_{23}}$$

Upon substitution of x_{12}, x_{13}, and x_{23}

$$x_{123} = \frac{p^2 x_1 (x_2 - x_3) + q^2 x_2 (x_3 - x_1) + k^2 x_3 (x_1 - x_2)}{p^2 (x_2 - x_3) + q^2 (x_3 - x_1) + k^2 (x_1 - x_2)}$$

Note: x_{123} is independent of x (tetrahedron condition)

Consistency around the cube is satisfied!

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- Step 2: Homogenization

Numerator and denominator of

$x_{13} = \frac{x_3 x - x x_1 + p^2 - k^2}{x_3 - x_1}$ and $x_{23} = \frac{x_3 x - x x_2 + q^2 - k^2}{x_3 - x_2}$

are linear in x_3

Substitute $x_3 = \frac{f}{g} \longrightarrow x_{13} = \frac{f_1}{g_1},\; x_{23} = \frac{f_2}{g_2}$.

From x_{13} : $\quad \frac{f_1}{g_1} = \frac{x f + (p^2 - k^2 - x x_1) g}{f - x_1 g}$

Hence, $f_1 = t\left(x f + (p^2 - k^2 - x x_1) g\right)$ and $g_1 = t\,(f - x_1 g)$

or, in matrix form

$$\begin{bmatrix} f_1 \\ g_1 \end{bmatrix} = t \begin{bmatrix} x & p^2 - k^2 - x x_1 \\ 1 & -x_1 \end{bmatrix} \begin{bmatrix} f \\ g \end{bmatrix}$$

.

Similarly, from x_{23} :

$$\begin{bmatrix} f_2 \\ g_2 \end{bmatrix} = s \begin{bmatrix} x & q^2 - k^2 - x x_2 \\ 1 & -x_2 \end{bmatrix} \begin{bmatrix} f \\ g \end{bmatrix}$$

Therefore,

$$\boxed{L = t\, L_c = t \begin{bmatrix} x & p^2 - k^2 - x x_1 \\ 1 & -x_1 \end{bmatrix}}$$

$$\boxed{M = s\, M_c = s \begin{bmatrix} x & q^2 - k^2 - x x_2 \\ 1 & -x_2 \end{bmatrix}}$$

- Step 3: Determine t and s

$\star$ Substitute $L = tL_c, M = sM_c$ into $\mathrm{L_2M - M_1L = 0}$

$\longrightarrow t_2 s (L_c)_2 M_c - s_1 t (M_c)_1 L_c = 0$

$\star$ Solve the equation coming from the (2-1)-element for

$$\frac{t_2}{t}\,\frac{s}{s_1} = f(x, x_1, x_2, p, q, \ldots)$$

$\star$ If f factors as

$$f = \frac{\mathcal{F}(x, x_1, p, q, \ldots)\mathcal{G}(x, x_1, p, q, \ldots)}{\mathcal{F}(x, x_2, q, p, \ldots)\mathcal{G}(x, x_2, q, p, \ldots)}$$

then $t = \frac{1}{\mathcal{F}(x,x_1,\ldots)}$ or $\frac{1}{\mathcal{G}(x,x_1,\ldots)}$ and $s = \frac{1}{\mathcal{F}(x,x_2,\ldots)}$ or $\frac{1}{\mathcal{G}(x,x_2,\ldots)}$

No square roots needed!

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Exceptions: For the A1 and A2 lattices

$$\frac{t_2}{t}\,\frac{s}{s_1} = \frac{\mathcal{F}(x, x_1, p, q, \ldots)\mathcal{G}(x, x_1, p, q, \ldots)}{\mathcal{F}(x, x_2, q, p, \ldots)\mathcal{G}(x, x_2, q, p, \ldots)}$$

Nevertheless, one needs to apply the determinant (which introduces square roots).

⋆ If f does not factor, apply determinant to get

$$\frac{t_2}{t}\,\frac{s}{s_1} = \sqrt{\frac{\det L_c}{\det\,(L_c)_2}}\,\sqrt{\frac{\det\,(M_c)_1}{\det M_c}}$$

⋆ A solution: $t = \frac{1}{\sqrt{\det L_c}}, \quad s = \frac{1}{\sqrt{\det M_c}}$

$\longrightarrow$ Introduces square roots!

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Try to avoid square roots!

Remedy 1: Work with the 3-leg form of the lattice

Remedy 2: Apply a change of variables $x = F(X)$

$$
\begin{aligned}
\frac{t_2}{t}\,\frac{s}{s_1} &= f(F(X), F(X_1), F(X_2), p, q, \ldots) \\
&= \frac{\mathcal{F}(X, X_1, p, q, \ldots)\mathcal{G}(X, X_1, p, q, \ldots)}{\mathcal{F}(X, X_2, q, p, \ldots)\mathcal{G}(X, X_2, q, p, \ldots)}
\end{aligned}
$$

Example: Q2 lattice. Replace $x = F(X) = X^2$

$$
\begin{aligned}
\frac{t_2}{t}\,\frac{s}{s_1} &= \frac{q\left((x-x_1)^2 - 2\delta p^2(x+x_1) + \delta^2 p^4\right)}{p\left((x-x_2)^2 - 2\delta q^2(x+x_2) + \delta^2 q^4\right)} \\
&= \frac{q\left((X+X_1)^2 - \delta p^2\right)\left((X-X_1)^2 - \delta p^2\right)}{p\left((X+X_2)^2 - \delta q^2\right)\left((X-X_2)^2 - \delta q^2\right)}
\end{aligned}
$$

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⋆ Different Lax pairs are equivalent (gauge invariance of the Lax equation):

$$\mathcal{L} = \mathrm{G}_1 \mathrm{L} \mathrm{G}^{-1}, \qquad \mathcal{M} = \mathrm{G}_2 \mathrm{M} \mathrm{G}^{-1}$$

where G is diagonal matrix (or scalar factor) and $\phi = \mathrm{G}\psi$

Proof: Trivial verification that
$(\mathcal{L}_2\mathcal{M} - \mathcal{M}_1\mathcal{L})\,\phi = 0 \leftrightarrow (\mathrm{L}_2\mathrm{M} - \mathrm{M}_1\mathrm{L})\,\psi = 0$

Software Demonstration

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Additional Examples

- Example 3: H1 equation (ABS classification)

$$(x - x_{12})(x_1 - x_2) + q - p = 0$$

- Lax operators:

$$L = t \begin{bmatrix} x & p - k - x x_1 \\ 1 & -x_1 \end{bmatrix}$$

$$M = s \begin{bmatrix} x & q - k - x x_2 \\ 1 & -x_2 \end{bmatrix}$$

with $t = s = 1$ or $t = \frac{1}{\sqrt{k-p}}$ and $s = \frac{1}{\sqrt{k-q}}$

Note: $\frac{t_2}{t} \frac{s}{s_1} = 1$

- Example 4: H2 equation (ABS 2003)

$$(x-x_{12})(x_1-x_2)+(q-p)(x+x_1+x_2+x_{12})+q^2-p^2=0$$

- Lax operators:

$$L = t\begin{bmatrix} p-k+x & p^2-k^2+(p-k)(x+x_1)-xx_1 \\ 1 & -(p-k+x_1) \end{bmatrix}$$

$$M = s\begin{bmatrix} q-k+x & q^2-k^2+(q-k)(x+x_2)-xx_2 \\ 1 & -(q-k+x_2) \end{bmatrix}$$

with $t = \frac{1}{\sqrt{2(k-p)(p+x+x_1)}}$ and $s = \frac{1}{\sqrt{2(k-q)(q+x+x_2)}}$

Note: $\frac{t_2}{t}\,\frac{s}{s_1} = \frac{p+x+x_1}{q+x+x_2}$

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- Example 5: H3 equation (ABS 2003)

$$p(xx_1 + x_2x_{12}) - q(xx_2 + xx_{12}) + \delta(p^2 - q^2) = 0$$

- Lax operators:

$$L = t \begin{bmatrix} kx & -\left(\delta(p^2-k^2) + pxx_1\right) \\ p & -kx_1 \end{bmatrix}$$

$$M = s \begin{bmatrix} kx & -\left(\delta(q^2-k^2) + qxx_2\right) \\ q & -kx_2 \end{bmatrix}$$

with $t = \frac{1}{\sqrt{(p^2-k^2)(\delta p + xx_1)}}$ and $s = \frac{1}{\sqrt{(q^2-k^2)(\delta q + xx_2)}}$

Note: $\frac{t_2}{t}\frac{s}{s_1} = \frac{\delta p + xx_1}{\delta q + xx_2}$

.

- Example 6: H3 equation with $\delta = 0$ (ABS 2003)

$$\boxed{p(xx_1 + x_2x_{12}) - q(xx_2 + xx_{12}) = 0}$$

- Lax operators:

$$\boxed{L = t \begin{bmatrix} kx & -pxx_1 \\ p & -kx_1 \end{bmatrix}}$$

$$\boxed{M = s \begin{bmatrix} kx & -qxx_2 \\ q & -kx_2 \end{bmatrix}}$$

with $t = s = \frac{1}{x}$ or $t = \frac{1}{x_1}$ and $s = \frac{1}{x_2}$

Note: $\frac{t_2}{t} \frac{s}{s_1} = \frac{x\,x_1}{x\,x_2} = \frac{x_1}{x_2}$

- Example 7: Q1 equation (ABS 2003)

$$p(x-x_2)(x_1-x_{12})-q(x-x_1)(x_2-x_{12})+\delta^2 pq(p-q)=0$$

- Lax operators:

$$L = t\begin{bmatrix}(p-k)x_1+kx & -p\left(\delta^2 k(p-k)+xx_1\right)\\ p & -\left((p-k)x+kx_1\right)\end{bmatrix}$$

$$M = s\begin{bmatrix}(q-k)x_2+kx & -q\left(\delta^2 k(q-k)+xx_2\right)\\ q & -\left((q-k)x+kx_2\right)\end{bmatrix}$$

with $t=\frac{1}{\delta p\pm(x-x_1)}$ and $s=\frac{1}{\delta q\pm(x-x_2)}$,
or $t=\frac{1}{\sqrt{k(p-k)\left((\delta p+x-x_1)(\delta p-x+x_1)\right)}}$ and

$$s=\frac{1}{\sqrt{k(q-k)\left((\delta q+x-x_2)(\delta q-x+x_2)\right)}}$$

Note: $\frac{t_2}{t}\,\frac{s}{s_1}=\frac{q\left(\delta p+(x-x_1)\right)\left(\delta p-(x-x_1)\right)}{p\left(\delta q+(x-x_2)\right)\left(\delta q-(x-x_2)\right)}$

- Example 8: Q1 equation with $\delta = 0$ (ABS 2003)

$$p(x - x_2)(x_1 - x_{12}) - q(x - x_1)(x_2 - x_{12}) = 0$$

which is the cross-ratio equation

$$\frac{(x - x_1)(x_{12} - x_2)}{(x_1 - x_{12})(x_2 - x)} = \frac{p}{q}$$

- Lax operators:

$$L = t \begin{bmatrix} (p-k)x_1 + kx & -pxx_1 \\ p & -((p-k)x + kx_1) \end{bmatrix}$$

$$M = s \begin{bmatrix} (q-k)x_2 + kx & -qxx_2 \\ q & -((q-k)x + kx_2) \end{bmatrix}$$

Here, $\frac{t_2}{t}\frac{s}{s_1} = \frac{q(x-x_1)^2}{p(x-x_2)^2}$. So, $t = \frac{1}{x-x_1}$ and $s = \frac{1}{x-x_2}$
or $t = \frac{1}{\sqrt{k(k-p)(x-x_1)}}$ and $s = \frac{1}{\sqrt{k(k-q)(x-x_2)}}$

- Example 9: Q2 equation (ABS 2003)

$$p(x-x_2)(x_1-x_{12})-q(x-x_1)(x_2-x_{12})+\delta pq(p-q)$$
$$(x+x_1+x_2+x_{12})-\delta^2 pq(p-q)(p^2-pq+q^2)=0$$

- Lax operators:

$$L=t\begin{bmatrix} (k-p)(\delta kp-x_1)+kx & \\ & -p\left(\delta k(k-p)(\delta k^2-\delta kp+\delta p^2-x-x_1)+xx_1\right) \\ p & -\left((k-p)(\delta kp-x)+kx_1\right) \end{bmatrix}$$

$$M=s\begin{bmatrix} (k-q)(\delta kq-x_2)+kx & \\ & -q\left(\delta k(k-q)(\delta k^2-\delta kq+\delta q^2-x-x_2)+xx_2\right) \\ q & -\left((k-q)(\delta kq-x)+kx_2\right) \end{bmatrix}$$

.

- with

$$t = \frac{1}{\sqrt{k(k-p)\big((x-x_1)^2-2\delta p^2(x+x_1)+\delta^2 p^4\big)}}$$

and

$$s = \frac{1}{\sqrt{k(k-q)\big((x-x_2)^2-2\delta q^2(x+x_2)+\delta^2 q^4\big)}}$$

Note:

$$\begin{aligned}\frac{t_2}{t}\,\frac{s}{s_1} &= \frac{q\left((x-x_1)^2-2\delta p^2(x+x_1)+\delta^2 p^4\right)}{p\left((x-x_2)^2-2\delta q^2(x+x_2)+\delta^2 q^4\right)} \\ &= \frac{p\left((X+X_1)^2-\delta p^2\right)\left((X-X_1)^2-\delta p^2\right)}{q\left((X+X_2)^2-\delta q^2\right)\left((X-X_2)^2-\delta q^2\right)}\end{aligned}$$

with $x = X^2$

- Example 10: Q3 equation (ABS 2003)

$$(q^2-p^2)(xx_{12}+x_1x_2)+q(p^2-1)(xx_1+x_2x_{12})$$
$$-p(q^2-1)(xx_2+x_1x_{12})-\frac{\delta^2}{4pq}(p^2-q^2)(p^2-1)(q^2-1)=0$$

- Lax operators:

$$L=t\begin{bmatrix} -4kp\left(p(k^2-1)x+(p^2-k^2)x_1\right) & -(p^2-1)(\delta k^2-\delta^2k^4-\delta^2p^2+\delta^2k^2p^2-4k^2pxx_1) \\ -4k^2p(p^2-1) & 4kp\left(p(k^2-1)x_1+(p^2-k^2)x\right) \end{bmatrix}$$

$$M=s\begin{bmatrix} -4kq\left(q(k^2-1)x+(q^2-k^2)x_2\right) & -(q^2-1)(\delta k^2-\delta^2k^4-\delta^2q^2+\delta^2k^2q^2-4k^2qxx_2) \\ -4k^2q(q^2-1) & 4kq\left(q(k^2-1)x_2+(q^2-k^2)x\right) \end{bmatrix}$$

.

- with

$$t = \frac{1}{2k\sqrt{p(k^2-1)(k^2-p^2)\left(4p^2(x-x_1)^2-4p(p-1)^2xx_1+\delta^2(1-p^2)^2\right)}}$$

and

$$s = \frac{1}{2k\sqrt{q(k^2-1)(k^2-q^2)\left(4q^2(x-x_2)^2-4q(q-1)^2xx_2+\delta^2(1-q^2)^2\right)}}$$

Note:

$$\frac{t_2}{t}\,\frac{s}{s_1}$$

$$= \frac{q(q^2-1)\left(4p^2(x-x_1)^2-4p(p-1)^2xx_1+\delta^2(1-p^2)^2\right)}{p(p^2-1)\left(4q^2(x-x_2)^2-4q(q-1)^2xx_2+\delta^2(1-q^2)^2\right)}$$

$$= \frac{q(q^2-1)\left(4p^2(x+x_1)^2-4p(p+1)^2xx_1+\delta^2(1-p^2)^2\right)}{p(p^2-1)\left(4q^2(x+x_2)^2-4q(q+1)^2xx_2+\delta^2(1-q^2)^2\right)}$$

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- Example 11: Q3 equation with $\delta = 0$ (ABS 2003)

$$(q^2 - p^2)(xx_{12} + x_1x_2) + q(p^2 - 1)(xx_1 + x_2x_{12})$$
$$-p(q^2 - 1)(xx_2 + x_1x_{12}) = 0$$

- Lax operators:

$$L = t\begin{bmatrix}(p^2-k^2)x_1+p(k^2-1)x & -k(p^2-1)xx_1 \\ (p^2-1)k & -\left((p^2-k^2)x+p(k^2-1)x_1\right)\end{bmatrix}$$

$$M = s\begin{bmatrix}(q^2-k^2)x_2+q(k^2-1)x & -k(q^2-1)xx_2 \\ (q^2-1)k & -\left((q^2-k^2)x+q(k^2-1)x_2\right)\end{bmatrix}$$

.

- with $t = \frac{1}{px - x_1}$ and $s = \frac{1}{qx - x_2}$

  or $t = \frac{1}{px_1 - x}$ and $s = \frac{1}{qx_2 - x}$

  or $t = \frac{1}{\sqrt{(k^2-1)(p^2-k^2)(px-x_1)(px_1-x)}}$

  and $s = \frac{1}{\sqrt{(k^2-1)(q^2-k^2)(qx-x_2)(qx_2-x)}}$

  Note: $\frac{t_2}{t}\,\frac{s}{s_1} = \frac{(q^2-1)(px-x_1)(px_1-x)}{(p^2-1)(qx-x_2)(qx_2-x)}$

.

- Example 12: (α,β)-equation (Tran)

$$\big((p-\alpha)x-(p+\beta)x_1\big)\big((p-\beta)x_2-(p+\alpha)x_{12}\big)$$
$$-\big((q-\alpha)x-(q+\beta)x_2\big)\big((q-\beta)x_1-(q+\alpha)x_{12}\big)=0$$

- Lax operators:

$$L=t\begin{bmatrix}(p-\alpha)(p-\beta)x+(k^2-p^2)x_1 & -(k-\alpha)(k-\beta)xx_1\\ (k+\alpha)(k+\beta) & -\big((p+\alpha)(p+\beta)x_1+(k^2-p^2)x\big)\end{bmatrix}$$

$$M=s\begin{bmatrix}(q-\alpha)(q-\beta)x+(k^2-q^2)x_2 & -(k-\alpha)(k-\beta)xx_2\\ (k+\alpha)(k+\beta) & -\big((q+\alpha)(q+\beta)x_2+(k^2-q^2)x\big)\end{bmatrix}$$

.

- with $t = \frac{1}{(\alpha-p)x+(\beta+p)x_1}$ and $s = \frac{1}{(\alpha-q)x+(\beta+q)x_2}$

  or $t = \frac{1}{(\beta-p)x+(\alpha+p)x_1}$ and $s = \frac{1}{(\beta-q)x+(\alpha+q)x_2}$

  or $t = \frac{1}{\sqrt{(p^2-k^2)\big((\beta-p)x+(\alpha+p)x_1\big)\big((\alpha-p)x+(\beta+p)x_1\big)}}$

  and $s = \frac{1}{\sqrt{(q^2-k^2)\big((\beta-q)x+(\alpha+q)x_2\big)\big((\alpha-q)x+(\beta+q)x_2\big)}}$

  Note: $\frac{t_2}{t}\,\frac{s}{s_1} = \frac{\big((\beta-p)x+(\alpha+p)x_1\big)\big((\alpha-p)x+(\beta+p)x_1\big)}{\big((\beta-q)x+(\alpha+q)x_2\big)\big((\alpha-q)x+(\beta+q)x_2\big)}$

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- Example 13: Discrete sine-Gordon equation

$$\boxed{xx_1x_2x_{12} - pq(xx_{12} - x_1x_2) - 1 = 0}$$

H3 with $\delta = 0$ via extended Möbius transformation:

$$x \longrightarrow x, x_1 \longrightarrow x_1, x_2 \longrightarrow \frac{1}{x_2}, x_{12} \longrightarrow -\frac{1}{x_{12}}, p \longrightarrow p, q \longrightarrow \frac{1}{q}$$

The discrete sine-Gordon equation is NOT consistent around the cube, but has a Lax pair.

- Lax operators:

$$\boxed{L = \begin{bmatrix} p & -kx_1 \\ -\frac{k}{x} & \frac{px_1}{x} \end{bmatrix}}$$

$$\boxed{M = \begin{bmatrix} \frac{qx_2}{x} & -\frac{1}{kx} \\ -\frac{x_2}{k} & q \end{bmatrix}}$$

.

Conclusions and Future Work

- *Mathematica* code works for **PΔEs** in 2D defined on quad-graphs (quadrilateral faces)
- Code can be used to test consistency around the cube
- Code can be used to test Lax pairs
- Avoid the determinant method to avoid square roots! Factorization plays an essential role!
- Work with equivalent representation of the lattice (3-leg form)
- Try to find the simplest Lax pair

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- Consistency around the cube $\Longrightarrow$ PΔE has Lax pair
- Lax pair reveals itself from the map (by eliminations)
- There are PΔEs with a Lax pair that are not consistent around the cube.
  Example: discrete sine-Gordon equation
- Hard case: $Q4$ equation (elliptic curves, Weierstrass functions) (Nijhoff, 2001)
- **P**ΔEs in 3D: Lax pair will consist of tensors.
  Examples: discrete Kadomtsev-Petviashvili (KP) equations (AKP, BKP, KP lattice)

Thank You