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A Completeness Study on Certain 2×2 Lax Pairs

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What is a Lax pair?

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A Lax pair is a pair of linear problems who's compatibility is associated with a nonlinear equation.

$$\begin{aligned} \bar{\phi} = \phi(l+1,m) &= L(l,m)\phi(l,m) \\ \hat{\phi} = \phi(l,m+1) &= M(l,m)\phi(l,m) \end{aligned}$$

$$\begin{aligned} \hat{\bar{\phi}} &= \widehat{L}\hat{\phi} = & \widehat{L}M\phi \\ \hat{\bar{\phi}} &= \overline{M}\bar{\phi} = & \overline{M}L\phi \end{aligned}$$

Compatibility condition: $\widehat{L}M = \overline{M}L$

A nonlinear equation is *integrable* if it has a Lax pair.

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LSG_2

The recently discovered, higher order version of the lattice sine-Gordon equation, LSG_2:

$$\begin{aligned}\mathrm{LSG}_2 : \qquad \frac{\rho}{\sigma}\frac{\hat{x}}{x} + \lambda_1\mu_1\hat{\bar{x}}\hat{y} &= \frac{\sigma}{\rho}\frac{\hat{\bar{x}}}{\bar{x}} + \frac{\lambda_2\mu_2}{x\bar{y}} \\ \frac{\sigma}{\rho}\frac{\hat{\bar{y}}}{\hat{y}} + \frac{\lambda_2\mu_2}{\hat{x}y} &= \frac{\rho}{\sigma}\frac{\bar{y}}{y} + \lambda_1\mu_1\bar{x}\hat{\bar{y}}\end{aligned}$$

So-called because using $x = y$ retrieves the well-known *lattice sine-Gordon equation*:

$$\mathrm{LSG} : \quad \hat{\bar{x}}x\left(\frac{\sigma}{\rho} - \lambda_1\mu_1\bar{x}\hat{x}\right) = \frac{\rho}{\sigma}\bar{x}\hat{x} - \lambda_2\mu_2$$

Elliptic Lax pair for LSG_2

The elliptic Lax pair for LSG_2 is

$$
\begin{aligned}
L &= \begin{pmatrix} \wp/\rho & \wp'\lambda_1\bar{x} \\ \wp'\lambda_2/x & \wp\rho\bar{y}/y \end{pmatrix} \\
M &= \begin{pmatrix} \wp'\hat{x}/(\sigma x) & \wp/y \\ \wp\mu_1\hat{y} & \wp'\sigma \end{pmatrix}
\end{aligned}
$$

where $\wp = \wp(n)$ is the Weierstrass ‘p’ elliptic function, and $\wp'$ is its derivative wrt. n.

We call terms that depend on the spectral variable, n, *spectral terms*. Those that depend on the lattice variables, l, m, are *lattice terms*.

The $(1,2)$ entry of the compatibility condition

Explicitly, the compatibility condition, $\widehat{L}M = \overline{M}L \quad \Rightarrow$

$$\begin{pmatrix} \wp\rho & \wp'\lambda_1\hat{\hat{x}} \\ \wp'\lambda_2/\hat{x} & \wp\hat{\hat{y}}/(\hat{y}\rho) \end{pmatrix}\begin{pmatrix} \wp'\hat{x}/(\sigma x) & \wp/y \\ \wp\mu_1\hat{y} & \wp'\sigma \end{pmatrix} =$$
$$\begin{pmatrix} \wp'\sigma\hat{\bar{x}}/\bar{x} & \wp/\bar{y} \\ \wp\mu_1\hat{\bar{y}} & \wp'/\sigma \end{pmatrix}\begin{pmatrix} \wp/\rho & \wp'\lambda_1\bar{x} \\ \wp'\lambda_2/x & \wp\rho\bar{y}/y \end{pmatrix}$$

The $(1,2)$ entry of which,

$$\wp^2\rho/y + \wp'^2\sigma\lambda_1\hat{\bar{x}} = \wp'^2\sigma\lambda_1\hat{\bar{x}} + \wp^2\rho/y$$

is clearly an identity.

Consider the starting point for the elliptic LSG$_2$ Lax pair:

$$
\begin{aligned}
L &= \begin{pmatrix} \wp a & \wp' b \\ \wp' c & \wp d \end{pmatrix} \\
M &= \begin{pmatrix} \wp' \alpha & \wp \beta \\ \wp \gamma & \wp' \delta \end{pmatrix}
\end{aligned}
$$

The important feature of the spectral dependence is that it groups the terms into equations that lead to LSG$_2$. The $(1,2)$ entry of the compatibility condition is

$$
\begin{matrix} \wp^2 \hat{a}\beta + \\ \wp'^2 \hat{b}\delta \end{matrix} \quad = \quad \begin{matrix} \wp^2 d\bar{\beta} + \\ \wp'^2 b\bar{\alpha} \end{matrix} \quad \Rightarrow \quad \begin{matrix} \hat{a}\beta = d\bar{\beta} \\ \hat{b}\delta = b\bar{\alpha} \end{matrix}
$$

Generalized Lax pair for LSG_2

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Now consider a general 2×2 Lax pair with *one separable term* in each entry

$$L = \begin{pmatrix} aA & bB \\ cC & dD \end{pmatrix}$$

$$M = \begin{pmatrix} \alpha\Lambda & \beta\Xi \\ \gamma\Gamma & \delta\Delta \end{pmatrix}$$

Lower cases: a=a(l,m), upper cases A=A(n)

We are free to choose a spectral dependence so that the sought after equations arise.

The $(1,2)$ entry is

$$\begin{matrix} \hat{a}\beta A\Xi + \\ \hat{b}\delta B\Delta \end{matrix} = \begin{matrix} d\bar{\beta}D\Xi + \\ b\bar{\alpha}B\Lambda \end{matrix} \Rightarrow \begin{matrix} A = D \\ \Delta = \Lambda \\ A\Xi \neq B\Delta \end{matrix}$$

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By analyzing all entries, we find that LSG_2 comes from any Lax pair with the following spectral dependence:

$$L = \begin{pmatrix} aA & bB \\ cC & dD \end{pmatrix}, \quad M = \begin{pmatrix} \alpha\Lambda & \beta\Xi \\ \gamma\Gamma & \delta\Delta \end{pmatrix}$$

where

$$\begin{array}{rclcrcl} A &=& F_1/\Xi & & \Lambda &=& F_2/B \\ C &=& B\Gamma/\Xi & & & & \\ D &=& F_1/\Xi & & \Delta &=& F_2/B \end{array} \quad \Rightarrow \quad \begin{array}{rcl} \hat{a}\alpha + \hat{b}\gamma &=& a\bar{\alpha} + c\bar{\beta} \\ \hat{d}\delta + \hat{c}\beta &=& d\bar{\delta} + b\bar{\gamma} \\ \hat{a}\beta &=& d\bar{\beta} \\ \hat{b}\alpha &=& b\bar{\delta} \\ \hat{c}\alpha &=& c\bar{\delta} \\ \hat{a}\gamma &=& d\bar{\gamma} \end{array}$$

$F_1 \neq kF_2$, k a constant

General solution of the compatibility condition

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Need to solve the compatibility condition in full generality.

All Lax pairs studied so far have compatibility conditions that are solvable by using the parametrization:

$$\bar{a} - k_1 a = \hat{b} - k_2 b \qquad \Rightarrow \qquad \begin{aligned} a &= \hat{v} - k_2 v + k_3 k_1^l \\ b &= \bar{v} - k_1 v + k_4 k_2^m \end{aligned}$$

Using this tool, often in an exponentiated form, there has been no need to appeal to assumptions about the lattice terms.

Other equations from this Lax pair

We have constructed a family of Lax pairs, all for LSG_2, by:

- *Grouping the lattice terms* into the appropriate set of equations
- *Constructing a spectral dependence* that gives rise to those equations

How else can the lattice terms be grouped?

(1,2)

$$\begin{array}{ccc} \hat{a}\beta A\Xi & & b\bar{\alpha}B\Lambda \\ \hat{b}\delta B\Delta & & d\bar{\beta}D\Xi \end{array}$$

What other integrable equations can we find Lax pairs for?

Grouping terms

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Arbitrarily, start with the spectral terms in the (1,2) entry of the compatibility condition

$$\boxed{(1,2)} \quad \begin{matrix} A\Xi & B\Lambda \\ B\Delta & D\Xi \end{matrix}$$

Consider the different combinations of different sized groups in this entry.

Any combination chosen has implications for the other entries of the compatibility condition.

How the choice affects the other entries

Let's use the following groups in the (1,2) entry:

$$\boxed{(1,2)} \quad \begin{array}{ccc} A\Xi & & B\Lambda \\ & ◸\,| & \\ B\Delta & & D\Xi \end{array} \quad \Rightarrow \quad \begin{array}{rcl} A & = & F_1/\Xi \\ \Delta & = & F_1/B \\ \Lambda & = & (F_1+F_2)/B \\ D & = & F_2/\Xi \end{array}$$

Turning to the (2,1) entry:

$$\boxed{(2,1)} \quad \begin{array}{cc} C\Lambda & A\Gamma \\ D\Gamma & C\Delta \end{array} \quad \Rightarrow \quad \begin{array}{ccc} (F_1+F_2)\dfrac{C}{B} & & F_1\dfrac{\Gamma}{\Xi} \\ & |\,◹ & \\ F_2\dfrac{\Gamma}{\Xi} & & F_1\dfrac{C}{B} \end{array}$$

The resulting lattice term equations

These groupings lead to the following lattice term equations:

$$
\begin{aligned}
\hat{a}\alpha &= a\bar{\alpha} \\
\hat{b}\gamma &= c\bar{\beta} \\
\hat{d}\delta + \hat{c}\beta &= d\bar{\delta} + b\bar{\gamma} \\
\hat{a}\beta + \hat{b}\delta &= b\bar{\alpha} \\
kb\bar{\alpha} &= -d\bar{\beta} \\
\hat{c}\alpha &= a\bar{\gamma} + c\bar{\delta} \\
k\hat{c}\alpha &= -\hat{d}\gamma
\end{aligned}
$$

which are associated with the trivial pair of equations:

$$
\begin{aligned}
\lambda_2(\hat{x} - x) &= k(\underline{\lambda}_2\lambda - \underline{\lambda}\lambda_2) \\
\hat{x} - x + k(\underline{\lambda} - \lambda_2) &= k\frac{x}{\hat{\bar{x}}}(\mu\lambda - \lambda_2)
\end{aligned}
$$

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Completeness survey

Only two Lax pairs of the form

$$L = \begin{pmatrix} aA & bB \\ cC & dD \end{pmatrix}, \quad M = \begin{pmatrix} \alpha\Lambda & \beta\Xi \\ \gamma\Gamma & \delta\Delta \end{pmatrix}$$

Lead to unconstrained, nontrivial lattice equations.

The equations are:

- LSG_2, higher order lattice sine-Gordon
- $LMKdV_2$, higher order lattice modified KdV

and no others

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One can, in principle, find all Lax pairs of a general form with a set number of terms.

We do this by considering all possible groupings of lattice terms that arise in each entry of the compatibility condition.

It is also possible to conduct this analysis with other types of Lax pairs including differential, differential-difference and isomonodromy Lax pairs.