Tau functions, random processes and fermions on a lattice*

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(* Based on joint work with A. Yu. Orlov)

Outline

1. **Fermionic Fock space and VEV's**
   - Fermionic Fock space
   - Vacuum expectation values (VEV)
2. **Partitions, Maya diagrams and Schur functions**
   - Partitions, Young Diagrams, Maya Diagrams
   - Schur functions
3. **Tau Functions as Generators of Random Processes**
   - Generating function for determinantal processes
   - Large time asymptotic distributions
   - Decay of step function
   - Other relations to integrable systems: Bethe ansatz
4. **Tau functions as partition functions for Fermions on a lattice**
   - Partition functions for Fermion models on a lattice
   - Other related work

Fermionic Fock space

The **Fermionic Fock space** (exterior space)

$$\mathcal{F} := \Lambda\mathcal{H}$$

is a Clifford module for $\mathbf{Cliff}(\mathcal{H} \oplus \mathcal{H}^*) \sim \mathbf{gl}(\Lambda\mathcal{H})$.
Orthonormal basis for $\mathcal{H}$, and **dual basis for** $\mathcal{H}^*$

$$\{e_i\}_{i\in\mathbf{Z}}, \qquad \{\tilde{e}_i\}_{i\in\mathbf{Z}}, \qquad \tilde{e}_i(e_j) = \delta_{ij}$$

Linear elements of $\mathbf{Cliff}(\mathcal{H} \oplus \mathcal{H}^*)$ act by interior and exterior multiplication (**creation and annihilation operators**)

$$f_i := i(\tilde{e}_i), \quad \bar{f}_i := e_i\wedge$$

Anticommutation relations

$$[f_n, f_m]_+ = [\bar{f}_n, \bar{f}_m]_+ = 0, \quad [f_n, \bar{f}_m]_+ = \delta_{nm}$$

Fermi fields

$$f(x) := \sum_{k\in\mathbf{Z}} f_k x^k, \quad \bar{f}(y) := \sum_{k\in\mathbf{Z}} \bar{f}_k y^{-k-1},$$

Vacuum expectation values and Wick's theorem

Vacuum and n-charged vacuum

$$
\begin{aligned}
|0\rangle &:= e_0 \wedge e_1 \wedge e_2 \wedge \cdots \\
|n\rangle &= f_{n-1} \cdots f_0 |0> \ \text{ for } n \geq 0, \\
|n\rangle &= \bar{f}_n \cdots \bar{f}_0 |0>, \ \text{ for n} < 0 \\
f_m|0\rangle = 0 \quad (m < 0), &\quad \bar{f}_m|0\rangle = 0 \quad (m \geq 0), \\
\langle 0|f_m = 0 \quad (m \geq 0), &\quad \langle 0|\bar{f}_m = 0 \quad (m < 0)
\end{aligned}
$$

Wick's theorem implies that for any $\{w_k \in \mathcal{H}, \bar{w}_k \in \mathcal{H}^*\}_{k=1,\ldots N}$

$$
\langle 0|w_1 \cdots w_N \bar{w}_N \cdots \bar{w}_1|0\rangle = \det\left(\langle 0|w_i \bar{w}_j|0\rangle\right)|_{i,j=1,\ldots,N}
$$

In particular, this implies (for $n, \ m \in \mathbf{N}$)

$$
\langle n-m|f(x_1)\cdots f(x_n)\bar{f}(y_1)\cdots \bar{f}(y_m)|0\rangle = \frac{\Delta_n(x)\Delta_m(y)}{\prod_{\substack{i=1,\ldots,n \\ j=1,\ldots,m}} (x_i - y_j)}
$$

Partitions, Young diagrams

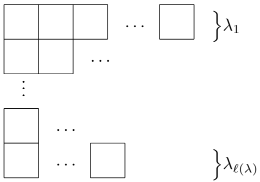
Partitions

$$\lambda = (\lambda_1, \ldots \lambda_{\ell(\lambda)}), \quad \lambda_1 \geq \cdots \geq \lambda_{\ell(\lambda)}, \quad \lambda_i \in \mathbf{N}^+$$

Length $\ell(\lambda)$ and weight

$$|\lambda| := \sum_{i=1}^{\ell(\lambda)} \lambda_i$$

Young diagram



Frobenius notation:

$$(\alpha_1, \ldots \alpha_k | \beta_1, \ldots \beta_k)$$

Partition basis and Maya diagrams

For each integer N, and partition λ of length $\ell(\lambda)$

$$\lambda := \lambda_1 \geq \lambda_2 \geq \dots \quad \lambda_\ell(\lambda) > 0 \in \mathbf{N} \qquad \lambda_i := 0 \forall i > \ell(\lambda),$$

define **particle positions** (levels): $\{l_i := \lambda_i - i + N, \}_{i=1,\dots\infty}$ to form a **basis vector**:

$$|\lambda, N\rangle := (-1)^{\sum_{i=1}^k \beta_i} f_{N+\alpha_k} \bar{f}_{N-1-\beta_k} \cdots f_{N+\alpha_1} \bar{f}_{N-1-\beta_1} |N\rangle$$
$$|N\rangle := |0, N\rangle \quad (\text{charge } N \ \text{ vacuum})$$

where $(\alpha_1, \dots \alpha_k | \beta_1, \dots \beta_k)$ is the **Frobenius notation** for a partition. ((α_i, β_i) = no. of blocks to the right, resp. beneath the diagonal block in the (i, i) position) and **Maya Diagram**

Maya diagrams

○ N+2	○ N+2
○ N+1	● N+1
○ N	○ N
● N-1	○ N-1
● N-2	● N-2
● N-3	● N-3

Fig.1 Dirac sea of level N. $|0;N\rangle$ Fig.2 Maya diagram for $|(2,1);N\rangle$

Schur functions

The Schur function

$$s_\lambda(\mathbf{t}) := \mathrm{tr}(\rho_\lambda(g)), \quad g \in GL(N)$$
$$\mathbf{t} := (t_1, t_2, \cdots), \quad t_i := \frac{1}{i}\mathrm{tr}(g^i), \quad g \in GL(N)$$

is the character of the irreducible representation

$$\rho_\lambda : GL(N) \longrightarrow \mathrm{End}(T^{(\lambda)} \subset (\mathbf{C}^N)^{\otimes|\lambda|})$$

obtained by restricting to tensors of symmetry type λ.
It is expressed as a determinant by the Jacobi-Trudy formula:

$$s_\lambda(\mathbf{t}) = \det(h_{\lambda_i - i + j})|_{1 \le i,j \le \ell(\lambda)},$$

where $h_j(\mathbf{t})$ is the elementary Schur function (complete symmetric function) determined by the generating function formula:

$$e^{\sum_{i=1}^{\infty} t_i z^i} = \sum_{j=0}^{\infty} h_j(\mathbf{t}) z^j$$

Fermionic form for KP and $2-D$ Toda Tau functions

KP τ-function

$$\tau_{N,g}(\mathbf{t}) := \langle N|\gamma(\mathbf{t})g|N\rangle$$
$$\gamma(\mathbf{t}) := e^{\sum_{i=1}^{\infty} t_i H_i}, \quad g = e^{\mathcal{A}} \in \mathbf{GL}(\infty)$$
$$H_i := \sum_{j\in\mathbf{Z}} f_i \bar{f}_{j+i}, \quad \mathcal{A} := \sum_{ij} a_{ij} f_i \bar{f}_j \in \mathbf{gl}(\infty)$$

2-D Toda τ-function

$$\tau_{N,g}(\mathbf{t},\tilde{\mathbf{t}}) := \langle N|\gamma(\mathbf{t})g\tilde{\gamma}(\tilde{\mathbf{t}})|N\rangle$$
$$\gamma(\mathbf{t}) := e^{\sum_{i=1}^{\infty} t_i H_i}, \quad \tilde{\gamma}(\tilde{\mathbf{t}}) := e^{\sum_{i=1}^{\infty} \tilde{t}_i H_{-i}}, \quad g = e^{\mathcal{A}} \in \mathbf{GL}(\infty)$$
$$H_i := \sum_{j\in\mathbf{Z}} f_i \bar{f}_{j+i}, \quad \mathcal{A} := \sum_{ij} a_{ij} f_i \bar{f}_j \in \mathbf{gl}(\infty)$$

Schur function as τ functions and special values

Schur functions as KP τ-function

$$\langle n|e^{H(\mathbf{t})} = \sum_{\lambda\in P}\langle\lambda, n|s_\lambda(\mathbf{t}), \qquad e^{\bar{H}(\tilde{\mathbf{t}})}|n\rangle = \sum_{\lambda\in P} s_\lambda(\tilde{\mathbf{t}})|\lambda, n\rangle$$

$$\langle n|e^{H(\mathbf{t})}|\lambda, n\rangle = s_\lambda(\mathbf{t}), \qquad \langle\lambda, n|e^{\bar{H}(\tilde{\mathbf{t}})}|n\rangle = s_\lambda(\tilde{\mathbf{t}})$$

For a partition $\lambda = (\lambda_1, \lambda_2, \dots)$, let

$$h_i := n + \lambda_i - i, \quad (1 \le i \le n)$$
$$\mathbf{t}_\infty := (1, 0, 0, \dots)$$

where $n \ge \ell(\lambda))$. Then

$$s_\lambda(\mathbf{t}_\infty) = \frac{\Delta(h)}{\prod_{i=1}^{n} h_i!} \ ,$$

Schur function expansions

KP τ-function expansion

$$\tau_{N,g}(\mathbf{t}) = \sum_{\substack{\lambda \\ \ell(\lambda) \leq N}} \pi_\lambda^N s_\lambda(\mathbf{t})$$

$$\text{where} \quad \pi_\lambda^{\mathrm{N}} = <\lambda, N \mid e^{\mathcal{A}} \mid \mu, N>$$

are the Plücker coordinates of the element $g(\mathcal{H}_+) \in Gr_{\mathcal{H}_+}(\mathcal{H})$ of the Hilbert space Grassmannian $Gr_{\mathcal{H}_+}(\mathcal{H})$.

2-D Toda τ-function

$$\tau_{N,g}(\mathbf{t}, \tilde{\mathbf{t}}) = \sum_{\substack{\lambda,\mu, \\ \ell(\lambda)\ell(\mu) \leq N}} B_{\lambda\mu}^N s_\lambda(\mathbf{t}) s_\mu(\tilde{\mathbf{t}})$$

where

$$B_{\lambda\mu}^N = <\lambda, N \mid e^{\mathcal{A}} \mid \mu, N>$$

Random processes on partitions

$gl(\infty)$ action on $\mathcal{F}$

$$gl(\infty) : \mathcal{F} \to \mathcal{F}$$
$$gl(\infty) = \mathrm{span}\{E_{ij} := f_i \bar{f}_j\}_{i,j \in \mathbf{Z}}$$

This determines weighted actions on **Maya diagrams**

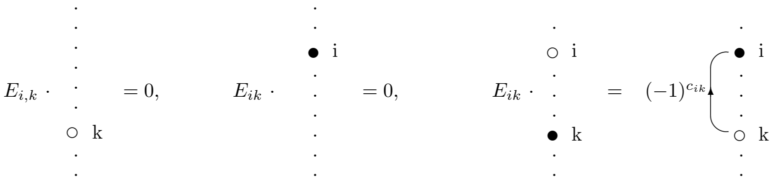
$$\mathcal{A} := \sum_{ij} a_{ij} f_i \bar{f}_j \in gl(\infty)$$
$$\mathcal{A} : |\lambda; N > \to \sum_{ij} a_{ij} f_i \bar{f}_j |\lambda; N >= \sum_{N', \mu} C^{N'N}_{\mu\lambda} |\mu, N' >$$

For positive coefficients a_{ij}, we can view

$$< \lambda, N' | \mathcal{A}^k | \mu, N >$$

as an (unnormalized) **transition weight** after k (discrete) time steps.

Action of $E_{i,k}$ on Maya diagrams



1. Elimination of Maya diagrams

2. Nontrivial action

The τ function as a generating function for transition probabilities

Assume $\mathcal{A}$ preserves N, and use

$$< N|\tilde{\gamma}(\tilde{\mathbf{t}})|\mu, N >= s_\mu(\tilde{\mathbf{t}}) \qquad < \lambda, N|\gamma(\mathbf{t})|N >= s_\lambda(\mathbf{t})$$

$$\tau_{N,g}(\mathbf{t}, \tilde{\mathbf{t}}) = \sum_{\lambda,\mu} \sum_{k=0}^{\infty} \frac{1}{k!} < \lambda, N|\mathcal{A}^k|\mu, N > s_\mu(\mathbf{t}) s_\lambda(\tilde{\mathbf{t}})$$

$$< \lambda, N|\mathcal{A}^k|\mu, N > := \frac{1}{k!} < \lambda, N|\mathcal{A}_+^k + \mathcal{A}_-^k|\mu, N > \qquad (\pm \text{ permutations})$$

The **transition probability** in k time steps is

$$P_k((\mu, N) \to (\lambda, N)) = \frac{W_{N,\mu\to\lambda}(k)}{\sum_\nu W_{N,\mu\to\nu}(k)},$$

$$W_{N,\mu\to\nu}(k) :=< \lambda, N|\mathcal{A}_+^k|\mu, N > - < \lambda, N|\mathcal{A}_-^k|\mu, N >$$

Random turn non-intersecting walkers

Example. Random turn non-intersecting walkers

$$\mathcal{A} := \sum_{i\in\mathbf{Z}} (p_{-,i} f_{i-1} \bar{f}_i + p_{+.i} f_{i+1} \bar{f}_i), \qquad p_l, p_r \geq 0,$$

For comparison with equilibrium Fermi models, we parametrize:

$$p_{+,i} = e^{-U_i + U_{i-1}}, \quad p_{-,i} e^{-U_{i-1}+U_i}$$

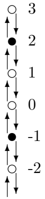
(Later, U_i will denote instead the **energy at lattice site $i \in \mathbf{Z}$**). Define

$$U_\lambda(n) := \sum_{i=1}^{\infty} \left(U_{\lambda_i - i + n} - U_{-i+n} \right)$$

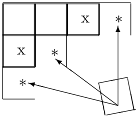
Mostly, we take the case of constant, symmetric rate

$$p_{+,i} = p_{-,i}^{-1} = e^{-U_i + U_{i-1}} = r(i) = r$$

Two realizations of a random turn step

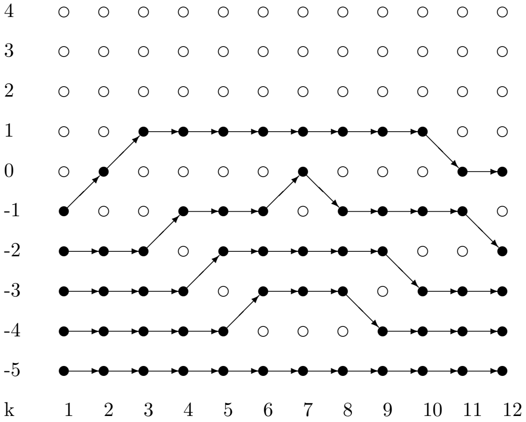


1. Random turn steps of particles in a Maya diagram.



2. Random adding or removing a box in a Young diagram corresponding to up or down hops of particles in the Maya diagram. At each time step a box is either added at any of the vacant places marked by a $*$, or a box marked by x is removed, with probabilities p_+ and p_- respectively.

Sample path of random turn non-intersecting walkers



$$\mathcal{A} = \sum_i (p_+ f_{i-1} \bar{f}_i + p_- f_{i+1} \bar{f}_i)$$

Random turn non-intersecting walkers

Transition rate

$$W_{\lambda' \to \lambda}(\mathrm{T}) = e^{U_{\lambda'} - U_{\lambda}} \mathrm{N}_{\lambda,\lambda'}(\mathrm{T})$$

where $\mathrm{N}_{\lambda,\lambda'}(\mathrm{T})$ is the number of paths from λ' to λ in T time steps.

In particular, for $\lambda' = 0$

Number of paths from $|0\rangle$ to $|\lambda\rangle$

$$\mathrm{N}_{\lambda,0}(\mathrm{T}) = 2^{\frac{|\lambda| - \mathrm{T}}{2}} \frac{\mathrm{T}!}{(\frac{\mathrm{T} - |\lambda|}{2})!} \cdot s_{\lambda}(\mathbf{t}_{\infty})$$

Random turn non-intersecting walkers

Total transition probability

$$Z_{\lambda'}(\mathrm{T}) = \sum_{\lambda} e^{U_{\lambda'} - U_{\lambda}} \mathrm{N}_{\lambda,\lambda'}(\mathrm{T})$$

Starting at $|\,0\rangle$, the probability of finding the hard-core particles $\{U_i, i \in \mathbb{Z}\}$) in a configuration $|\lambda\rangle$ after T, steps is

$$P_{0\to\lambda}(\mathrm{T}) = Z_0(\mathrm{T})^{-1} e^{-U_\lambda} \mathrm{N}_{\lambda,0}(\mathrm{T})$$

For time duration $\mathrm{T} = 2m + |\lambda|$ we obtain

$$W_{0\to\lambda}(\mathrm{T}) = \mathrm{T}! \left(\frac{1}{2^m m!}\right) e^{-U_\lambda} s_\lambda(\mathbf{t}_\infty) = \mathrm{T}! 2^{\frac{|\lambda| - \mathrm{T}}{2}} \frac{1}{(\frac{\mathrm{T} - |\lambda|}{2})!} \cdot e^{-U_\lambda} s_\lambda(\mathbf{t}_\infty)$$

Large time asymptotics in the continuum limit

Let R be the **asymptotic length** after T time steps, h the **particle location** and $\sigma(h)$

$$0 \le \sigma(h) \le 1$$

the **density of particles** in the continuum limit. Then

$$\int_{-R}^{\infty} \sigma(h) dh = R$$

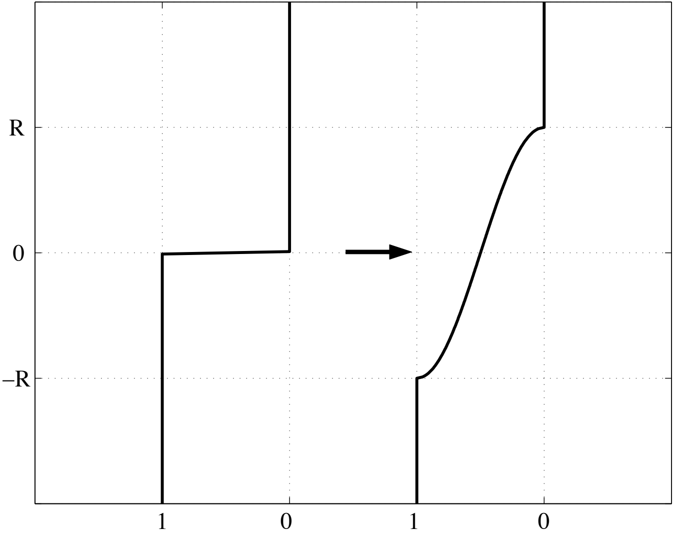
and

$$R \sim 2\sqrt{\frac{\mathrm{T}}{1 + r^{-2}}}, \quad \mathrm{T} \to \infty$$

Asymptotic density distribution (cf. Vershik-Kerov (1977)

$$\sigma(h) = \frac{1}{2} - \frac{1}{\pi} \arcsin\left(\frac{h}{R}\right), \quad h \in [-R, R]$$
$$\sigma(h) = 1, \quad h < -R; \qquad \sigma(h) = 0, \quad h > R$$

Decay of step function for constant hopping rate as $T \to \infty$



Asymmetric Exclusion Process (ASEP)

Other relations to integrable systems: **Bethe ansatz solution of ASEP**, using equivalence with integrable spin models

K-H Gwa and H. Spohn
"Six-vertex model, roughened surfaces and an asymmetric spin hamiltonian", *Phys. Rev. Let.* **68**, 725 - 728 (1992); "Bethe solution for the dynamical-scaling exponent of the noisy Burgers equation", *Phys. Rev.* **A 46**, 844-854 (1992)

G.M. Schütz
"Exact solution of the master equation for the asymmetric exclusion process", *J. Stat. Physics* **88**, 427-445 (1997)

C. Tracy and H. Widom
"Integral formulas for the simple asymmetric exclusion process", *Commun. Math. Phys.* **279** 815-844 (2008)

Bethe ansatz solution of ASEP

Continuous time limit (ASEP), for a finite number of particles.
Particle positions: $X = (x_1, \dots x_n), \quad Y = (y_1, \dots y_n)$

Let $u_Y(X;t) :=$ probability of being in state X at time t, given they are in state Y at $t = 0$.

Master equation:

$$\frac{du_Y}{dt} = \sum_{i=1}^{n} \left(p_r u_Y(X_i^-;t) + p_l u_Y(X_i^+;t) - u(X;t) \right)$$
$$X_i^{\pm} := (x_1, \dots x_{i-1}, x_i \pm 1, \dots, x_n)$$

Initial and **boundary** conditions:

$$\begin{aligned} u_Y(X;0) &= \delta_{X,Y} \\ u_Y((x_1, \dots x_i, x_i + 1, \dots x_n);t) &= p_r u_Y((x_1, \dots, x_i, x_i, \dots x_n);t) \\ &\quad + p_l u_Y((x_1, \dots, x_i + 1, x_i + 1, \dots x_n);t) \end{aligned}$$

Bethe ansatz solution

Integral formula for transition probabilities

(Tracy, Widom (2007))

$$u_Y(X;t) = \sum_{\sigma \in S_n} \left(\frac{1}{2\pi i}\right)^n \prod_{j=1}^{n} \oint_{\xi_j=0} A_\sigma \xi_{\sigma(j)}^{x_j - y_{\sigma(j)} - 1} e^{\sum_{j=1}^{n} \epsilon(\xi_j) t} d\xi_j$$

where

$$A_\sigma := \prod_{\mathrm{inversions}(\alpha,\beta)\subset\sigma} \left(-\frac{p_r + p_l \xi_\alpha \xi_\beta - \xi_\alpha}{p_r + p_l \xi_\alpha \xi_\beta - \xi_\beta}\right)$$

$$\epsilon(\xi) := p\xi^{-1} + q\xi - 1$$

Fermi models on a lattice

Energy at lattice site $i \in \mathbf{Z}$: U_i
Total energy of noninteracting configuration (λ, n)

$$U_\lambda(n) = \sum_{i=1}^{\infty} \left(U_{\lambda_i - i + n} - U_{-i+n} \right)$$

Partition function without interactions

$$e^{-\mathrm{F}_0} = \sum_\lambda e^{-U_\lambda(n)}$$

Lattice Coulomb gas interaction (L occupied sites, unit charge)

$$\mathrm{E}_\lambda = -\log s_\lambda(\mathbf{t}_\infty) := \log \frac{\prod_{i<j}^{L}(h_i - h_j)}{\prod_{i=1}^{L} h_i!}, \quad \mathbf{t}_\infty := (1, 0, 0, \dots)$$
$$h_i = \lambda_i - i + L,\ i = 1, \dots, L,\ L \geq \ell(\lambda)$$

Fermi models on a lattice (A. Yu. Orlov and J.H. (2007))

Partition function for interacting Coulomb gas, charge $= q$

$$e^{-\mathrm{F}_q(n,t_1,\bar{t}_1)} = \sum_\lambda e^{-U_\lambda - q^2\,\mathrm{E}_\lambda + q^2|\lambda|\log(t_1\bar{t}_1)}$$

$$= \sum_\lambda e^{-U_\lambda}\left(s_\lambda(\mathbf{t}_\infty)\right)^{q^2}(t_1\bar{t}_1)^{|\lambda|}$$

Partition function as a τ function for $q^2 = 2$

$$e^{-\mathrm{F}_2(n,t_1,\bar{t}_1)} := \sum_\lambda e^{-U_\lambda - 2\,\mathrm{E}_\lambda + |\lambda|\log(t_1\bar{t}_1)}$$

$$= \sum_\lambda e^{-U_\lambda}(t_1\bar{t}_1)^{|\lambda|}\left(s_\lambda(\mathbf{t}_\infty)\right)^2$$

$$= c_n^{-1}\langle n|e^{t_1 H_1}e^{\sum_{i\geq 0}U_i f_i\bar{f}_i - \sum_{i<0}U_i\bar{f}_i f_i}e^{\bar{t}_1 H_{-1}}|n\rangle$$

Fermi models on a lattice

More generally

$$\tau(n, \mathbf{t}, U, \bar{\mathbf{t}}) = \langle n | e^{H(\mathbf{t})} e^{\sum_{i\geq 0} U_i f_i \bar{f}_i - \sum_{i<0} U_i \bar{f}_i f_i} e^{\bar{H}(\bar{\mathbf{t}})} | n \rangle$$
$$= c_n \sum_{\lambda} e^{-U_\lambda} s_\lambda(\mathbf{t}) s_\lambda(\bar{\mathbf{t}})$$

corresponds to **interaction energy**

$$I_\lambda = \log s_\lambda(\mathbf{t}) + \log s_\lambda(\bar{\mathbf{t}})$$

Remark: If

$$U_i = \sum_{m\neq 0}^{\infty} i^m \tilde{t}_m, \quad i \in \mathbb{Z}$$

$e^{-\mathrm{F}_2(n, t_1, \bar{t}_1)}$ is a TL τ-function.

Fermi models on a lattice

Partition function as a τ function for $q^2 = 1$

$$
\begin{aligned}
e^{-\mathrm{F}_1(n)} &:= \frac{1}{n!} \sum_{h_1,\dots,h_n \geq 0} \prod_{i=1}^{n} \frac{e^{-U_{h_i}+U_{-i+n}}}{h_i!} \prod_{i<j}^{n} |h_i - h_j| \\
&= c_n \sum_{\substack{\lambda \\ \ell(\lambda)\leq n}} e^{-U_\lambda(n)} s_\lambda(\mathbf{t}_\infty) \\
&= c_n \sum_{k=1}^{n} \sum_{h_1>\cdots>h_n\geq 0} \prod_{i=1}^{n} \frac{e^{-U_{h_i}+U_{-i+n}}}{h_i!} \prod_{i<j}^{n} (h_i - h_j) \\
&= \sum_{\substack{\lambda \\ \ell(\lambda)\leq n}} \langle \lambda, n | e^{\sum_{i\geq 0} U_i f_i \bar{f}_i - \sum_{i<0} U_i \bar{f}_i f_i} e^{H_{-1}} | 0, n \rangle
\end{aligned}
$$

Remark: The **grand partition function** $\mathbf{Z}^G(n,\mu) := \sum_{n=0}^{\infty} e^{n\mu}$ is an *infinite soliton* BKP τ function (Kac, Van de Leur)

Further relations of random processes to integrable systems

Tau function as partition function for Fermion statistical ensembles

- J. Harnad and A. Yu. Orlov, “Fermionic construction of tau functions and random processes”, *Physica* **D235** 168-206 (2007) arXiv:0704.1157

Tau functions as weights on 2-D partitions (path space weight for 1-D partitions)

- A. Okounkov and N. Reshetikhin, “Correlation function of Schur process with application to local geometry of a random 3-dimensional Young diagram ”*J. Amer. Math. Soc.* **16** 581-603 (2003); “Random skew plane partitions and the Pearcey process ” *Commun. Math. Phys.* **269**, (2007)

Asymptotics of random partitions, growth problems, limiting shapes

- A. Borodin and G. Olshanski, “Z-measures on partitions and their scaling limits ” *Eyr. J. Comb.* **26**, 795-834 (2005); “Random partitions and the gamma kernel” *Adv. Math.* **194** , 141-202 (2005)
- R. Kenyon, A. Okounkov and S. Sheffield, “Dimers and amoebae ” *Ann. Math.* **163** , 1019-1056 (2006)