A matrix valued version of the bispectral problem

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Thanks to the organizers and to the audience for sticking to the last talk.

Pictures for motivation.

My theme: some of the mathematical explicit results that come from areas such as the bispectral problem (or your favorite area: integrable systems, etc. etc.) can be used to formulate and sometimes solve inverse problems that are motivated by areas such as medical imaging, geophysical imaging, etc.

In particular, the use of low energy probes (optical or infrared tomography), leads to the need to use some stochastic processes and their corresponding inverse problems.

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An Identification Problem for Multiterminal Networks: Solving for the Traffic Matrix from Input-Output Measurements

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Abstract. We consider the problem of determining the unknown characteristics of a *random routing strategy* from *end-to-end* measurements. More specifically, we construct a Markov chain that models the traffic of messages in a multiterminal network consisting of *input*, *intermediate*, and *output* terminals. The topology of the network is assumed to be known, but the Markovian routing strategy is not known. We solve the problem of determining the unknown one-step transition probability matrix of our random walk from input-output measurements of "travel time." We give explicit inversion formulas (up to a natural gauge) in a nontrivial example. The result holds for a large (but not arbitrary) class of multiterminal networks, many of which are indicated here. The networks that we display here are constructed in a canonical fashion from certain graphs. Some of these graphs as well as the way to go from graphs to networks are also displayed. One example of a graph for which our method works is the edge graph of a hypercube in any dimension.

1. Introduction

Consider the following exceedingly simple network, consisting of an input terminal I, a intermediate terminal H, and an output terminal O, as seen in Figure 1. Messages can be sent from I to O in two ways: either directly or by means of the intermediate terminal H. The probability of a direct transmission is denoted

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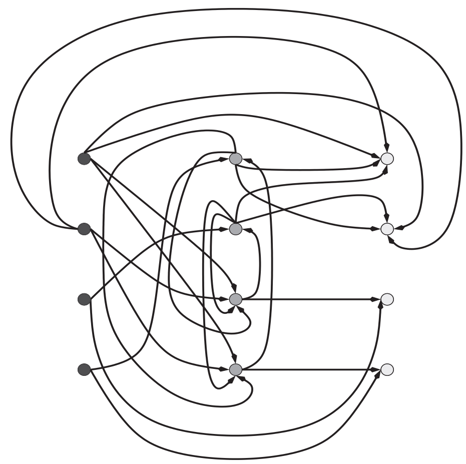


Figure 8. The network arising from a two-point star.

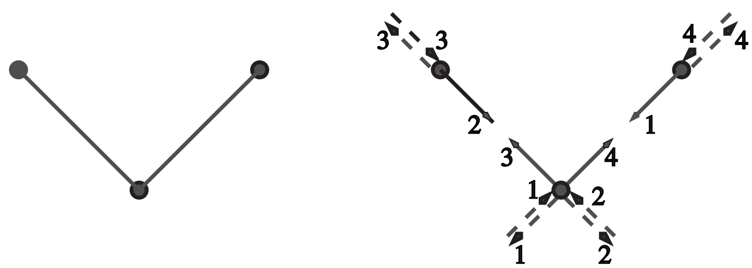


Figure 9. The two-point star graph on the left. On the right, a diagram representing the states: dotted arrows are incoming or outgoing depending on how they point; solid arrows are intermediate states.

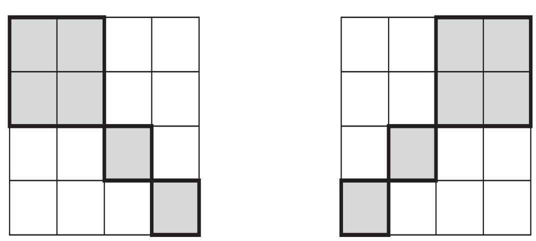


Figure 10. The block structure of P_{IO} and P_{HO} on the left, and that of P_{HH} and P_{IH} on the right. These correspond to the network arising from the two-point star.

The scalar bispectral problem, considered in this form in a joint paper with Hans Duistermaat, 1986.

Find all nontrivial cases where a function $\varphi(x,k)$ satisfies

$$L\left(x, \frac{d}{dx}\right)\varphi(x,k) \equiv (-D^2 + V(x))\varphi(x,k) = k\varphi(x,k)$$

as well as

$$B\left(k, \frac{d}{dk}\right)\varphi(x,k) \equiv \left(\sum_{i=0}^{M} b_i(k)\left(\frac{d}{dk}\right)^i\right)\varphi(x,k) = \Theta(x)\varphi(x,k).$$

All the functions $V(x), b_i(k), \Theta(x)$ are, in principle, arbitrary except for smoothness assumptions. Notice that here M is arbitrary (finite).

The complete answer to the problem is given as follows:

If $M = 2$, then $V(x)$ is (except for translation) either c/x^2 or ax, i.e., we have a Bessel or an Airy case. If $M > 2$, there are two families of solutions.

a) L is obtained from $L_0 = -D^2$ by a finite number of Darboux transformations ($L = AA^* \to \tilde{L} = A^*A$). In this case V is a rational solution of the Korteweg-deVries equation and all rational solutions of KdV decaying at infinity show up in this fashion.

b) L is obtained from $L_0 = -D^2 + \frac{1}{4x^2}$ after a finite number of rational Darboux transformations.

It was later observed by Magri and Zubelli that in the second case we are dealing with rational solutions of the Virasoro or master symmetries of KdV.

My reasons for originally asking this question, 1980, can be traced back to my effort to understand some work on “time-and-band-limiting” that had led me to isolate certain properties of well known special functions. This was motivated by my work on the "limited angle problem" in medical imaging using X-rays, i.e. CAT scanners.

Slepian, Landau and Pollack (following work by Claude Shannon)

There is a strong connection between the bispectral problem and the issue of time-and-band-limiting but I will not allude to it here.

The work with Hans gave rise to a large number of papers by other people, In particular A. Veselov, M. Gekhtman, J. Harnard, A. Zhedanov and many others, such as George Wilson, Luc Haine, Alex Kasman, M. Rothstein, Emil Horozov, Milen Yakimov, Boshko Bakalov, Jose Liberati, George Latham, Vyacheslav Spiridonov, Plamen Iliev, Franco Magri, Jorge Zubelli, Yuri Berest, Michael Reach, Paul Wright, Paul Terwilliger, Oleg Chalykh, Frank Nijhoff.

It may be appropriate to observe that what we are calling the Darboux process has been reinvented many times, including in the work of some rather well known people, such as E. Schroedinger himself. In some very nice papers A. Zhedanov talks about the Geronimus transformation, from 1940, and its inverse the Christoffel transform. It is clear that the first one has a lot in common with what we are calling the Darboux transformation.

The scalar Bochner Krall problem, another form of the bispectral problem.

In a series of papers with Luc Haine, and then with Luc and Emil Horozov, we noticed that a large class of polynomials

$$p_n(k)$$

that satisfy three term recursion relations in the variable n, as well as differential equations in the variable k can be obtained by an application of a similar Darboux transformation starting from the so called classical orthogonal polynomials of Jacobi, Laguerre and Hermite.

In this case one goes from a tridiagonal matrix

$$L_0 = AB$$

to a new tridiagonal matrix L by factorizing the first one (or a function of it) as a product of two bidiagonal matrices. This form of the Darboux process had been used and proposed by V. Matveev.

In this case one runs into the Toda flows and its master symmetries.

Moving into a matrix valued version of the bispectral problem (Bochner-Krall).

Matrix valued orthogonal polynomials

Given a self-adjoint positive definite matrix valued weight function $W(t)$, Krein, 1949, considers the skew symmetric bilinear form defined for any pair of matrix valued functions $P(t)$ and $Q(t)$ by the numerical matrix

$$\langle P, Q\rangle = \langle P, Q\rangle_W = \int_{\mathbb{R}} P(t)W(t)Q^*(t)dt,$$

where $Q^*(t)$ denotes the conjugate transpose of $Q(t)$.

Proceeding as in the case of a scalar valued inner product Krein proves that there exists a sequence $(P_n)_n$ of matrix polynomials, orthogonal with respect to W, with P_n of degree n and monic.

Krein goes on to prove that any sequence of monic orthogonal matrix valued polynomials $(P_n)_n$ satisfies a three term recurrence relation

$$A_n P_{n-1}(t) + B_n P_n(t) + P_{n+1}(t) = t P_n(t), \tag{1}$$

where P_{-1} is the zero matrix and P_0 is the identity matrix. These coefficient matrices enjoy certain properties: in particular the A_n are nonsingular.

The equation above can be rewritten as

$$\mathscr{L} P_n(t) = t P_n(t)$$

with a matrix $\mathscr{L}$ such as the one that has appeared in previous sections.

To place ourselves in the context of the bispectral problem we consider matrix valued polynomials $(P_n)_n$ satisfying not only the equation above but also "right hand side" differential equations of the form:

$$P_n D = \Lambda_n P_n \qquad \text{for all} \quad n \geq 0 \tag{2}$$

with Λ_n a matrix valued eigenvalue and D a differential operator of order s with matrix coefficients given by

$$D = \sum_{i=0}^{s} \partial^i F_i(x), \qquad \partial = \frac{d}{dx},$$

which acts on $P(x)$ by means of

$$PD = \sum_{i=0}^{s} \partial^i (P)(x) F_i(x).$$

One can see that the differential operators D that correspond to a fixed family of polynomials form an associative algebra which in general is non-commutative, and which I denote by $\mathcal{D}(W)$.

A result from a joint paper with J. Alfredo Tirao

Let $W(x)$ be a weight matrix on the real line, $\{P_n\}_{n\geq 0}$ the corresponding sequence of monic orthogonal polynomials and L the block tridiagonal matrix that gives $LP = xP$. If D in $\mathcal{D}(W)$ and Λ is the block diagonal matrix as above with $\Lambda_n = \Lambda_n(D)$ we have

$$(\text{ad } L)^{m+1}(\Lambda) = 0 \tag{3}$$

for some m. Conversely, if Λ is a block diagonal matrix satisfying this condition for some $m \geq 0$, then there is a unique differential operator D in $\mathcal{D}(W)$ such that $\Lambda_n = \Lambda_n(D)$ for all $n \geq 0$. Moreover the order of D is equal to the minimum m satisfying the condition above.

Our proof produces the differential operator D in $\mathcal{D}(W)$ explicitly.

The structure of some of these algebras $\mathcal{D}(W)$ has been conjectured in a paper of M. Castro and myself, and a first such instance has been settled completly by J. Alfredo Tirao.

The problem of exhibiting elements of this algebra that have a minimal order will occupy us in a few examples in the next two sections.

The problem of constructing the BASIC cases where this holds is not simple.

There are two main sourecs of examples so far

the theory of matrix valued spherical functions, developed a while back by J. Alfredo Tirao gives rise to many such examples (papers by Grünbaum, Pacharoni and Tirao).

the solution of a system of matrix valued differential equations, has led myself and Antonio Duran to find a small collection of examples.

There is also work by other people, such as P. Roman, M. de la Iglesia,...

Instead of showing many examples now I will concentrate on

showing how the Darboux method can be applied in this matrix valued case

showing that some of the examples can be used to provide explict solutions to some problems involving natural Markov chains.

A matrix valued version of the Darboux process for a difference operator

Consider the block tridiagonal matrix L_0

$$L_0 = \begin{pmatrix} B_0 & I & & \\ A_1 & B_1 & I & \\ & \ddots & \ddots & \ddots \end{pmatrix}$$

where all the matrices A_i, B_i are of size $N \times N$ and I denotes the $N \times N$ identity matrix.

If we try to factorize this in the form

$$L_0 = \alpha\beta$$

where

$$\alpha = \begin{pmatrix} \alpha_0 & I & & \\ 0 & \alpha_1 & I & \\ & \ddots & \ddots & \ddots \end{pmatrix}$$

and

$$\beta = \begin{pmatrix} I & & \\ \beta_1 & I & \\ & \ddots & \ddots \end{pmatrix}$$

with all the matrices α_i, β_i are of size $N \times N$, and then define the matrix

$$L = \beta\alpha$$

we have that

$$L = \begin{pmatrix} \tilde{B}_0 & I & & \\ \tilde{A}_1 & \tilde{B}_1 & I & \\ & \ddots & \ddots & \ddots \end{pmatrix}$$

where all the matrices $\tilde{A}_i, \tilde{B}_i$ are of size $N \times N$, and the following relations hold

$$\beta_n = B_{n-1} - \alpha_{n-1}, \alpha_n = A_n \beta_n^{-1}$$

which then gives

$$\tilde{B}_n = B_n - \beta_{n+1} + \beta_n = B_{n-1} - \alpha_{n-1} + \alpha_n$$

and

$$\tilde{A}_n = \beta_n A_{n-1} \beta_{n-1}^{-1} = \alpha_n^{-1} A_n \alpha_{n-1}.$$

These expressions are valid for $n = 2, 3, \ldots$ in the case of $\tilde{A}_n$ and for $n = 1, 2, \ldots$ in the case of $\tilde{B}_n$.

Above we take $\beta_0 = 0$, so that $\tilde{B}_0 = \alpha_0 = B_0 - \beta_1$. We also need to take $\tilde{A}_1 = (B_0 - \alpha_0)\alpha_0$.

A moment's thought gives that once L_0 is given, the only free parameter is the matrix α_0.

A few examples

Here we consider in detail few examples of the matrix version of the basic Darboux process described above.

For $\lambda > 3/2$ consider the monic matrix valued polynomials which are orthogonal with respect to the weight matrix

$$W(x) = ((2-x)x)^{\lambda-3/2} \begin{pmatrix} 1 & x-1 \\ x-1 & 1 \end{pmatrix}, \ x \in [0,2].$$

Let

$$L_0 = \begin{pmatrix} B_0 & I & & \\ A_1 & B_1 & I & \\ & \ddots & \ddots & \ddots \end{pmatrix}$$

be the corresponding block tridiagonal matrix with

$$B_n = \frac{1}{2}\frac{\lambda - 1}{(n+\lambda)(n+\lambda-1)}S + I$$

$$A_n = \frac{n(n+2\lambda-2)}{4(n+\lambda-1)^2}I.$$

Here $S = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

These polynomials can be seen to be joint eigenfunctions of a first order differential operator, an observation that was made for a special value of λ in papers with M. Castro.

If α_0 is an arbitrary matrix we can consider the monic polynomials that result from one application of the Darboux process to the block tridiagonal matrix L_0 with free parameter α_0.

We can see that for an invertible symmetric α_0 the new orthogonality weight is given by

$$\widetilde{W}(x) = (2-x)^{\lambda-3/2}x^{\lambda-5/2}\begin{pmatrix} 1 & x-1 \\ x-1 & 1 \end{pmatrix}$$

$$- \frac{-2^{2\lambda}Be\left(\frac{2\lambda-1}{2}, \frac{2\lambda-1}{2}\right)}{4(2\lambda-3)}\left(\begin{pmatrix} 2\lambda-2 & -1 \\ -1 & 2\lambda-2 \end{pmatrix} - (2\lambda-3)\alpha_0^{-1}\right)\delta_0(x).$$

Here Be stands for the usual Beta function.

We show below some examples that illustrate that for appropriate values of λ the new polynomials are joint eigenfunctions of some higher order differential operators, i.e., we get new bispectral situations. This appears to have little to do with α_0 being symmetric.

Example 2.

$$\lambda = 5/2 \quad \alpha_0 = \begin{pmatrix} 5 & 2 \\ 3 & 1 \end{pmatrix}.$$

Here we find one differential operator D satisfying

$$P_n D = \Lambda_n P_n$$

with

$$D = \sum_{r=0}^{4} \partial^r B_r$$

and

$$B_4 = (t-2)^2 t^2 \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$$

$$B_3 = 4(t-2)t(3t-2) \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$$

$$B_2 = \frac{24}{5} t(7t-9) \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$$

$$B_1 = \frac{8}{5} \begin{pmatrix} 5t+6 & -8t \\ -2(5t+3) & 13t \end{pmatrix}$$

$$B_0 = \begin{pmatrix} -\frac{8\times 11}{5} & -8 \\ -\frac{32}{5} & 0 \end{pmatrix}$$

$$\Lambda_n = \begin{pmatrix} \frac{(n+2)(5n^3+20n^2+3n-44)}{5} & -\frac{5n^4+30n^3+43n^2-14n+40}{5} \\ -\frac{5n^4+30n^3+43n^2+2n+32}{5} & \frac{n(5n^3+30n^2+43n+26}{5} \end{pmatrix}.$$

There are no operators of lower order in the algebra.

A few Jacobi type examples

A different avenue for exploring the similarities as well as the differences between the use of the Darboux process in the scalar and the matrix valued case is given by looking at some examples.

First recall that in the scalar case it follows from results by Koekoek, Zhedanov, Haine and Grünbaum, Yakimov that the polynomials orthogonal to the weight $\mu(x)$ consisting of a Jacobi density plus two possible delta masses of nonnegative strengths W, V at the ends of the interval, i.e.,

$$\mu(x) = (1-x)^{\alpha}(1+x)^{\beta} + W\delta_1(x) + V\delta_{-1}(x)$$

satisfy differential equations when α and β satisfy certain natural integrality conditions. The simplest example is given by the so called Koorwinder polynomials, corresponding to $\alpha = \beta = 0$. If the weight at 1 is the only one that is present then the order is $2\alpha + 4$. If both delta weights are thrown in, then the order is $2\alpha + 2\beta + 6$.

The results can be obtained by an application of the Darboux process as shown in the work of Haine, Yakimov and myself.

We consider now a small colection of situations analogous to the ones above.

The weight matrices will, as before, consist of a matrix weight density plus a pair of deltas at the end points weighted by certain matrices W, V, i.e., we have

$$\widetilde{W}(x) = (1-x)^{\alpha}(1+x)^{\beta} \begin{pmatrix} 1 & x \\ x & 1 \end{pmatrix} + W\delta_1(x) + V\delta_{-1}(x).$$

For the first batch of examples we will assume that α, β are both 0.

If V and W coincide with the matrix $\begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}$ then we find two linearly independent operators of order 5 and one of order 6 as well as other operators of higher order. There are no other operators of lower order.

If V is the matrix $\begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}$ and W is the matrix $\begin{pmatrix} 0 & 0 \\ 0 & b \end{pmatrix}$ or the matrix $\begin{pmatrix} b & 0 \\ 0 & 0 \end{pmatrix}$ then we find two linearly independent operators of order 6 as well as other operators of higher order. There are no other operators of lower order.

More generally if V is the matrix $\begin{pmatrix} a^2 & ab \\ ab & b^2 \end{pmatrix}$ and W is the matrix $\begin{pmatrix} c^2 & cd \\ cd & d^2 \end{pmatrix}$ then we have the same situation as in the last example.

In general if V and W are of the form $\begin{pmatrix} a & b \\ b & c \end{pmatrix}$ and $\begin{pmatrix} d & e \\ e & f \end{pmatrix}$ then the lowest order operator in the algebra is just one operator of order 8.

We come now to a different sort of examples.

Assume that α and β (> -1) are arbitrary, but insist in picking W and V to be arbitrary (and non-necessarily equal) nonnegative multiples of the matrix $\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$.

In this case there is a very nice second order differential operator in the algebra which is independent of the choice of the scalar factors that appear in front of the matrix above to give W and V. There is no lower order operator in the algebra. When the deltas are both missing then the algebra contains an operator of order 1.

The right handed differential operator alluded to above is a scalar operator of the usual Jacobi type, with coefficients $(1-x^2)$ and $(\alpha+\beta-1)-x(\alpha+\beta+3)$ multiplied on the right by the matrix

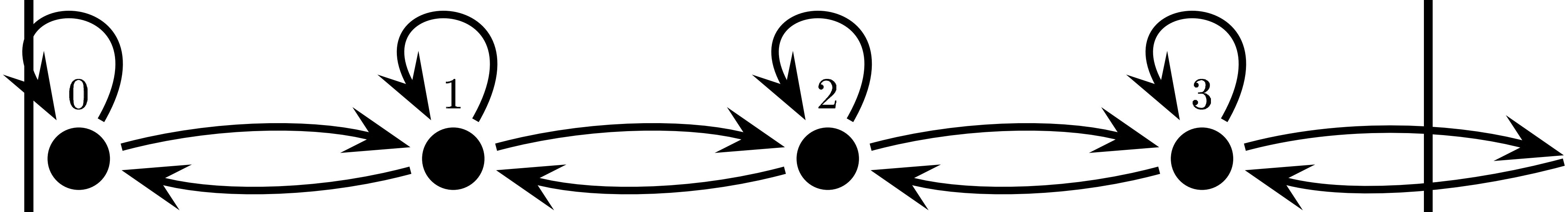
$$\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}.$$

The eigenvalue is $-n(n+\alpha+\beta+2)$ multiplied by this same matrix.

Applications to birth and processes and more complicated Markov chains.

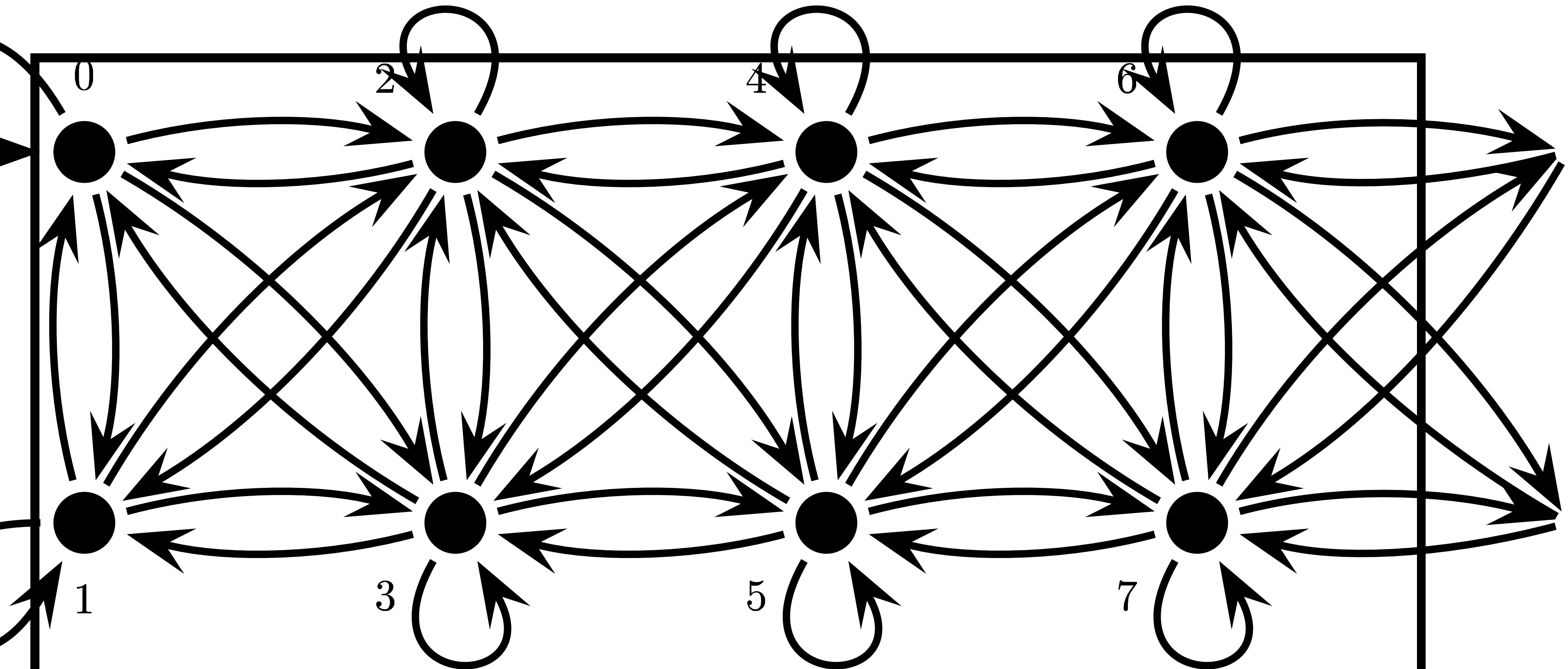
The models that we can solve are simpler than the ones considered at the beginning.....

In the case of a birth and death process it is useful to think of a graph like



If we want to deal with a nearest neighbourhood random walk on the set of all integers it is convenient to fold the integers to get a two stranded version of the nonnegative integers.

In general we may want to consider a state space which is product of the non-negative integers by a finite (or infinite) set.



This graph clearly corresponds to a general block tridiagonal matrix, with blocks of size 2×2.

For the general graph above if $\mathbb{P}_{i,j}$ denotes the i, j block of $\mathbb{P}$ we can generate a sequence of 2×2 matrix valued polynomials $Q_i(t)$ by imposing the three term recursion (1) of Krein. By using the earlier notation we would have

$$\mathbb{P}Q(x) = xQ(x)$$

where the entries of the column vector $Q(x)$ are now 2×2 matrices. Proceeding as in the scalar case, this relation can be iterated to give

$$\mathbb{P}^n Q(x) = x^n Q(x)$$

and if we assume the existence of a weight matrix $W(x)$ as in section 2, with the property

$$(Q_j, Q_j)\delta_{i,j} = \int_{\mathbb{R}} Q_i(x) W(x) Q_j^*(x) dx,$$

it is then clear that one can get an expression for the (i, j) entry of the block matrix $\mathbb{P}^n$ that would look exactly as in the scalar case, namely

$$(\mathbb{P}^n)_{ij}(Q_j, Q_j) = \int x^n Q_i(x) W(x) Q_j^*(x) dx.$$

Just as in the scalar case, this expression becomes useful when we can get our hands on the matrix valued polynomials $Q_i(x)$ and the orthogonality measure $W(x)$. Notice that we have not discussed conditions on the matrix $\mathbb{P}$ to give rise to such a measure.

In summary the spectral theory of a scalar double-infinite tridiagonal matrix leads naturally to a 2×2 semi-infinite matrix.

This is a matrix valued extension of the Karlin-McGregor formula.

Markov chains

I want to show how some examples of important Markov chains can be studied properly by using matrix valued orthogonal polynomials.

The first example is ordinary random walk on the integers which can (of course) be studied at least with two other methods

counting paths

Fourier analysis (exploits translation invariance)

The example in Karlin-McGregor

The probabilities of going right or left are p and q, respectively, with $p+q=1$.

The block tridiagonal matrix $\mathbb{P}$ is given by

$$\mathbb{P} = \begin{pmatrix} 0 & q & p & 0 & 0 & 0 & \dots & \dots & \dots \\ p & 0 & 0 & q & 0 & 0 & \dots & \dots & \dots \\ q & 0 & 0 & 0 & p & 0 & \dots & \dots & \dots \\ 0 & p & 0 & 0 & 0 & q & \dots & \dots & \dots \\ 0 & 0 & q & 0 & 0 & 0 & p & 0 & \dots \\ 0 & 0 & 0 & p & 0 & 0 & 0 & q & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \end{pmatrix}$$

Here is the weight matrix, which was already computed by Karlin and McGregor:

In the interval $|x| \leq \sqrt{4pq}$

$$\frac{1}{\sqrt{4pq - x^2}} \begin{pmatrix} 1 & x/2q \\ x/2q & p/q \end{pmatrix}$$

In this example we have,

$$B_0 = \begin{pmatrix} 0 & q \\ p & 0 \end{pmatrix}, \quad B_k = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \; k \geq 1$$

we also have

$$A_k = \begin{pmatrix} q & 0 \\ 0 & p \end{pmatrix}, k \geq 1$$

and

$$C_k = \begin{pmatrix} p & 0 \\ 0 & q \end{pmatrix}, k \geq 0$$

The orthogonal polynomials given by

$$A_k P_{k-1}(x) + B_k P_k(x) + C_k P_{k+1}(x) = x P_k(x)$$

with $P_{-1}(x) = 0, P_0(x) = I$, can be readily expressed in terms of Chebyshev polynomials.

Let us denote by $U_n(x)$ the Chebyshev polynomials of the second kind, which satisfy

$$U_{n+1}(x) + U_{n-1}(x) = 2xU_n(x), \quad \text{with} \quad U_{-1}(x) = 0 \quad \text{and} \quad U_0(x) = 1.$$

The relation with the Chebyshev polynomials $U_n(x)$, is given by

$$P_k(x) = \begin{pmatrix} (q/p)^{k/2} U_k(x^*) & -(q/p)^{(k+1)/2} U_{k-1}(x^*) \\ -(p/q)^{(k+1)/2} U_{k-1}(x^*) & (p/q)^{k/2} U_k(x^*) \end{pmatrix}$$

with $x^* = x/(2\sqrt{pq})$.

A modification of the example in Karlin-McGregor with probabilities p of going away from the center (located at 1/2) and q of going towards the center.

$$L = \begin{pmatrix} 0 & q & p & 0 & \dots & \dots & \dots & \dots & \dots \\ q & 0 & 0 & p & \dots & \dots & \dots & \dots & \dots \\ q & 0 & 0 & 0 & p & 0 & \dots & \dots & \dots \\ 0 & q & 0 & 0 & 0 & p & \dots & \dots & \dots \\ 0 & 0 & q & 0 & 0 & 0 & p & 0 & \dots \\ 0 & 0 & 0 & q & 0 & 0 & 0 & p & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \end{pmatrix}$$

$p + q = 1$

Weight matrix:

In the interval $|t| \leq \sqrt{4pq}$

$$\frac{\sqrt{4pq - t^2}}{1 - t^2} \begin{pmatrix} 1 & t \\ t & 1 \end{pmatrix}$$

and if $p < 1/2$ one adds the "point masses"

$$(1 - 2p)\pi \left[\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \delta_{-1} + \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \delta_1 \right]$$

Allowing for a "defect" at the origin.

$x_1 + x_2 = 1.$

$$L = \begin{pmatrix} 0 & x_2 & x_1 & 0 & \dots & \dots & \dots & \dots & \dots \\ q & 0 & 0 & p & \dots & \dots & \dots & \dots & \dots \\ q & 0 & 0 & 0 & p & 0 & \dots & \dots & \dots \\ 0 & q & 0 & 0 & 0 & p & \dots & \dots & \dots \\ 0 & 0 & q & 0 & 0 & 0 & p & 0 & \dots \\ 0 & 0 & 0 & q & 0 & 0 & 0 & p & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \end{pmatrix}$$

Weight matrix:

In the interval $|t| \leq \sqrt{4pq}$

$$W = \frac{\sqrt{4pq - t^2}}{1 - t^2} \begin{pmatrix} p(1 - x_1) & p(1 - x_1)t \\ p(1 - x_1)t & (1 - p)x_1 + (p - x_1)t^2 \end{pmatrix}$$

If $p < 1/2$ one needs to add "point masses".

$$(1 - x_1)(1 - 2p)\pi \left[\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \delta_{-1} + \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \delta_1 \right]$$

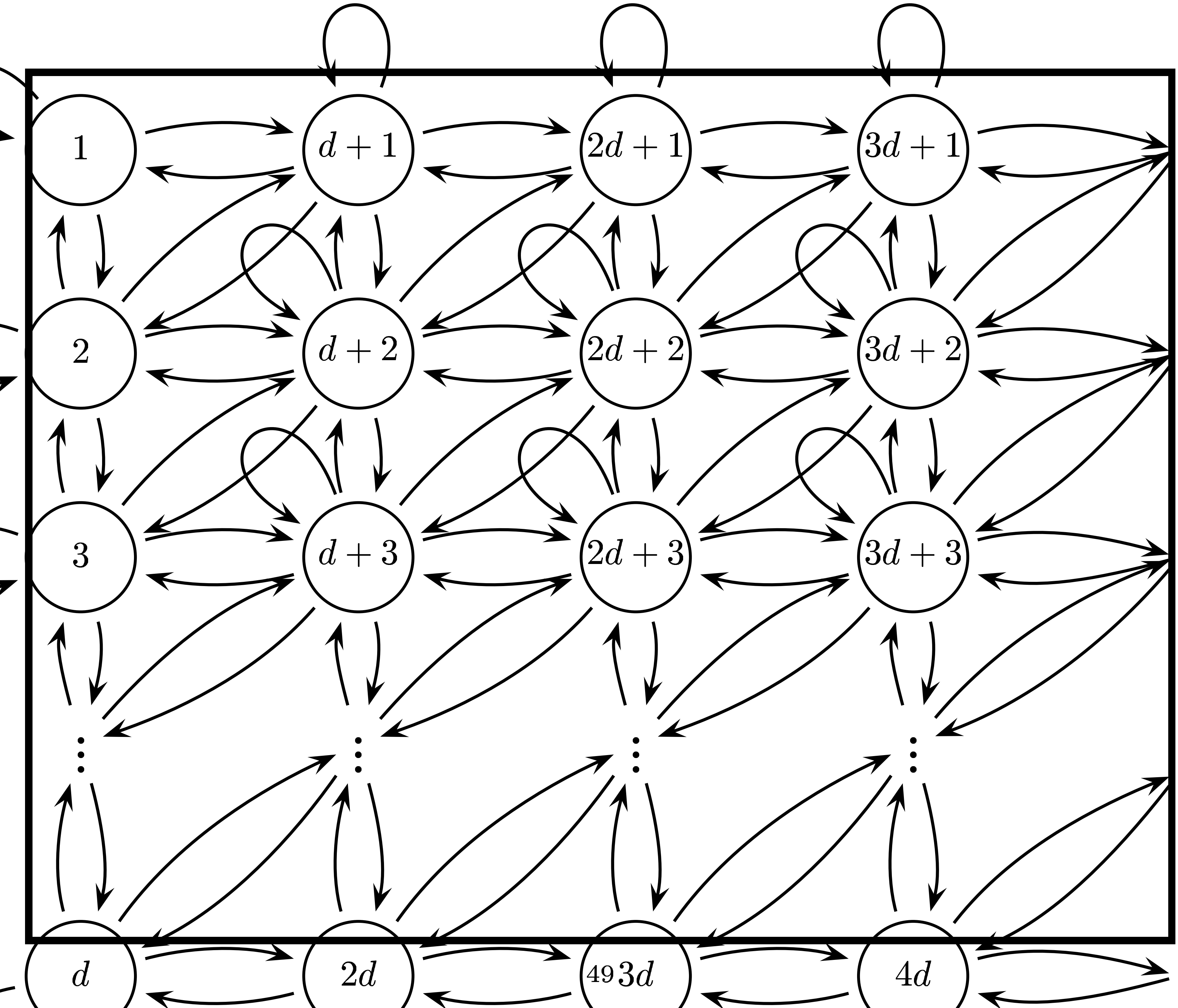
The multivariable case

I mention that in the paper

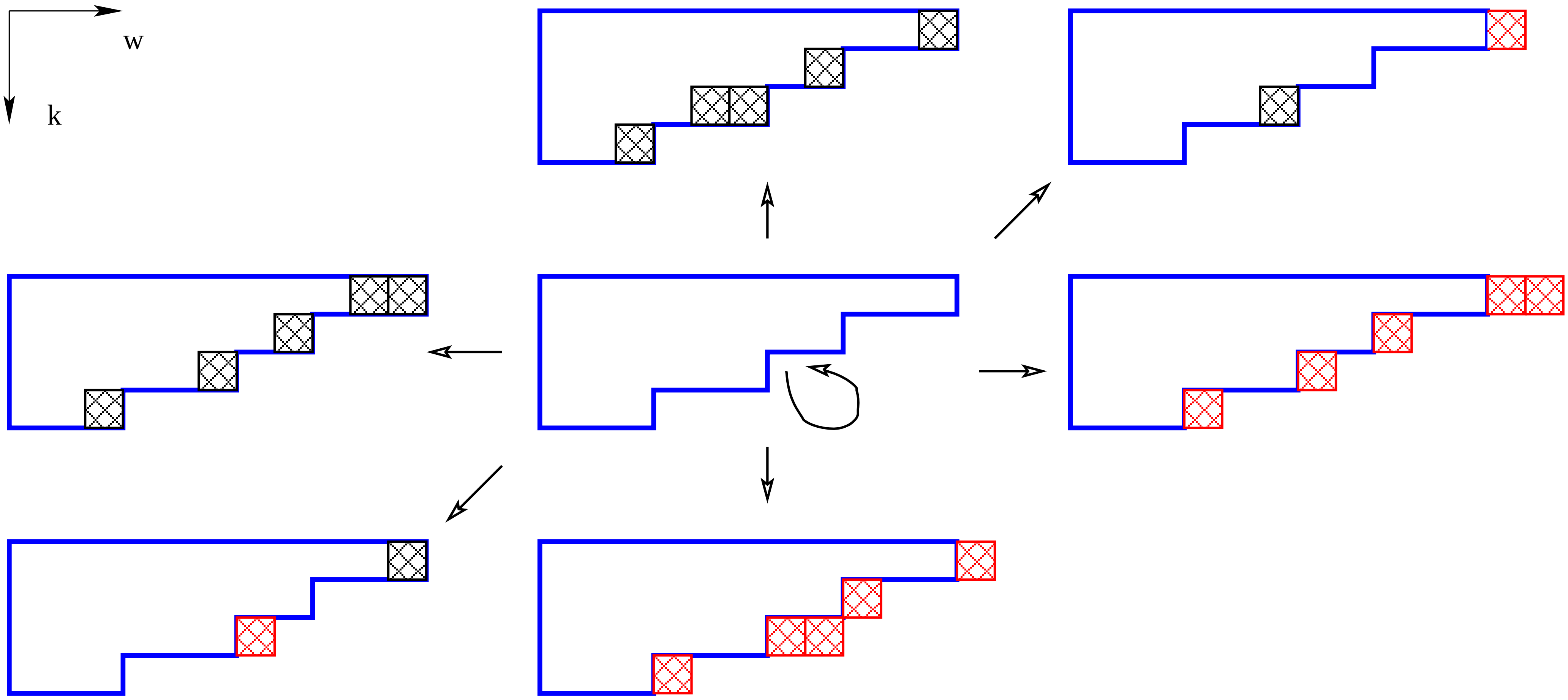
The Rahman polynomials are bispectral

one finds a specific random walk introduced by Hoare and Rahman, which we show leads to a bispectral situation in terms of polynomials of two variables. I also want to mention that in the multivariable case one finds a version of the Darboux process to obtain interesting deformations of the two dimensional Chebyshev measure in some recent work of Plamen Iliev and J. Geronimo.

Some concrete examples of matrix valued spherical functions , for $SU(3), U(2)$ give rise to a Markov chain whose state space can be depicted by


1
d + 1
2d + 1
3d + 1
2
d + 2
2d + 2
3d + 2
3
d + 3
2d + 3
3d + 3
d
2d
3d
4d

Or even better by the following picture which was kindly produced for me during this conference by Alexei Borodin.


w
k