Geometric approach to discrete isomonodromy transformations

Dima Arinkin
(joint work with Alexei Borodin)

University of North Carolina

SIDE 8. June 26, 2008

Linear difference equations

$$y(z+1) = A(z) \cdot y(z)$$

$A(z)$ is an $r \times r$ matrix whose entries are ($\mathbb{C}$-valued) rational functions;
$\det(A(z)) \neq 0$

Linear difference equations

$$y(z+1) = A(z) \cdot y(z)$$

$A(z)$ is an $r \times r$ matrix whose entries are ($\mathbb{C}$-valued) rational functions;
$\det(A(z)) \neq 0$

Isomonodromy transformation

$$A(z) \mapsto \widetilde{A}(z) = R(z+1)^{-1} A(z) R(z)$$

shifts the singularities of $A(z)$

$R(z)$ is a rational $r \times r$ matrix

Linear difference equations

$$y(z+1) = A(z) \cdot y(z)$$

$A(z)$ is an $r \times r$ matrix whose entries are ($\mathbb{C}$-valued) rational functions;
$\det(A(z)) \neq 0$

Isomonodromy transformation

$$A(z) \mapsto \widetilde{A}(z) = R(z+1)^{-1} A(z) R(z)$$

shifts the singularities of $A(z)$

$R(z)$ is a rational $r \times r$ matrix

Isomonodromy transformations are described by difference non-linear equations, such as difference Painlevé equations.

Goal

Geometric approach to

$$y(z+1) = A(z) \cdot y(z)$$

Plan:

1. Difference connections
2. Isomonodromy transformation
3. Tau-function

Other worlds

- Difference world

$$y(z+1) = A(z)y(z)$$

Isomonodromy $\longrightarrow$ difference Painlevé equations

- Continuous world

$$\frac{dy}{dz} = B(z)y(z)$$

Isomonodromy $\longrightarrow$ Painlevé equations (Fuchs, Okamoto)

Other worlds

- Difference world

$$y(z+1) = A(z)y(z)$$

Isomonodromy $\longrightarrow$ difference Painlevé equations

- Continuous world

$$\frac{dy}{dz} = B(z)y(z)$$

Isomonodromy $\longrightarrow$ Painlevé equations (Fuchs, Okamoto)

- q-world

$$y(qz) = A(z)y(z)$$

Isomonodromy $\longrightarrow$ q-Painlevé equations (Sakai)

Other worlds

- Difference world

$$y(z+1) = A(z)y(z)$$

Isomonodromy $\longrightarrow$ difference Painlevé equations

- Continuous world

$$\frac{dy}{dz} = B(z)y(z)$$

Isomonodromy $\longrightarrow$ Painlevé equations (Fuchs, Okamoto)

- q-world

$$y(qz) = A(z)y(z)$$

Isomonodromy $\longrightarrow$ q-Painlevé equations (Sakai)

- Elliptic world

Isomonodromy $\longrightarrow$ elliptic Painlevé equations (joint with Borodin, Rains)

Outline

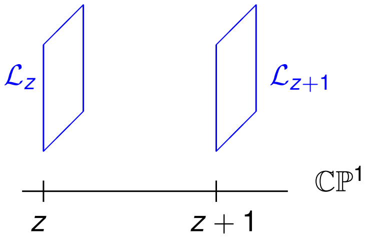
Difference connections

Isomonodromic transformation

Tau-function

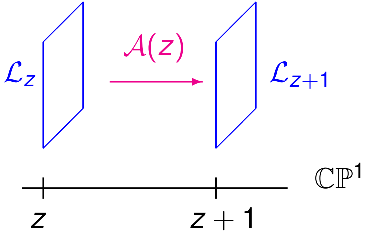
Difference connections

- $\mathcal{L}$ = rank r vector bundle on $\mathbb{CP}^1$



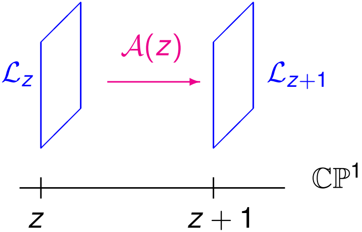
Difference connections

- $\mathcal{L}$ = rank r vector bundle on $\mathbb{CP}^1$
- $\mathcal{A}$ = linear operator $\mathcal{L}_z \to \mathcal{L}_{z+1}$ that depends rationally on z, $\det(\mathcal{A}) \neq 0$
  $\mathcal{A}$ = d-connection on $\mathcal{L}$



Difference connections

- $\mathcal{L}$ = rank r vector bundle on $\mathbb{CP}^1$
- $\mathcal{A}$ = linear operator $\mathcal{L}_z \to \mathcal{L}_{z+1}$ that depends rationally on z, $\det(\mathcal{A}) \neq 0$
  $\mathcal{A}$ = d-connection on $\mathcal{L}$



Remark

$\mathcal{A}$ or $\mathcal{A}^{-1}$ might be undefined at finitely many z:
d-connection has singularities

Example

$\mathcal{L}$ = trivial:
$\mathcal{L}_z = \mathbb{C}^r$
$\mathcal{A}(z) = r \times r$ matrix with rational entries.

- d-connection on trivial $\mathcal{L}$ = difference equation
  $y(z+1) = A(z)y(z)$

Example

$\mathcal{L} =$ trivial:
$\mathcal{L}_z = \mathbb{C}^r$
$\mathcal{A}(z) = r \times r$ matrix with rational entries.

- d-connection on trivial $\mathcal{L}$ = difference equation $y(z+1) = A(z)y(z)$
- d-connection on any $\mathcal{L}$ = difference equation on section $y(z)$ of $\mathcal{L}$

Example

$\mathcal{L} =$ trivial:
$\mathcal{L}_z = \mathbb{C}^r$
$\mathcal{A}(z) = r \times r$ matrix with rational entries.

- d-connection on trivial $\mathcal{L}$ = difference equation $y(z+1) = A(z)y(z)$
- d-connection on any $\mathcal{L}$ = difference equation on section $y(z)$ of $\mathcal{L}$

Remark

Bundle $\mathcal{L}$ on $\mathbb{CP}^1$ has a topological invariant

$$\deg(L) = c_1(L) \in \mathbb{Z}$$

Generic $\mathcal{L}$ with $\deg(\mathcal{L}) = 0$ is trivial

Outline

Difference connections

Isomonodromic transformation

Tau-function

Isomonodromic transformation of $(\mathcal{L}, \mathcal{A})$

- Modify $\mathcal{L}$ at finitely many points:
  Extend $\mathcal{L}$ from $\mathbb{CP}^1 - \{\text{points}\}$ to a different bundle $\widetilde{\mathcal{L}}$ on $\mathbb{CP}^1$

Isomonodromic transformation of $(\mathcal{L}, \mathcal{A})$

- Modify $\mathcal{L}$ at finitely many points:
  Extend $\mathcal{L}$ from $\mathbb{CP}^1 - \{\text{points}\}$ to a different bundle $\widetilde{\mathcal{L}}$ on $\mathbb{CP}^1$
- $\widetilde{\mathcal{L}}$ acquires a d-connection $\widetilde{\mathcal{A}}$
  $(\widetilde{\mathcal{L}}, \widetilde{\mathcal{A}})$ is an isomonodromic transformation of $(\mathcal{L}, \mathcal{A})$.

Isomonodromic transformation of $(\mathcal{L}, \mathcal{A})$

- Modify $\mathcal{L}$ at finitely many points:
  Extend $\mathcal{L}$ from $\mathbb{CP}^1 - \{\text{points}\}$ to a different bundle $\widetilde{\mathcal{L}}$ on $\mathbb{CP}^1$
- $\widetilde{\mathcal{L}}$ acquires a d-connection $\widetilde{\mathcal{A}}$
  $(\widetilde{\mathcal{L}}, \widetilde{\mathcal{A}})$ is an isomonodromic transformation of $(\mathcal{L}, \mathcal{A})$.

Remark

Why 'isomonodromic'? Whatever monodromy is, it is global: it feels no difference between $\mathcal{L}$ and $\widetilde{\mathcal{L}}$

Transformation shifts singularities

$$(\mathcal{L}, \mathcal{A}) \mapsto (\widetilde{\mathcal{L}}, \widetilde{\mathcal{A}})$$

$\widetilde{\mathcal{L}}$ must agree with $\mathcal{A}$ to avoid introducing new singularities

Transformation shifts singularities

$$(\mathcal{L}, \mathcal{A}) \mapsto (\widetilde{\mathcal{L}}, \widetilde{\mathcal{A}})$$

$\widetilde{\mathcal{L}}$ must agree with $\mathcal{A}$ to avoid introducing new singularities

z = singularity of $\mathcal{A}$ (so $\mathcal{A}(z)$ is undefined or degenerate)
$z + 1$ = not a singularity of $\mathcal{A}$:

$$\mathcal{L}_z \dashrightarrow \mathcal{L}_{z+1} \rightarrow \mathcal{L}_{z+2}$$

Transformation shifts singularities

$$(\mathcal{L}, \mathcal{A}) \mapsto (\widetilde{\mathcal{L}}, \widetilde{\mathcal{A}})$$

$\widetilde{\mathcal{L}}$ must agree with $\mathcal{A}$ to avoid introducing new singularities

z = singularity of $\mathcal{A}$ (so $\mathcal{A}(z)$ is undefined or degenerate)
$z+1$ = not a singularity of $\mathcal{A}$:

$$\mathcal{L}_z \dashrightarrow \mathcal{L}_{z+1} \to \mathcal{L}_{z+2}$$

Proposition

There is a unique way to modify $\mathcal{L}$ at $z+1$ to match $\mathcal{L}_z$:

$$\widetilde{\mathcal{L}}_z = \mathcal{L}_z \to \widetilde{\mathcal{L}}_{z+1} \dashrightarrow \widetilde{\mathcal{L}}_{z+2} = \mathcal{L}_{z+2}$$

Singularity of $\mathcal{A}$ at z shifts to singularity of $\widetilde{\mathcal{A}}$ at $z+1$

Transformation shifts singularities

$$(\mathcal{L}, \mathcal{A}) \mapsto (\widetilde{\mathcal{L}}, \widetilde{\mathcal{A}})$$

$\widetilde{\mathcal{L}}$ must agree with $\mathcal{A}$ to avoid introducing new singularities

z = singularity of $\mathcal{A}$ (so $\mathcal{A}(z)$ is undefined or degenerate)
$z+1$ = not a singularity of $\mathcal{A}$:

$$\mathcal{L}_z \dashrightarrow \mathcal{L}_{z+1} \rightarrow \mathcal{L}_{z+2}$$

Proposition

There is a unique way to modify $\mathcal{L}$ at $z+1$ to match $\mathcal{L}_z$:

$$\widetilde{\mathcal{L}}_z = \mathcal{L}_z \rightarrow \widetilde{\mathcal{L}}_{z+1} \dashrightarrow \widetilde{\mathcal{L}}_{z+2} = \mathcal{L}_{z+2}$$

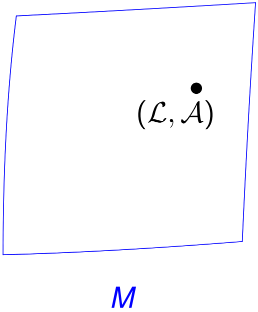
Singularity of $\mathcal{A}$ at z shifts to singularity of $\widetilde{\mathcal{A}}$ at $z+1$

Isomonodromic transformations shift singularities by integers

Moduli spaces

M = moduli space of $(\mathcal{L}, \mathcal{A}) = \{(\mathcal{L}, \mathcal{A})\}/\text{isomorphisms}$

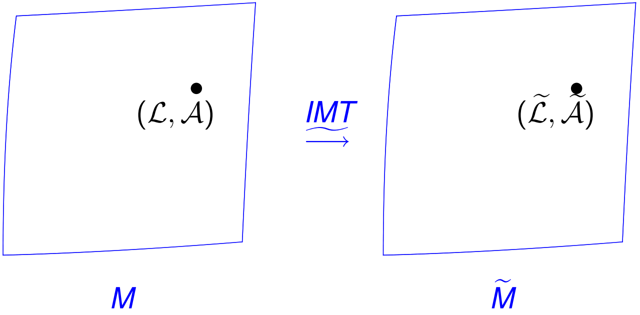
Types and locations of singularities of $\mathcal{A}$ are fixed



Moduli spaces

M = moduli space of $(\mathcal{L}, \mathcal{A}) = \{(\mathcal{L}, \mathcal{A})\}/\text{isomorphisms}$
Types and locations of singularities of $\mathcal{A}$ are fixed

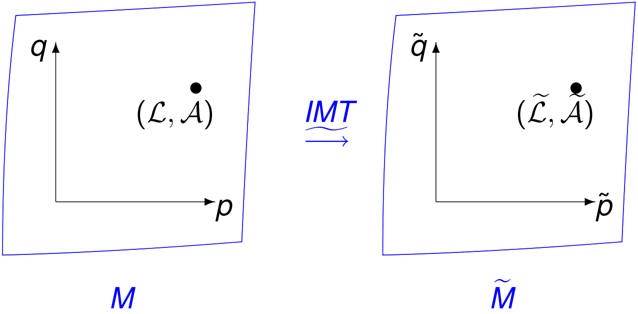
$IMT : (\mathcal{L}, \mathcal{A}) \mapsto (\widetilde{\mathcal{L}}, \widetilde{\mathcal{A}})$ shifts singularities



Moduli spaces

M = moduli space of $(\mathcal{L}, \mathcal{A})$ = $\{(\mathcal{L}, \mathcal{A})\}/$isomorphisms
Types and locations of singularities of $\mathcal{A}$ are fixed

$IMT : (\mathcal{L}, \mathcal{A}) \mapsto (\widetilde{\mathcal{L}}, \widetilde{\mathcal{A}})$ shifts singularities

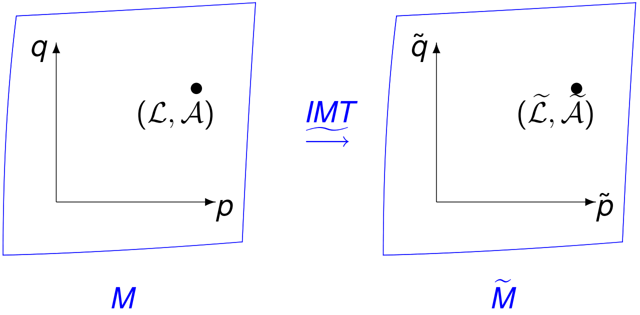


In coordinates (p, q) on M, $IMT = \begin{cases} \tilde{p} = F(p, q), \\ \tilde{q} = G(p, q) \end{cases}$

Moduli spaces

M = moduli space of $(\mathcal{L}, \mathcal{A})$ = $\{(\mathcal{L}, \mathcal{A})\}/\text{isomorphisms}$
Types and locations of singularities of $\mathcal{A}$ are fixed

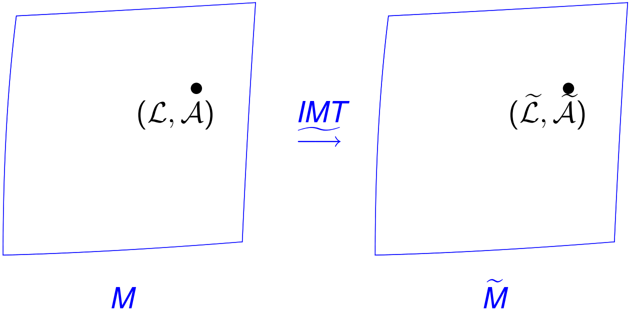
$IMT : (\mathcal{L}, \mathcal{A}) \mapsto (\widetilde{\mathcal{L}}, \widetilde{\mathcal{A}})$ shifts singularities



In coordinates (p, q) on M, $IMT = \begin{cases} \tilde{p} = F(p, q), \\ \tilde{q} = G(p, q) \end{cases}$

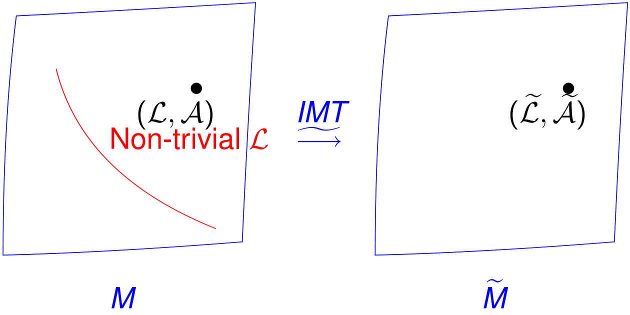
Example: difference Painlevé equations

Moduli spaces = spaces of initial conditions



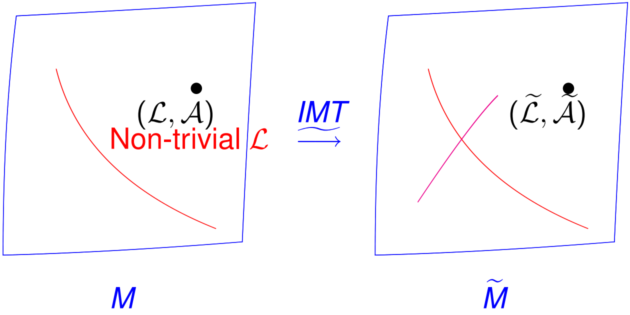
Assume $\deg(\mathcal{L}) = \deg(\widetilde{\mathcal{L}}) = 0$

Moduli spaces = spaces of initial conditions



Assume $\deg(\mathcal{L}) = \deg(\widetilde{\mathcal{L}}) = 0$
There is open subset $M_0 \subset M$ of $(\mathcal{L}, \mathcal{A})$ with trivial $\mathcal{L}$
M_0 = space of difference equations

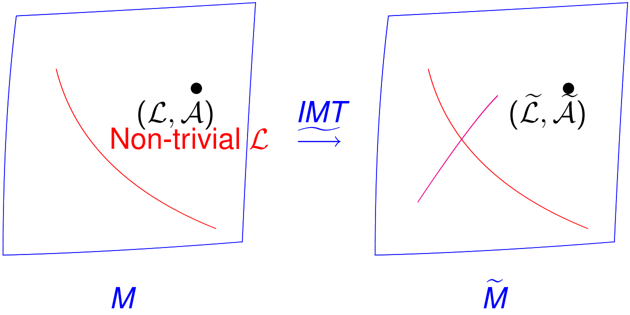
Moduli spaces = spaces of initial conditions



Assume $\deg(\mathcal{L}) = \deg(\widetilde{\mathcal{L}}) = 0$
There is open subset $M_0 \subset M$ of $(\mathcal{L}, \mathcal{A})$ with trivial $\mathcal{L}$
M_0 = space of difference equations

But: M_0 is not preserved by *IMT*

Moduli spaces = spaces of initial conditions

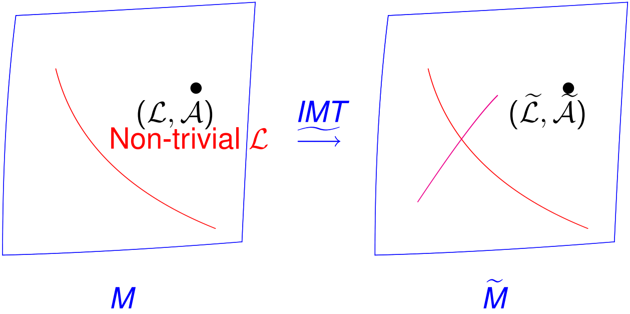


Assume $\deg(\mathcal{L}) = \deg(\widetilde{\mathcal{L}}) = 0$
There is open subset $M_0 \subset M$ of $(\mathcal{L}, \mathcal{A})$ with trivial $\mathcal{L}$
M_0 = space of difference equations

But: M_0 is not preserved by IMT

On M_0, IMT is only a rational map
Going to M from M_0 removes singularities of IMT

Moduli spaces = spaces of initial conditions



Theorem

M is the space of initial conditions of the isomonodromic transformation

Outline

Difference connections

Isomonodromic transformation

Tau-function

Tau-function

$\mathcal{L}$ = rank r vector bundle on $\mathbb{CP}^1$, $\deg(\mathcal{L}) = 0$

$$\Gamma(\mathbb{CP}^1, \mathcal{L}) = \text{space of global holomorphic sections of } \mathcal{L}$$

Assumption: $\dim \Gamma(\mathbb{CP}^1, \mathcal{L}) = r$
(Can be dropped, holds for discrete Painlevé)

Tau-function

$\mathcal{L}$ = rank r vector bundle on $\mathbb{CP}^1$, $\deg(\mathcal{L}) = 0$

$$\Gamma(\mathbb{CP}^1, \mathcal{L}) = \text{space of global holomorphic sections of } \mathcal{L}$$

Assumption: $\dim \Gamma(\mathbb{CP}^1, \mathcal{L}) = r$
(Can be dropped, holds for discrete Painlevé)

$$ev : \Gamma(\mathbb{CP}^1, \mathcal{L}) \to \mathcal{L}_\infty : y \mapsto y(\infty)$$

Definition

$\tau(\mathcal{L}) = \det(ev)$

Tau-function

$\mathcal{L}$ = rank r vector bundle on $\mathbb{CP}^1$, $\deg(\mathcal{L}) = 0$

$$\Gamma(\mathbb{CP}^1, \mathcal{L}) = \text{space of global holomorphic sections of } \mathcal{L}$$

Assumption: $\dim \Gamma(\mathbb{CP}^1, \mathcal{L}) = r$
(Can be dropped, holds for discrete Painlevé)

$$ev : \Gamma(\mathbb{CP}^1, \mathcal{L}) \to \mathcal{L}_\infty : y \mapsto y(\infty)$$

Definition

$\tau(\mathcal{L}) = \det(ev)$
is not a number
unless there are bases in $\Gamma(\mathbb{CP}^1, \mathcal{L})$ and $\mathcal{L}_\infty$

Tau-function

$\mathcal{L}$ = rank r vector bundle on $\mathbb{CP}^1$, $\deg(\mathcal{L}) = 0$

$$\Gamma(\mathbb{CP}^1, \mathcal{L}) = \text{space of global holomorphic sections of } \mathcal{L}$$

Assumption: $\dim \Gamma(\mathbb{CP}^1, \mathcal{L}) = r$
(Can be dropped, holds for discrete Painlevé)

$$ev : \Gamma(\mathbb{CP}^1, \mathcal{L}) \to \mathcal{L}_\infty : y \mapsto y(\infty)$$

Definition

$$\tau(\mathcal{L}) \in \delta \qquad \dim_{\mathbb{C}} \delta = 1$$

$$\delta = \delta(\mathcal{L}) = \bigwedge^r \mathcal{L}_\infty \otimes \left(\bigwedge^r \Gamma(\mathbb{CP}^1, \mathcal{L}) \right)^*$$

Zeros of tau-function

$$\tau(\mathcal{L}) \in \delta(\mathcal{L}) \qquad \dim_{\mathbb{C}} \delta(\mathcal{L}) = 1.$$

No use talking about τ's value

Zeros of tau-function

$$\tau(\mathcal{L}) \in \delta(\mathcal{L}) \qquad \dim_{\mathbb{C}} \delta(\mathcal{L}) = 1.$$

Proposition

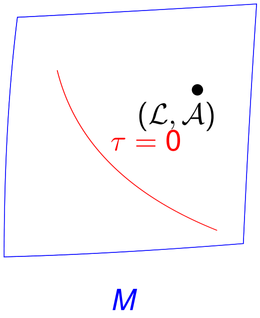
$\tau(\mathcal{L}) \neq 0$ iff $\mathcal{L}$ is trivial

Zeros of tau-function

$$\tau(\mathcal{L}) \in \delta(\mathcal{L}) \qquad \dim_{\mathbb{C}} \delta(\mathcal{L}) = 1.$$

Proposition

$\tau(\mathcal{L}) \neq 0$ iff $\mathcal{L}$ is trivial



Derivatives of tau-function

$$\tau(\mathcal{L}) \in \delta(\mathcal{L}) \qquad \text{dim}_{\mathbb{C}}\, \delta(\mathcal{L}) = 1.$$

Derivatives of tau-function

$$\tau(\mathcal{L}) \in \delta(\mathcal{L}) \qquad \dim_{\mathbb{C}} \delta(\mathcal{L}) = 1.$$

Isomonodromic transformations of $(\mathcal{L}, \mathcal{A}) = (\mathcal{L}_0, \mathcal{A}_0)$:

$$\ldots \mapsto (\mathcal{L}_0, \mathcal{A}_0) \mapsto (\mathcal{L}_1, \mathcal{A}_1) \mapsto (\mathcal{L}_2, \mathcal{A}_2) \mapsto \ldots$$

Idea: relate $\delta(\mathcal{L}_i)$'s to each other

Derivatives of tau-function

$$\tau(\mathcal{L}) \in \delta(\mathcal{L}) \qquad \dim_{\mathbb{C}} \delta(\mathcal{L}) = 1.$$

Isomonodromic transformations of $(\mathcal{L}, \mathcal{A}) = (\mathcal{L}_0, \mathcal{A}_0)$:

$$\ldots \mapsto (\mathcal{L}_0, \mathcal{A}_0) \mapsto (\mathcal{L}_1, \mathcal{A}_1) \mapsto (\mathcal{L}_2, \mathcal{A}_2) \mapsto \ldots$$

Idea: relate $\delta(\mathcal{L}_i)$'s to each other

Theorem

$\delta(\mathcal{L}_{n+1}) = \delta(\mathcal{L}_n) \otimes \ell$

One-dimensional space ℓ does not depend on n

Derivatives of tau-function

$$\tau(\mathcal{L}) \in \delta(\mathcal{L}) \qquad \dim_{\mathbb{C}} \delta(\mathcal{L}) = 1.$$

Isomonodromic transformations of $(\mathcal{L}, \mathcal{A}) = (\mathcal{L}_0, \mathcal{A}_0)$:

$$\ldots \mapsto (\mathcal{L}_0, \mathcal{A}_0) \mapsto (\mathcal{L}_1, \mathcal{A}_1) \mapsto (\mathcal{L}_2, \mathcal{A}_2) \mapsto \ldots$$

Idea: relate $\delta(\mathcal{L}_i)$'s to each other

Theorem

$\delta(\mathcal{L}_{n+1}) = \delta(\mathcal{L}_n) \otimes \ell$

One-dimensional space ℓ does not depend on n

Corollary

$$\frac{\tau(\mathcal{L}_{n+1})}{\tau(\mathcal{L}_n)} \in \ell$$

$$\frac{\tau(\mathcal{L}_{n+1})\tau(\mathcal{L}_{n-1})}{\tau(\mathcal{L}_n)^2} \in \mathbb{C}$$

Hirota's equation

Theorem

Tau-function of the isomonodromic transformation satisfies various identities of the Hirota type.

Summary

- Vector bundles with discrete connections provide a uniform approach to discrete isomonodromy deformations: they supply spaces of initial conditions and tau-functions

Summary

- Vector bundles with discrete connections provide a uniform approach to discrete isomonodromy deformations: they supply spaces of initial conditions and tau-functions
- They live in the hierarchy of ‘worlds’ (elliptic, difference, differential...)

Summary

- Vector bundles with discrete connections provide a uniform approach to discrete isomonodromy deformations: they supply spaces of initial conditions and tau-functions
- They live in the hierarchy of ‘worlds’ (elliptic, difference, differential...)
- They explain symmetries between the isomonodromic deformations (such as Painlevé equations) as operations on vector bundles