Embeddings of affine space using Chebyshev polynomials and their successors

Gene Freudenburg

Department of Mathematics
Western Michigan University
1903 W. Michigan Ave
Kalamazoo, MI 49008-5248
USA

gene.freudenburg@wmich.edu

Part I.
Key algebraic properties of the Chebyshev polynomials \( T_i(x) \in \mathbb{Z}[x] \) are established, followed by a discussion of the curves \( T_i(x) = T_j(y) \) over any field of characteristic 0, where \( \gcd(i, j) = 1 \). These are polynomial versions of the classical Lissajous curves. Then, mappings \( \mathbb{A}^1 \to \mathbb{A}^3 \) defined by \( (T_i(t), T_j(t), T_k(t)) \) are explored. In particular, it is determined which of these are embeddings. Moreover, for fixed \( i, j \) with \( \gcd(i, j) = 1 \), it is shown that there are at most a finite number of algebraically non-equivalent embeddings of the form \( (T_i, T_j, T_k) \). For the knot enthusiast, when the ground field is real, this procedure supplies a vast array of knots parametrized by polynomials. A table of knot types associated with low-degree embeddings is given.

Part II.
It is natural to ask: what are the analogues of Chebyshev polynomials in two or more variables? Candidates \( F_n(x, y) \in \mathbb{Z}[x, y] \) are defined and their algebraic content is discussed. These are used to define other affine space embeddings, including \( \mathbb{A}^2 \to \mathbb{A}^5 \).