Comparison geometry

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Abstract

Comparison geometry has its roots in global Riemannian geometry, where it took off in the 1930's, through the work of Hopf, Morse, Schoenberg, Myers, and Synge. The real breakthrough came in the 1950's with the pioneering work of Rauch and the foundational work of Alexandrov, Toponogov and Bishop. Since then, the simple idea of comparing the geometry of an arbitrary Riemannian manifold with the geometries of constant curvature spaces has witnessed a tremendous evolution: First in conjunction with Morse theory and convexity, then with critical point theory for distance functions, with the Gromov–Hausdorff topology on spaces of Riemannian manifolds, and the geometry of singular spaces, and most recently in the presence of symmetries. As a result, our understanding of relations between the geometry and topology of Riemannian manifolds has gained remarkable breadth and depth, and is currently experiencing exciting progress.

As indicated above, curvature is the most central aspect in this area. There are, however, many notions of curvature all derived from the Riemann curvature tensor. The classical and most important ones are scalar curvature, Ricci curvature, and sectional curvature. Here scalar curvature carries the least information and sectional curvature the most. For this reason, and for lack of space, we will focus our attention to sectional curvature. Even here, upper and lower curvature bounds have vastly different impact. For example, without a lower curvature bound, a positive upper curvature bound is generally unobstructed, but manifolds with nonnegative curvature have contractible

universal covers, and their own theory. We will therefore concentrate primarily on lower curvature bounds.

Our point of departure will be the presentation and discussion of basic comparison theorems. Our aim and emphasis is to use these results in each of the above-mentioned contexts: Morse theory, convexity, critical point theory, Gromov–Hausdorff distance, Alexandrov geometry, and equivariant comparison geometry.

Prerequisites: Basic differential geometry and topology corresponding to a first year graduate course in geometry and topology.