From the curve to its Jacobian and back

Christophe Ritzenthaler

Institut de Mathématiques de Luminy, CNRS

Montréal 04-10

e-mail: ritzenth@iml.univ-mrs.fr
web: http://iml.univ-mrs.fr/∼ritzenth/

Why do we care ?

CM method: CM-type + fundamental unit $\leadsto$ lattice + polarization $\leadsto$ period matrix $\leadsto$ ThetaNullwerte $\leadsto$ $\begin{cases}\text{the curve over } \mathbb{C} \\ \text{invariants}\end{cases}$ $\leadsto$ curve $/\mathbb{F}_q$.

AGM for point counting: curve $/\mathbb{F}_q$ $\leadsto$ lift $\leadsto$ quotients of ThetaNullwerte $\leadsto$ canonical lift + info on Weil polynomial $\leadsto$ Weil polynomial.

Other applications: fast computation of modular polynomials, class polynomials, isogenies ...

Caution: work over $\mathbb{C}$ but try to show why it works in general.

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2. Period matrices and ThetaNullwerte
   - Period matrices
   - ThetaNullwerte
   - From the ThetaNullwerte to the Riemann matrix
   - From the Riemann matrix to the (quotients of) ThetaNullwerte
3. From the curve to its Jacobian
   - Hyperelliptic case and the first tool: s_ε
   - Non hyperelliptic case and the second tool: Jacobian Nullwerte
4. From the Jacobian to its curve
   - Even characteristics
   - Odd characteristics

Definitions

Let C be a curve over $k \subset \mathbb{C}$ of genus $g > 0$.

The Jacobian of C is a torus $\mathrm{Jac}(C) \simeq \mathbb{C}^g/\Lambda$ where

- the lattice $\Lambda = \Omega\mathbb{Z}^{2g}$,
- the matrix $\Omega = [\Omega_1,\ \Omega_2] \in \mathrm{M}_{g,2g}(\mathbb{C})$ is a period matrix and
- $$\tau = \Omega_2^{-1}\Omega_1 \in \mathbb{H}_g = \{M \in \mathrm{GL}_g(\mathbb{C}),\ {}^tM = M,\ \mathrm{Im}\, M > 0\}$$

  is a Riemann matrix.

Construction

- $v_1, \ldots, v_g$ be a k-basis of $H^0(C, \Omega^1)$,
- $\delta_1, \ldots, \delta_{2g}$ be generators of $H_1(C, \mathbb{Z})$ such that $(\delta_i)_{1 \ldots 2g}$ form a symplectic basis for the intersection pairing on C.

$$\Omega := [\Omega_1, \ \Omega_2] = \left[\int_{\delta_j} v_i \right]_{\substack{i = 1, \ldots, g \\ j = 1, \ldots, 2g}} .$$

- Magma (Vermeulen): can compute Ω for a hyperelliptic curve.
- Maple (Deconinck, van Hoeij) can compute Ω for any plane model.

Rem: there is a polarization j involved in the definition of Ω with Chern class

$$2i \left(\bar{\Omega} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} {}^t\Omega \right)^{-1} .$$

Example

Ex: $E : y^2 = x^3 - 35x - 98 = (x-7)(x-a)(x-\bar{a})$ which has complex multiplication by $\mathbb{Z}[\alpha]$ with $\alpha = \frac{-1-\sqrt{-7}}{2}$ and $a = \frac{-7}{2} - \frac{\sqrt{-7}}{2}$.

$$\Omega = \left[\, 2\int_a^{\bar{a}} \frac{dx}{2y}, 2\int_a^{7} \frac{dx}{2y} \right] = c \cdot [\alpha,\ 1].$$

(Chowla, Selberg 67) formula gives

$$c = \frac{1}{8\pi\sqrt{7}} \cdot \Gamma(\frac{1}{7}) \cdot \Gamma(\frac{2}{7}) \cdot \Gamma(\frac{4}{7})$$

with

$$\Gamma(x) = \int_0^{\infty} t^{z-1} \exp(-t)\, dt.$$

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Projective embedding

The polarization j comes from an ample divisor D on $\mathrm{Jac}(C)$ (defined up to translation).

Theorem (Lefschetz, Mumford, Kempf)

For $n \geq 3$, nD is very ample, i.e. one can embed $\mathrm{Jac}(C)$ *in a* $\mathbb{P}^{n^g-1}$ *with a basis of sections of* $\mathcal{L}(nD)$.

For $n = 4$, the embedding is given by intersection of quadrics, whose equations are completely determined by the image of 0.

ThetaNullwert

A basis of sections of $\mathcal{L}(4D)$ is given by theta functions $\theta[\varepsilon](2z,\tau)$ with integer characteristics $[\varepsilon]=(\epsilon,\epsilon')\in\{0,1\}^{2g}$ where

$$\theta\begin{bmatrix}\epsilon\\ \epsilon'\end{bmatrix}(z,\tau)=\sum_{n\in\mathbb{Z}^g}\exp\left(i\pi\ (n+\frac{\epsilon}{2})\tau^t(n+\frac{\epsilon}{2})+2i\pi\ (n+\frac{\epsilon}{2})^t(z+\frac{\epsilon'}{2})\right).$$

When $\epsilon^t\epsilon'\equiv 0 \pmod 2$, $[\varepsilon]$ is said even and one calls ThetaNullwert

$$\theta\begin{bmatrix}\epsilon\\ \epsilon'\end{bmatrix}(0,\tau)=\theta\begin{bmatrix}\epsilon\\ \epsilon'\end{bmatrix}(\tau)=\theta[\varepsilon](\tau)=\theta_{ab}$$

where the binary representations of a and b are ϵ,ϵ'.

Example

Let $q = \exp(\pi i \tau)$. There are 3 genus 1 ThetaNullwerte:

$$\theta_{00} = \theta\begin{bmatrix}0\\0\end{bmatrix}(0,\tau) = \sum_{n\in\mathbb{Z}} q^{n^2},$$

$$\theta_{10} = \theta\begin{bmatrix}1\\0\end{bmatrix}(0,\tau) = \sum_{n\in\mathbb{Z}} q^{\left(n+\frac{1}{2}\right)^2},$$

$$\theta_{01} = \theta\begin{bmatrix}0\\1\end{bmatrix}(0,\tau) = \sum_{n\in\mathbb{Z}} (-1)^n q^{n^2}.$$

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Case $g = 1$ Gauss, Cox 84, Dupont 07

- Let $z = \theta_{01}(\tau)^2/\theta_{00}(\tau)^2$.
- Duplication formulae vs AGM formulae :

$$\begin{array}{rcl|rcl}
\theta_{00}(2\tau)^2 & = & \frac{\theta_{00}(\tau)^2+\theta_{01}(\tau)^2}{2} & a_n & = & \frac{a_{n-1}+b_{n-1}}{2}, \\
\theta_{01}(2\tau)^2 & = & \theta_{00}(\tau)\cdot\theta_{01}(\tau) & b_n & = & \sqrt{a_{n-1}\cdot b_{n-1}}, \\
\theta_{10}(2\tau)^2 & = & \frac{\theta_{00}(\tau)^2-\theta_{01}(\tau)^2}{2} & & &
\end{array}$$

$$\Rightarrow AGM(\theta_{00}(\tau)^2, \theta_{01}(\tau)^2) = \lim \theta_{00}(2^n\tau)^2 = 1 \Rightarrow AGM(1, z) = \frac{1}{\theta_{00}(\tau)^2}.$$

$$\Rightarrow \theta_{10}(\tau)^2 = \sqrt{\theta_{00}(\tau)^4 - \theta_{01}(\tau)^4}.$$

- Transformation formula :

$$\theta_{00}(\tau)^2 = \frac{i}{\tau} \cdot \theta_{00}\left(\frac{-1}{\tau}\right)^2, \quad \theta_{10}(\tau)^2 = \frac{i}{\tau} \cdot \theta_{01}\left(\frac{-1}{\tau}\right)^2.$$

$\Rightarrow AGM(\theta_{00}(\tau)^2, \theta_{10}(\tau)^2) = \frac{i}{\tau} \cdot \lim \theta_{00}(2^n \cdot \frac{-1}{\tau})^2 = \frac{i}{\tau} \cdot 1$

$\Rightarrow AGM(1, \sqrt{1-z^2}) = \frac{i}{\tau} \cdot \frac{1}{\theta_{00}(\tau)^2}.$

Proposition

$$\frac{i \cdot AGM(1,z)}{AGM(1,\sqrt{1-z^2})} = \tau.$$

Difficulty: define the correct square root when the values are complex.

Rem: one cannot get Ω from the ThetaNullwerte. But from the curve:

Theorem (Gauss, Cox 84)

If $E : y^2 = x(x-a^2)(x-b^2)$ *then* $[\omega_1,\ \omega_2] = [\frac{\pi}{AGM(a,b)},\ \frac{i\pi}{AGM(a+b,a-b)}]$ *is a period matrix relative to* dx/y.

Use the same ingredients as above and, as first step, the Thomae's formulae

$$\omega_2 \cdot a = \pi \cdot \theta_{00}(\tau)^2,\ \ \omega_2 \cdot b = \pi \cdot \theta_{01}(\tau)^2.$$

Case $g \geq 2$

Particular case: real Weierstrass points and $g = 2$ (Bost-Mestre 88).

General case (Dupont 07): under some (experimentally verified) conjectures.

Proposition

One can compute τ in terms of $\theta[\varepsilon](\tau)^2/\theta[0](\tau)^2$ in time

$$O(g^2 \cdot 2^g \cdot M(n) \cdot \log n)$$

for n digits of precision ($M(n)$ is the complexity of the binary multiplication).

Question: what about the period matrix ?

The work of (Dupont 07)

Naive method: $O(M(n)\sqrt{n})$ for $g = 1$ and $O(n^{2+\epsilon})$ for $g = 2$.

New method: invert the AGM. Complexity for n bits of precision on the quotients

- $O(M(n)\log n)$ for $g = 1$,
- $O(n^{1+\epsilon})$ for $g = 2$ (conjectural algorithm).

Main idea for $g = 1$: let

$$f(z) = i \cdot AGM(1, z) - \tau \cdot AGM(1, \sqrt{1 - z^2}).$$

Then $f(\theta_{01}(\tau)^2/\theta_{00}(\tau)^2) = 0$. Do a Newton algorithm on f.

- can we get rid of the conjectures ?
- can we generalize to all genera ?
- can we compute the ThetaNullwerte alone ?

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Thomae's formulae

Let C be a hyperelliptic curve $C : y^2 = \prod_{i=1}^{2g+1}(x-\lambda_i)$.

Theorem (Thomae's formulae)

$$\theta[\varepsilon](\tau)^4 = \pm \left(\frac{\det \Omega_2}{\pi^g}\right)^2 \prod_{(i,j)\in I} (\lambda_i - \lambda_j)$$

with the choice of the basis of differentials $x^i dx/y$ (the set I depends on $[\varepsilon]$ and on the basis of $H_1(C,\mathbb{Z})$).

Proof: see (Fay 73) using a variational method.

Proof for the quotients:

- study the zeroes of the section

$$s_\varepsilon(P) = \theta[\varepsilon](\phi_{P_0}(P))$$

  where $P_0 \in C$ and $\phi_{P_0}(P) = P - P_0 \in \mathrm{Jac}(C)$.
- $c \cdot f(P) = \frac{s_\varepsilon(P)^2}{s_{\varepsilon'}(P)^2}$ for an explicit $f \in \mathbb{C}(C)$.
- $c = \frac{s_\varepsilon(P_1)^2}{s_{\varepsilon'}(P_1)^2 f(P_1)} = \frac{s_\varepsilon(P_2)^2}{s_{\varepsilon'}(P_2)^2 f(P_2)}$ for P_1, P_2 such that $\frac{s_\varepsilon(P_2)^2}{s_{\varepsilon'}(P_2)^2} = \frac{s_{\varepsilon'}(P_1)^2}{s_\varepsilon(P_1)^2}$.

Rem: work in progress by Cosset for non-integral $[\varepsilon]$.

Main result on $s_\varepsilon(P)$

Let C be a curve of genus $g > 0$, $P_0 \in C$.

Theorem

If $s_\varepsilon(P)$ is not identically zero, then $s_\varepsilon(P)$ has g zeroes $P_1, \dots, P_g$ such that the divisor $D = P_1 + \ldots + P_g$ is characterized by $D - gP_0 \sim \varepsilon + \kappa$ where κ is a constant depending only on the homology basis and on P_0.

Geometrically, let $\Theta = \{z,\ \theta[0](z,\tau) = 0\} \subset \mathrm{Jac}(C)$, L be the corresponding ample line bundle.

- Poincaré's formula $\Rightarrow (\phi_{P_0}(C) \cdot \Theta) = g$.
- s_ε is a section of the line bundle $\mathcal{L}_\varepsilon = \phi_{P_0}^* t_\varepsilon^* L = \mathcal{O}_C(P_1 + \ldots + P_g)$.

- Riemann's theorem: $\exists \kappa_0$ a theta characteristic such that

$$\mathbf{Sym}^{g-1} C - \kappa_0 = \Theta.$$

$\Rightarrow$ $\mathcal{L}_0 = \phi_{P_0}^* L = \mathcal{O}_C(P_1^0 + \ldots + P_g^0)$ with $P_1^0 + \ldots + P_g^0 \sim \kappa_0 + P_0$. Indeed $P_i^0 - P_0 \sim -(\sum_{j \neq i} P_j^0 - \kappa_0) \in -\Theta = \Theta$.

- Canonical isomorphism:

$$\phi_{P_0}^* : \mathrm{Pic}^0(\mathrm{Jac}(C)) = \widehat{\mathrm{Jac}(C)} \to Pic^0(C) = \mathrm{Jac}(C)$$

is an isomorphism with inverse $-\phi_L$.

$\Rightarrow$

$$\begin{aligned} \phi_{P_0}^*(t_\varepsilon^* L \otimes L^{-1}) &= -\phi_L^{-1} \circ \phi_L(\varepsilon) = \mathcal{O}_C(\varepsilon) \\ &= \mathcal{O}_C(\sum P_i - P_0 - \kappa_0) = \mathcal{O}_C(\sum P_i - \kappa) \end{aligned}$$

where $\kappa = \kappa_0 - (g-1)P_0$.

Lemma

$s_\varepsilon \equiv 0 \iff \varepsilon + \kappa \sim D - gP_0$ *with* $D \in \mathbf{Sym}^g(C)$ *and* $i(D) > 0$.

$$\begin{aligned} s_\varepsilon \equiv 0 \quad &\iff \quad \forall P,\ P - P_0 - \varepsilon \in \Theta \\ &\iff \quad P - P_0 - D + gP_0 + \kappa_0 - (g-1)P_0 \in \mathbf{Sym}^{g-1}(C) - \kappa_0 \\ &\iff \quad P - D + 2\kappa_0 \in \mathbf{Sym}^{g-1}(C) \\ &\iff \quad D - P \in \mathbf{Sym}^{g-1}(C) \iff i(D) > 0 \end{aligned}$$

Corollary

The zero divisor D of s_ε is completely determined by the equivalence $D - gP_0 \sim \varepsilon + \kappa$.

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Non hyperelliptic case genus 3

Let C be a smooth plane quartic.

Theorem (Weber 1876)

$$\left(\frac{\theta[\varepsilon](\tau)}{\theta[\varepsilon'](\tau)}\right)^4 = \frac{[b_i, b_j, b_{ij}][b_{ik}, b_{jk}, b_{ij}][b_j, b_{jk}, b_k][b_i, b_{ik}, b_k]}{[b_j, b_{jk}, b_{ij}][b_i, b_{ik}, b_{ij}][b_i, b_j, b_k][b_{ik}, b_{jk}, b_k]}$$

where the b_i, b_{ij} are linear equations of certain bitangents of C and $[b_i, b_j, b_k]$ is the determinant of the matrix of the coefficients of (once for all fixed) equations of the bitangents.

- Weber's proof uses $s_\varepsilon(P)$.
- Question: can we find a formula for a Thetanullwert alone like in the hyperelliptic case ?

Derivative of theta functions

When $\epsilon^t \epsilon' \equiv 1 \pmod 2$, $[\varepsilon]$ is said odd and we write $[\boldsymbol{\mu}]$ instead.

Definition

The theta gradient (with odd characteristic $[\boldsymbol{\mu}]$) is the vector

$$\nabla\theta[\boldsymbol{\mu}] := \left(\frac{\partial\theta[\boldsymbol{\mu}](z,\tau)}{\partial z_1}(0,\tau), \ldots, \frac{\partial\theta[\boldsymbol{\mu}](z,\tau)}{\partial z_g}(0,\tau) \right).$$

The theta hyperplane is the projective hyperplane

$$\nabla\theta[\boldsymbol{\mu}] \cdot (X_1, \ldots, X_g) = 0$$

of $\mathbb{P}^{g-1}$ defined by a theta gradient.
We denote the matrix

$$J[\boldsymbol{\mu}_1, \ldots, \boldsymbol{\mu}_g] := (\nabla\theta[\boldsymbol{\mu}_1], \ldots, \nabla\theta[\boldsymbol{\mu}_g])$$

and $[\boldsymbol{\mu}_1, \ldots, \boldsymbol{\mu}_g]$ its determinant (called Jacobian Nullwerte).

Case of Riemann-Mumford-Kempf singularity theorem

Let C be any curve of genus $g > 0$.

Theorem

Let ϕ be the canonical map

$$\phi : C \to \mathbb{P}^{g-1},\ P \mapsto (\omega_1(P), \ldots, \omega_g(P)).$$

Let D be an effective divisor of degree $g-1$ on C such that $h^0(D) = 1$. Then

$$\left(\frac{\partial\theta(z,\tau)}{\partial z_1}(D-\kappa_0,\tau), \frac{\partial\theta(z,\tau)}{\partial z_g}(D-\kappa_0,\tau)\right) \Omega_2^{-1}\ {}^t(X_1,\ldots,X_g) = 0$$

is an hyperplan of $\mathbb{P}^{g-1}$ which cuts out the divisor $\phi(D)$ on the curve $\phi(C)$.

Rem: this can also be re-interpreted geometrically in terms of Gauss maps.

Generalization of Jacobi's derivative formula

$$\frac{d\theta\begin{bmatrix}1\\1\end{bmatrix}(z,\tau)}{dz}(0,\tau)=\pi\cdot\theta\begin{bmatrix}0\\0\end{bmatrix}(0,\tau)\cdot\theta\begin{bmatrix}0\\1\end{bmatrix}(0,\tau)\cdot\theta\begin{bmatrix}1\\0\end{bmatrix}(0,\tau).$$

Theorem (Igusa 80)

Let $[\boldsymbol{\mu}_1],\ldots,[\boldsymbol{\mu}_g]$ be distinct odd theta characteristics such that the function $[\boldsymbol{\mu}_1,\ldots,\boldsymbol{\mu}_g](\tau)$ is contained in the $\mathbb{C}$-algebra $\mathbb{C}[\theta]$ generated by the functions $\theta[\varepsilon](\tau)$ for all even characteristics $[\varepsilon]$. Then

$$[\boldsymbol{\mu}_1,\ldots,\boldsymbol{\mu}_g](\tau)=\pi^g\sum_{[\boldsymbol{\varepsilon}_{g+1}],\ldots,[\boldsymbol{\varepsilon}_{2g+2}]\in\mathcal{S}}\pm\prod_{i=g+1}^{2g+2}\theta[\varepsilon_i](\tau),$$

where $\mathcal{S}$ is the set of all $g+2$-tuples $\{[\varepsilon_{g+1}],\ldots,[\varepsilon_{2g+2}]\}$ even theta characteristics such that $\{[\boldsymbol{\mu}_1],\ldots,[\boldsymbol{\mu}_g],[\varepsilon_{g+1}],\ldots,[\varepsilon_{2g+2}]\}$ forms a fundamental system.

Sketch of the proof of Weber's formula (Nart, R. unpublished)

Let $[\varepsilon], [\varepsilon']$ be two even characteristics in genus 3.

- create two fundamental systems of the form
  $$\{[\boldsymbol{\mu}_1],[\boldsymbol{\mu}_2],[\boldsymbol{\mu}_3],[\varepsilon],[\varepsilon_4],[\varepsilon_5],[\varepsilon_6],[\varepsilon_7]\},\{[\boldsymbol{\mu}_1'],[\boldsymbol{\mu}_2'],[\boldsymbol{\mu}_3'],[\varepsilon'],[\varepsilon_4],[\varepsilon_5],[\varepsilon_6],[\varepsilon_7]\}.$$
- $\#\mathcal{S} = 1$ and
  $$\frac{[\boldsymbol{\mu}_1,\boldsymbol{\mu}_2,\boldsymbol{\mu}_3]}{[\boldsymbol{\mu}_1',\boldsymbol{\mu}_2',\boldsymbol{\mu}_3']} = \frac{\theta[\varepsilon]}{\theta[\varepsilon']}.$$
- An odd 2-torsion point μ is given by $D - \kappa_0$ where D is a degree 2 divisor, support of a bitangent of equation $b_\mu = 0$.
- $$[b_{\boldsymbol{\mu}_1}, b_{\boldsymbol{\mu}_2}, b_{\boldsymbol{\mu}_3}] = \det(\Omega_2)^{-1}\cdot(\lambda_{\boldsymbol{\mu}_1}\lambda_{\boldsymbol{\mu}_2}\lambda_{\boldsymbol{\mu}_3})\cdot[\boldsymbol{\mu}_1,\boldsymbol{\mu}_2,\boldsymbol{\mu}_3]$$
  where λ_i are constants depending on the choice of scalar multiplier for b_i and of τ.
- use several quotients to get rid of the λ_i.

Remarks

- For $g = 4$, $\#\mathcal{S} = 2$ and for $g = 7$, $\#\mathcal{S} = 960$.
- For $g \leq 5$ it is known that $[\boldsymbol{\mu}_1, \ldots, \boldsymbol{\mu}_g]$ is in $\mathbb{C}[\theta]$.In general, it is not true but $[\boldsymbol{\mu}_1, \ldots, \boldsymbol{\mu}_g]$ can be expresses as a quotient of two polynomials in the ThetaNullwerte. There is also a precise conjectural formula (Igusa 80).
- Could we directly invert the formula, i.e. express a ThetaNullwert is terms of Jacobian Nullwerte (at least for $g \leq 5$) ?
- (Nakayashili 97, Enolski, Grava 06): Thomae's formula for $y^n = \prod_{i=1}^{m} (x - \lambda_i)^{n-1} \cdot \prod_{i=m+1}^{2m} (x - \lambda_i)$.
- a general theory exists (Klein vol.3 p.429, Matone-Volpato 07 over $\mathbb{C}$, Shepherd-Barron preprint 08 over any field). Their expressions involve determinants of bases of $H^0(C, \mathcal{L}(2K_C + \boldsymbol{\mu}))$. But no formula or implementation has been done.

Torelli theorem: classical versions

Let C/k be a curve of genus $g > 0$.

Theorem

C is uniquely determined up to k-isomorphism by $(\mathrm{Jac}(C), j)$.

Corollary

C is uniquely determined up to $\mathbb{C}$-isomorphism by Ω or by the ThetaNullwerte.

From the Jacobian to its curve: hyperelliptic case

$$C : y^2 = x(x-1)\prod_{i=1}^{2g-1}(x-\lambda_i).$$

Idea: invert quotient Thomae's formulae (Mumford Tata II p.136, Takase 96, Koizumi 97)

$$\frac{\lambda_k-\lambda_l}{\lambda_k-\lambda_m} = i^c\cdot\frac{\theta[\varepsilon_1]^2\cdot\theta[\varepsilon_2]^2}{\theta[\varepsilon_3]^2\cdot\theta[\varepsilon_4]^2},\quad c\in\{0,1,2,3\}.$$

- For genus 1: $\lambda_1 = \theta_1^4/\theta_0^4$.
- For genus 2 (Rosenhain formula):

$$\lambda_1 = -\frac{\theta_{01}^2\theta_{21}^2}{\theta_{30}^2\theta_{10}^2},\quad \lambda_2 = -\frac{\theta_{03}^2\theta_{21}^2}{\theta_{30}^2\theta_{12}^2},\quad \lambda_3 = -\frac{\theta_{03}^2\theta_{01}^2}{\theta_{10}^2\theta_{12}^2}.$$

- For genus 3 (Weng 01):

$$\lambda_{\mathbf{1}} = \frac{(\theta_{\mathbf{15}}\theta_{\mathbf{3}})^{\mathbf{4}}+(\theta_{\mathbf{12}}\theta_{\mathbf{1}})^{\mathbf{4}}-(\theta_{\mathbf{14}}\theta_{\mathbf{2}})^{\mathbf{4}}}{2(\theta_{\mathbf{15}}\theta_{\mathbf{3}})^{\mathbf{4}}},\lambda_{\mathbf{2}} = \frac{(\theta_{\mathbf{4}}\theta_{\mathbf{9}})^{\mathbf{4}}+(\theta_{\mathbf{6}}\theta_{\mathbf{11}})^{\mathbf{4}}-(\theta_{\mathbf{13}}\theta_{\mathbf{8}})^{\mathbf{4}}}{2(\theta_{\mathbf{4}}\theta_{\mathbf{9}})^{\mathbf{4}}},\ldots$$

From the Jacobian to its curve : non hyperelliptic genus 3

(Weber 1876) shows how to find the Riemann model:

$$C : \sqrt{x(a_1 x + a_1' y + a_1'' z)} + \sqrt{y(a_2 x + a_2' y + a_2'' z)} + \sqrt{z(a_3 x + a_3' y + a_3'' z)} = 0$$

with

$$a_1 = i\frac{\theta_{41}\theta_{05}}{\theta_{50}\theta_{14}}, \quad a_1' = i\frac{\theta_{05}\theta_{66}}{\theta_{33}\theta_{50}}, \quad a_1'' = -\frac{\theta_{66}\theta_{41}}{\theta_{14}\theta_{33}},$$

$$a_2 = i\frac{\theta_{25}\theta_{61}}{\theta_{36}\theta_{70}}, \quad a_2' = i\frac{\theta_{61}\theta_{02}}{\theta_{57}\theta_{34}}, \quad a_2'' = \frac{\theta_{02}\theta_{25}}{\theta_{70}\theta_{57}},$$

$$a_3 = i\frac{\theta_{07}\theta_{43}}{\theta_{16}\theta_{52}}, \quad a_3' = i\frac{\theta_{60}\theta_{20}}{\theta_{75}\theta_{16}}, \quad a_3'' = \frac{\theta_{20}\theta_{07}}{\theta_{52}\theta_{75}}.$$

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Torelli theorems: odd versions

Theorem (Grushevsky, Salvati Manni 04)

A generic abelian variety of dimension $g \geq 3$ *is uniquely determined by its theta gradients.*

Theorem (Caporaso, Sernesi 03)

A general curve C *of genus* $g \geq 3$ *is uniquely determined by its theta hyperplanes.*

Rem: the second result is not a corollary of the first.

Hyperelliptic case: genus 2 example (Guàrdia 01,07)

Let $[\mu_1], \ldots, [\mu_6]$ be the odd characteristics. Then C admits a symmetric model

$$y^2 = x\left(x - \frac{[\mu_1,\mu_3]}{[\mu_2,\mu_3]}\right)\left(x - \frac{[\mu_1,\mu_4]}{[\mu_2,\mu_4]}\right)\left(x - \frac{[\mu_1,\mu_5]}{[\mu_2,\mu_5]}\right)\left(x - \frac{[\mu_1,\mu_6]}{[\mu_2,\mu_6]}\right).$$

Remarks:

- his theory of symmetric models has nice invariants, nice reduction properties.
- he (also in Shimura's book 98 p.192) shows how to find algebraic differentials: $\Omega_2 = \frac{1}{2i\pi\theta[\varepsilon]} J(\mu_1, \ldots, \mu_g)$ is such that $(dz_1, \ldots, dz_g)\Omega_2$ are algebraic over k if τ comes from an abelian variety A defined over k.

Non hyperelliptic curves of genus 3: Guàrdia 09

Refinement of Riemann model: a smooth plane quartic over k is k-isomorphic to

$$\sqrt{\frac{[b_7 b_2 b_3][b_7 b'_2 b'_3]}{[b_1 b_2 b_3][b'_1 b'_2 b'_3]} X_1 X'_1} + \sqrt{\frac{[b_1 b_7 b_3][b_7 b'_1 b'_3]}{[b_1 b_2 b_3][b'_1 b'_2 b'_3]} X_2 X'_2} + \sqrt{\frac{[b_1 b_2 b_7][b_7 b'_1 b'_2]}{[b_1 b_2 b_3][b'_1 b'_2 b'_3]} X_3 X'_3} = 0$$

where X_i, X'_i are the equations of the bitangents b_i, b'_i.

Ex: Take $A = E^3$ where E has CM by $\sqrt{-19}$ + the unique undecomposable principal polarization. Then $A = \mathrm{Jac}(C)$ where

$$\begin{aligned} C \quad : \quad & U^4 + 2U^3V - 2U^3W + \left(6 - 3i\sqrt{19}\right) U^2V^2 + 18U^2VW + \left(6 + 3i\sqrt{19}\right) U^2W^2 \\ & + \left(5 - 3i\sqrt{19}\right) UV^3 + \left(15 + 3i\sqrt{19}\right) UV^2W + \left(-15 + 3i\sqrt{19}\right) UVW^2 \\ & + \left(-5 - 3i\sqrt{19}\right) UW^3 + \frac{1}{2}\left(3 - 3i\sqrt{19}\right) V^4 + \left(12 + 4i\sqrt{19}\right) V^3W - 30V^2W^2 \\ & + \left(12 - 4i\sqrt{19}\right) VW^3 + \frac{1}{2}\left(3 + 3i\sqrt{19}\right) W^4 = 0. \end{aligned}$$

C descends over $\mathbb{Q}$ as

$$C : x^4 + (1/9)y^4 + (2/3)x^2y^2 - 190y^2 - 570x^2 + (152/9)y^3 - 152x^2y - 1083 = 0$$

Summary

	$g = 1$	$g = 2$	$g \geq 3$ h.	$g = 3$ n.h.	$g > 3$ n.h.
$\theta \rightarrow \tau$	fast	fast conj.	fast conj.	fast conj.	fast conj.
$\tau \rightarrow \theta$	algo fast quotient	algo fast quot.	algo	algo	algo
$C \rightarrow \Omega$	fast	(free) algo	algo	algo	plane model
$C \rightarrow \theta$	fast	algo	algo	algo quot.	theory
$\theta \rightarrow C$	fast	fast	fast	fast	?
$\nabla\theta \rightarrow C$	fast	fast	fast	fast	?

algo: there exists an algorithm but slow.
fast (conj.): there exists a fast (conjectural) algorithm.
quot.: for the quotient of ThetaNullwerte.
theory: the theory is done but no implementation has been done.
?: nothing is done.