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A characterization of balanced episturmian sequences

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Abstract

Sturmian sequences are exactly the aperiodic balanced sequences over a 2-letter alphabet. A sequence s is balanced if for every letter a, the number of a's in any two n-length factors differ by at most 1, for any n. Sturmian sequences are also characterized by their number of n-length factors: they always have $(n+1)$ factors of length n, for every n. For sturmian sequences, the two conditions are equivalent. There are two different generalization of Sturmian sequences for alphabets of cardinality $k \geq 3$. The first one is the set of balanced sequences and the second one is the set of sequences which are closed under reversal and have at most one right special factor for each length. These are called episturmian sequences, and have been extensively studied by Justin and Pirillo. It is interesting to note that the two notions coincide for Sturmian sequences, which are both aperiodic episturmian and aperiodic balanced sequences over a 2-letter alphabet. Nevertheless, when the alphabet has 3 letters or more, the two notions no longer coincide. In particular, episturmian sequences are generally unbalanced over a k-letter alphabet, for $k \geq 3$. Thus, a natural question is to characterize which sequences are both episturmian and balanced. We pursue this question here.

We show that there are exactly three different kinds of balanced episturmian sequences, and among them, only one has different letters frequencies. Moreover, this characterization gives a partial proof of the Fraenkel's conjecture. He has conjectured that for a fixed $k > 2$, there is only one covering of $\mathbb{Z}$ by k Beatty sequences of the form $\lfloor an + b \rfloor$ with two by two distinct rates. In combinatorics on words, the conjecture can be stated as follows: there is only one balanced sequence over

a k-letter alphabet with different letter frequencies, up to letter permutation. In particular, we prove Fraenkel's conjecture in this paper for the class of episturmian sequences. Fraenkel's sequence $(Fr_3)^\omega$ for a 3-letter alphabet is $Fr_3 = 1213121$ and Fraenkel's sequence $(Fr_k)^\omega$ for a k-letter alphabet $\mathcal{A} = \{1, 2, \ldots, k\}$ is $Fr_k = Fr_{k-1}kFr_{k-1}$. The conjecture is verified for $k = 3, 4, 5, 6$ according to the work of Altman, Gaujal, Hordijk and Tijdeman. The case $k = 7$ has been recently settled by Barát and Varjú.

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